From Riemannian metrics to volume form: Heron's formula

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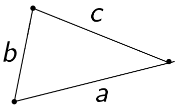
Presented (online) at
Open House in Category Theory
Mexico City, Nov. 2021

Thank you to the organizers !

Heron's formula:



$$Area = \sqrt{t(t-a)(t-b)(t-c)}$$

where

$$t := \frac{a+b+c}{2}$$

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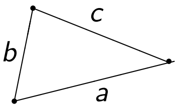

$$Area = \sqrt{t(t-a)(t-b)(t-c)}$$

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By squaring:

$$Area^2 = t(t-a)(t-b)(t-c)$$

Multiplying out gives

$$Area^2 = -16^{-1}(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2)$$

With $\alpha := a^2$, $\beta := b^2$, $\gamma := c^2$, this is the determinant of the symmetric matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \alpha & \beta \\ 1 & \cdot & 0 & \gamma \\ 1 & \cdot & \cdot & 0 \end{bmatrix}$$

(with the dots so as to make the matrix symmetric)
called the (Heron-)Cayley-Menger matrix/determinant C of the triangle

Generalizes from 2-simplices (triangles) to k-simplices in any e.g. metric space: symmetric $(k+2) \times (k+2)$ matrices/determinants with top line and leftmost column $(0, 1, \ldots, 1)$ and 0s down the diagonal. The remaining entries are the square-distances between the vertices.

In a Euclidean space, say R^n, the square-volume of such k simplex is then $-(-2)^k \cdot (k!)^2$ times $\det(C)$. (For $k = 2$, this constant is -16^{-1}.)

With $g(ij)$ denoting the square length of the edge between x_i and x_j in a 3-simplex (tetrahedron) (x_0, x_1, x_2, x_3), the Cayley-Menger matrix is the 5×5 matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & g(01) & g(02) & g(03) \\ 1 & \cdot & 0 & g(12) & g(13) \\ 1 & \cdot & \cdot & 0 & g(23) \\ 1 & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

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The square volume of a k-simplex $X = (x_0, x_1, \ldots, x_k)$ (as defined by Heron-Cayley-Menger) is $(k+1)!$ symmetric.
And it gives 0 when two vertices are equal.

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For $k = 1$, the square volume is square-length g.
For area of triangles: Heron, A.D. 100
For volume of tetrahedra: Piero della Francesca, A.D. 1450

Propaganda: ...

Euclidean geometry: fixed inner product G

Square distance between x and z:

$$(x - z)^T \cdot G \cdot (x - z)$$

(x and z in R^n, written as column vectors)

Heron-Gram comparison (G positive definite)

$$\text{Heron}(x_0, x_1, \ldots, x_k) = \frac{1}{k!^2}\det(Y^T \cdot G \cdot Y) = \frac{1}{k!^2}\text{Gram}_G(Y)$$

(where Y is the $n \times k$ matrix whose jth column is $x_j - x_0$)

Riemannian metric: variable $G(x)$

For each $x \in M \subseteq R^n$, a (positive definite) quadratic form $G(x)$ on R^n, the "metric tensor". $G(x)$ a symmetric $n \times n$ matrix).
square distance between x and z:

$$(x - y)^T \cdot G(x) \cdot (x - y)$$

or?

$$(y - x)^T \cdot G(y) \cdot (y - x)$$

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Taylor expand $G(y)$ from x in direction $y-x$:

$$G(y) = [G(x) + dG(x; y-x) + \ldots]$$

substitute, get

$$(y-x)^T \cdot G(y) \cdot (y-x) = (y-x)^T [\cdot G(x) + dG(x; y-x)] \cdot (y-x),$$

multiply out, get discrepancy with $(x-y)^T \cdot G(x) \cdot (x-y)$

$$(y-x)^T \cdot dG(x; y-x) \cdot (y-x)$$

trilinear in $(y-x)$

The "neigbourhood" relations $\sim_r$ on R^n

For $r = 1$:
$x \sim_1 y$ means

$$\phi(x-y, x-y) = 0 \text{ for all bilinear } \phi : R^n \times R^n \to R$$

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$$x = y \Rightarrow x \sim_1 y \Rightarrow x \sim_2 y \ldots$$

So if $x \sim_2 y$,

$$(x - y)^T \cdot G(x) \cdot (x - y) = (y - x)^T \cdot G(y)(y - x),$$

which is the symmetry needed for the properties of the Heron area formula

r-infinitesimal k-simplices

A k-simplex $X = (x_0, x_1, \ldots, x_k)$ is r-infinitesimal if the vertices are mutual r-neighbours, $x_i \sim_r x_j$.
Terminology: k-square densities on M: R-valued functions defined on set of 2-infinitesimal k-simplices in M, which are $(k+1)!$-symmetric, and normalized).

Theorem (Heron et al) The Heron-Cayley-Menger formula provides out of a 1-square density a k-square density.

SDG:
A 1-square density on $M \subseteq R^n$ is the same as a ps.-Riemannian metric G on M.

("There are enough infinitesimals")

Differential k-forms

Terminology: differential k-form on M: R-valued functions defined on set of 1-infinitesimal k-simplices in M, which are $(k+1)!$-alternating, and normalized).

Example with $k = n$ in R^n:

$$\omega(x_0, \ldots, x_n) := \det(x_1 - x_0, \ldots, x_n - x_0).$$

Also $f \cdot \omega$, i.e.

$$(f \cdot \omega)(x_0, \ldots, x_n) := f(x_0) \cdot \omega(x_0, \ldots, x_n).$$

From k-forms to k-square densities

1) Extend ω so as to be defined also on 2-infinitesimal k-simplices.

2) Square the extended form.

Step 1) is coordinate dependent, but Step 2) eliminates the indeterminateness

Heron_G for variable G

E.g for $k = 3$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & g(01) & g(02) & g(03) \\ 1 & \cdot & 0 & g(12) & g(13) \\ 1 & \cdot & \cdot & 0 & g(23) \\ 1 & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

with $g(ij)$ replaced by

$$(x_i - x_j)^T \cdot G(x_i) \cdot (x_i - x_j)$$

(for short: $G(x_i)(x_i - x_j)$). The Cayley-Menger determinant is a sum of terms, a typical one being the k-ary product ($k = 2$): $g(01) \cdot g(12)$, in full

$$[G(x_0)(x_0 - x_1)] \cdot [G(x_1)(x_1 - x_2)]$$

Write

$$G(x_1) = G(x_0) + dG(x_0; x_1 - x_0) + \text{higher terms}$$

Get

$$[G(x_0)(x_0 - x_1)] \cdot [G(x_1)(x_1 - x_2)]$$

$$= [G(x_0)(x_0 - x_1))] \cdot [G(x_0)(x_1 - x_2) + dG(x_0; x_1 - x_0)(x_1 - x_2) + \ldots]$$

Multiplying out, the product with dG as a factor vanishes: trilinear in $(x_0 - x_1)$, so since $x_0 \sim_2 x_1$,
we may replace the $G(x_1)$ by $G(x_0)$ in the product.

Let $X = (x_0, \ldots, x_k)$ be a 2-infinitesimal k-simplex in R^n

Proposition: In $\mathrm{Heron}_G(X)$ for variable G, all the $G(x_i)$s ocurring may be replaced by the single $G(x_0)$; so

$$\mathrm{Heron}_G(X) = \mathrm{Heron}_{G(x_0)}(X).$$

Volume form characterized in Heron terms

$G(x)$ a variable positive definite quadratic form on $M \subseteq R^n$. Then have function $M \to R$,

$$x \mapsto \frac{1}{n!}\sqrt{\det(G(x)},$$

hence a differential n-form ω on M with value at $(x_0, x_1, \ldots, x_n)$ given by

$$\omega(X) := \frac{1}{n!}\sqrt{\det(G(x_0)} \cdot \det(x_1 - x_0, \ldots, x_n - x_0)$$

for $X = (x_0, \ldots, x_n)$ a 1-infinitesimal n-simplex

May apply the formula to any 2-infinitesimal n-simplex X, and then square; then we get an n-square-density ω^2 and

$$\omega^2(X) = \frac{1}{n!^2} \det G(x_0) \cdot \det(Y)^2$$

(where we write $y_i := x_i - x_0$ $(i = 1, \ldots, n)$ and Y is the $n \times n$ matrix Y with the y_is as columns). But now Y is a square matrix! So continue (product rule for determinants!)

$$= \frac{1}{n!^2} \det(Y^T \cdot G(x_0) \cdot Y),$$

by Gram-Heron comparison for constant quadratic forms like $G(x_0)$,

$$= \text{Heron}_{G(x_0)}(X),$$

and by the Proposition above

$$= \text{Heron}_G(X).$$

Thank you !

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https://math.au.dk/~kock/heron8.pdf