Synthetic theory of geometric distributions

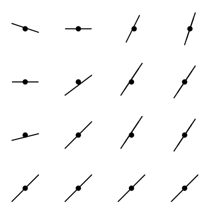
Anders Kock
University of Aarhus

Presented at the 2nd "Seminar on the Interaction between Discrete Geometry and Combinatorics on Words", Luminy 7.-9. June 2010.

I want to thank CIRM (Luminy) and Srecko Brlek for giving me this opportunity to explain some aspects of synthetic differential geometry to an audience of discrete geometers. I hope there will be some cross-fertilization.

The talk will be concerned with some of the geometry of *geometric distributions*, by use of what may be called the (or a) synthetic method.

An example of a geometric distribution is the *slope field* which we draw in the calculus class to give a geometric picture of the problem posed by a first order differential equation $y' = F(x, y)$: through "each" point (x, y) of the plane R^2, we draw a "little (pointed) piece of a line" with slope $F(x, y)$



Then the solutions to the differential equation are functions whose graphs "are made up of" small lines from the slope field.

For differential equations

$$\frac{\partial z}{\partial x} = F(x, y, z), \frac{\partial z}{\partial y} = G(x, y, z),$$

the corresponding geometric distribution consists in: at each point $(x, y, z) \in R^3$, we draw a "little" (pointed) piece of a plane, whose slope in the directions x (resp. y) is given by $F(x, y, z)$ (resp. $G(x, y, x)$). Here the integration problem may have no solution, as one can see geometrically; or, analytically, a necessary condition for solvability is that

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$$

because of equality of the mixed partial derivatives of a solution $z(x, y)$.

We address the question: how "little" should the "little piece" of line (resp. plane) be ? The synthetic method which I shall expound (called synthetic differential geometry, SDG) will give a precise answer. It is an *axiomatic* method. It has as one of its aspects that certain differential-geometric notions and results get a purely combinatorial formulation. This in particular applies to the solvability/integrability of geometric distributions, as exemplified by slope fields, or plane fields, as above. And these combinatorial notions may have geometric interpretations in more discrete contexts.

The axiomatics deals with a commutative ring R, whose elements we call *numbers*, because we should think of R as a model of the number line. Thus, we think of R^2 as the (coordinatized) plane, etc. The essential combinatorial structure on R^n is the kth order *neighbour* relation $\sim_k$ (for each $k = 0, 1, \ldots$). It is defined from the subset

$$D_k(n) := \{(x_1, \ldots, x_n) \mid \text{ any product of } k+1 \text{ of the } x_i\text{s is } 0\}.$$

(Repetitions are allowed.) If R were a field (say, $\mathbb{R}$), $D_k(n)$ would just consist of the zero vector. The axiomatics (to be partially presented), on the other hand, implies that there are sufficiently many nilpotent elements in the number line R, i.e. numbers $x \in R$ such that $x^{k+1} = 0$ for suitable natural number k.

It is clear that the subset $D_k(n) \subseteq R^n$ contains the zero vector $\underline{0}$, and is stable under additive inversion, $\underline{x} \in D_k(n)$ implies $-\underline{x} \in D_k(n)$ (where $\underline{x}$ denotes an n-tuple of numbers). Therefore, the binary relation $\sim_k$ on R^n defined by

$$\underline{x} \sim_k \underline{y} \text{ iff } \underline{x} - \underline{y} \in D_k(n)$$

is reflexive, $\underline{x} \sim_k \underline{x}$, and symmetric, $\underline{x} \sim_k \underline{y}$ implies $\underline{y} \sim_k \underline{x}$.

We say that $\underline{x}$ and $\underline{y}$ are *kth order neighbours* if $\underline{x} \sim_k \underline{y}$. It is useful to have a notation and name for the set of kth order neighbours of $\underline{x}$; it is called the kth order *monad* around $\underline{x}$ and denoted $\mathfrak{M}_k(\underline{x})$. Note that $\mathfrak{M}_k(\underline{0}) = D_k(n)$. The set $D_1(n)$ is also denoted $D(n)$.

The axiomatics to be used has for one of its main pillars the KL axiom scheme, of which one instance is that for any $\underline{x} \in R^n$,

any function $g : \mathfrak{M}_k(\underline{x}) \to R$ *extends uniquely to a polynomial function* $R^n \to R$ *of degree* $\leq k$.

This (family of) axioms implies a large amount of differential calculus. For instance, given a function $f : R^n \to R$; let $g : \mathfrak{M}_k(\underline{x}) \to R$ denote its restriction to $\mathfrak{M}_k(x)$. The unique extension of g to a polynomial function $R^n \to R$ of degree $\leq k$ is then the *Taylor polynomial of* f *of order* k *at* $\underline{x}$.

So the axiomatics also implies that every function $f : R^n \to R$ is differentiable.

We shall almost exclusively be concerned with the 1st order neighbour relation $\sim_1$, which we also denote just $\sim$; likewise, $\mathfrak{M}_1(\underline{x})$ is denoted $\mathfrak{M}(\underline{x})$.

For instance, in dimension 1, we have $\mathfrak{M}(0) = \{d \in R \mid d^2 = 0\}$, This set ($= D_1(1)$) is also denoted D. We note that the axiomatics does not imply that the neighbour relations $\sim_k$ are transitive. In one dimension, and with $k = 1$, transitivity of $\sim$ would be equivalent to the assertion that D is stable under addition, and this does not follow from the axiomatics:

$$(d_1 + d_2)^2 = d_1^2 + d_2^2 + 2d_1d_2 = 0 + 0 + 2d_1d_2,$$

and there is no reason why d_1d_2 should be 0. On the other hand, binomial expansion of $(d_1 + d_2)^3$ reveals that $d_1 + d_2$ belongs to $D_2(1)$.

We leave as an exercise to the reader to expand these latter considerations and prove that in any R^n,

$$\underline{x} \sim_k \underline{y} \text{ and } \underline{y} \sim_l \underline{z} \text{ imply } \underline{x} \sim_{k+l} \underline{z}.$$

Also clearly, if $l \geq k$, we have that $\underline{x} \sim_k \underline{y}$ implies $\underline{x} \sim_l \underline{y}$.

In R^n, we now have the reflexive symmetric relation $\sim$ (i.e. $\sim_1$, the first-order neighbour relation) – a purely combinatorial structure ("graph"). For the higher order relations $\sim_k$, a relevant intuition is that it is the relation: "$\underline{x} \sim_k \underline{y}$ if $\underline{y}$ can be reached from $\underline{x}$ in k (or fewer) "steps" $\sim$."

There is a notion of n-dimensional manifold, which is any set which can "locally" be coordinatized with R^n, in a suitable sense of "locally" (and which is also essentially to be given axiomatically). For any n-dimensional manifold M, the neighbour relations $\sim_k$ can be "imported" from R^n, and are independent of the coordinate charts used for the import. We shall present the following in terms of such manifolds M, rather than R^n, to stress the coordinate free nature of the $\sim_k$ relation, and the combinatorics arising from it.

After this "foundational" discussion, we shall turn to the geometric subject matter: the (geometric) distributions. By "manifold" M, we may here understand any set M equipped with a reflexive symmetric relation $\sim$ (so M is the set of *vertices* of a graph, with *edges*: pairs $\underline{x} \sim \underline{y}$).

Definition 0.1 *A* geometric distribution *on a manifold M is a reflexive symmetric refinement $\approx$ of $\sim$, i.e. for $\underline{x}, \underline{y}$ in M*

$$\underline{x} \approx \underline{y} \textit{ implies } \underline{x} \sim \underline{y}.$$

I want to stress that this is a purely combinatorial notion, and that it may be possible that aspects of this notion may have significance in context in pure graph theory, say, (without any reference to the KL axiomatics); this could in particular apply to the combinatorial notion of *involutive* distribution described below.

First, however, we have to describe how slope fields fit into the picture in the context of SDG. Just as we have the monads $\mathfrak{M}(\underline{x})$ for points $\underline{x} \in M$, as an alternative encoding of the first order neighbourhood structure $\sim$, we have, given a distribution $\approx$, an alternative encoding of it in terms of the "strong" monads $\mathfrak{M}_{\approx}(\underline{x}) := \{\underline{y} \in M \mid \underline{x} \approx \underline{y}\}$. We have $\mathfrak{M}_{\approx}(\underline{x}) \subseteq \mathfrak{M}(\underline{x})$ because $\approx$ is assumed to refine $\sim$.

The contention is that the "little lines" that make up the slope field of a differential equation are exactly the strong monads for a distribution $\approx$. Given a differential equation $y' = F(x, y)$, let us define, for $(x_1, y_1) \sim (x_2, y_2)$ in R^2

$$(x_1, y_1) \approx (x_2, y_2) \quad \text{iff} \quad y_2 - y_1 = F(x_1, y_1) \cdot (x_2 - x_1).$$

This is clearly a reflexive relation, but why is it symmetric ? The equation defining $(x_1, y_1) \approx (x_2, y_2)$ clearly implies

$$(y_1 - y_2) = F(x_1, y_1) \cdot (x_1 - x_2),$$

but $(x_2, y_2) \approx (x_1, y_1)$ would require that the F-factor were $F(x_2, y_2)$, not $F(x_1, y_1)$. Now the difference of these two F-factors can be described exactly: for, as mentioned, differential calculus (Taylor expansion) is available, on basis of the KL axiomatics, and therefore we can describe the discrepancy between the F-factors:

$$F(x_2, y_2) - F(x_1, y_1) = \frac{\partial F}{\partial x}(x_1, y_1) \cdot (x_2 - x_1) + \frac{\partial F}{\partial y}(x_1, y_1) \cdot (y_2 - y_1)$$

by Taylor expansion of F from (x_1, y_1). Note that there are no "higher" terms in the expansion because (x_2, y_2) is a first order neighbour of (x_1, y_1). The discrepancy between what we have and what we want is therefore

$$[\frac{\partial F}{\partial x}(x_1, y_1) \cdot (x_2 - x_1) + \frac{\partial F}{\partial y}(x_1, y_1) \cdot (y_2 - y_1)] \cdot (x_1 - x_2);$$

this, however, is 0, because $(x_1 - x_2) \cdot (x_1 - x_2)$ and $(x_1 - x_2) \cdot (y_1 - y_2$, by virtue of $(x_1, y_1) \sim (x_2, y_2)$. This proves the symmetry of the $\approx$ defined. (Put briefly, the argument may summarized that the discrepancy between what we have and what we want is of *second* order in $(x_1 - x_1, y_1 - y_2)$, and therefore vanishes because the points (x_1, y_1) and (x_2, y_2) are *first* order neighbours.)

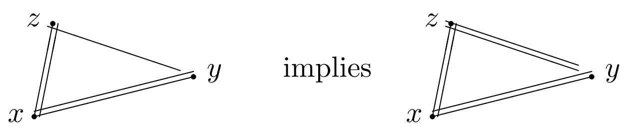
Similarly, the 3-dimensional example given provides a distribution in R^3, whose strong monads are the little "planes" occurring in that example. (Both these distributions are of codimension one; there are of course also distributions in higher codimension, e.g. line fields in R^3.)

Now the possibility of a plane field which cannot be integrated leads to another purely combinatorial notion, giving necessary conditions for integrability, namely

Definition 0.2 *A distribution $\approx$ on a manifold M is* involutive *if for all $\underline{x}, \underline{y}, \underline{z} \in M$,*

$$\underline{x} \approx \underline{y}, \quad \underline{x} \approx \underline{z} \quad \textit{and} \quad \underline{y} \sim \underline{z} \quad \textit{imply} \quad \underline{y} \approx \underline{z}.$$

A relevant picture is the following; single lines indicate the neighbour relation $\sim$, double lines indicate the assumed "strong" neighbour relation $\approx$:



(When differential calculus is available, it can be proved that 1-dimensional distributions are always involutive. The simplest examples of non-involutive distributions occur in dimension 3.)

The next notion is a combinatorial rendering of one of the aspects of an "integral" for a distribution, e.g. the curves that one sketches using a slope field in the plane:

Definition 0.3 *Given a distribution $\approx$ on a manifold M. Then a subset $F \subseteq M$ is an* integral subset *for $\approx$ if for all $\underline{x}$ and $\underline{y}$ in M, we have*

$$\underline{x} \in F, \underline{y} \in F \text{ and } \underline{x} \sim \underline{y} \quad \text{implies} \quad \underline{x} \approx \underline{y}.$$

Clearly a subset of an integral set is integral; in particular, singleton subsets are integral. Obviously, one is interested in *large* integral subsets. On the other hand, integral subsets cannot be too large; for clearly M itself is integral iff $\approx$ and $\sim$ agree.

We invite the reader to prove the following about certain "small" subsets, the strong monads:

Proposition 1 *A distribution $\approx$ is involutive iff all sets of the form $\mathfrak{M}_{\approx}(\underline{x})$ are integral.*

In the applications, one has the notion of when a subset $F \subseteq M$ is *connected*: in differential topology, this could typically mean "path connected"; in the purely graph theoretic world, this could mean: "can be connected by a finite chain of edges of the graph". The following notion is not purely combinatorial, but depend on the choice of the meaning of "connected".

Definition 1.1 *Given a distribution $\approx$ on a manifold M. Then a subset $F \subseteq M$ is a* leaf *for $\approx$ if its integral, connected, and if it is maximal with these properties.*

When differential topology is available, the fundamental Frobenius Thorem says that if $\approx$ is an involutive distribution on M, then every $\underline{x} \in M$ is contained in a unique leaf. (The leaves then form a foliation of M.)

Question: if "connected" is taken in the purely graph theoretic sense, is the conclusion of the Frobenius Theorem hold ?

We are now ready to make some mathematical arguments of purely combinatorial nature.

Proposition 2 *Assume $\approx$ is an involutive distribution on a manifold M, and that the conclusion of the Frobenius Theorem is valid (every $\underline{x}$ is contained in a leaf). Assume that all strong monads $\mathfrak{M}_{\approx}(\underline{x})$ are connected. Then*

1) every $\underline{x}$ is contained in a unique leaf $Q(x)$, and the relation $\underline{y} \equiv \underline{x}$ if $\underline{y} \in Q(\underline{x})$ is an equivalence relation.

2) for any leaf F, and any $\underline{x} \in F$

$$\mathfrak{M}_{\approx}(\underline{x}) = \mathfrak{M}(\underline{x}) \cap F.$$

Proof. The first assertion is a simple exercise using the assumed maximality of leaves. (In the second assertion, therefore, F is $Q(\underline{x})$.) To prove the second assertion: we have $\mathfrak{M}_{\approx}(\underline{x}) \subseteq \mathfrak{M}(\underline{x})$, since $\approx$ refines $\sim$. Also $\mathfrak{M}_{\approx}(\underline{x}) \subseteq F$, since $\mathfrak{M}_{\approx}(\underline{x})$ is integral (by involutivity of $\approx$, cf. Proposition 1) and connected, and because F is maximal with these properties. This proves the inclusion $\subseteq$. Conversely, let $\underline{y} \in \mathfrak{M}(\underline{x}) \cap F$. Then $\underline{y} \sim \underline{x}$ and $\underline{y} \in F$, and by integrality of F, we conclude $\underline{y} \approx \underline{x}$, i.e. $\underline{y} \in \mathfrak{M}_{\approx}(\underline{x})$.

This proves the Proposition. The second part may be seen asserting that the $\mathfrak{M}_{\approx}(\underline{x})$ have *precisely* the right size. As "small lines" in the case of a slope field, they are *contained in* the integral curves, which in turn is made up of these.

Where do distributions come from, analytically? This can also be modelled in a purely combinatorial way by the notion of *combinatorial differential 1-form* which we shall now expound. We consider a manifold $M, \sim$, and the number line R. A differential 1-form ω on M is a law which to any pair of neighbour points $\underline{x} \sim \underline{y}$ in M associates a number, subject to the normalization condition $\omega(\underline{x}, \underline{x}) = 0$. It can be proved on basis of the KL axiomatics, that this implies $\omega(\underline{y}, \underline{x}) = -\omega(\underline{x}, \underline{y})$. (Outside the SDG context, this equation should probably be assumed.)

Note that there is no linearity assumption; in the context of SDG, such ω extends uniquely to a fibrewise linear $T(M) \to R$. This is essentially because a map $\omega : D(n) \to R$ which satisfies the normalization condition $\omega(\underline{0}) = 0$, by KL extends uniquely to a *linear* map $R^n \to R$. So in this context, the combinatorial 1-forms are in bijective correspondence with the classical 1-forms

One says that the 1-form is *closed* if for all 3-tuple of mutual neighbours $\underline{x}, \underline{y}, \underline{z}$ in M, we have

$$\omega(\underline{x}, \underline{y}) + \omega(\underline{y}, \underline{z}) + \omega(\underline{z}, \underline{x}) = 0. \tag{1}$$

A combinatorial 1-form ω on M gives rise to a distribution $\approx$,

$$\underline{x} \approx \underline{y} \quad \text{iff} \quad \omega(\underline{x}, \underline{y}) = 0.$$

Clearly, if ω is closed, the corresponding distribution is involutive; for, if (1) holds, and two of the three terms on the left of (1) are 0, then so is the third. A 1-form may give rise to an involutive distribution without being closed; for, if $f : M \to R$ is a function with invertible values, we have another 1-form ω' given by $\omega'(\underline{x}, \underline{y}) := f(\underline{x}) \cdot \omega(\underline{x}, \underline{y})$, defining the same distribution as ω, but ω' will in general not be closed.

There is also a notion of combinatorial k-form for $k \geq 2$. We shall need it for $k = 2$: a combinatorial 2-form on $(M, \sim)$ is a law θ which to any 3-tuple $\underline{x}, \underline{y}, \underline{z}$ of mutual neighbours in M associates a number $\theta(\underline{x}, \underline{y}, \underline{z}) \in R$, subject to the normalization condition that the value is 0 if two of the three points are equal, (e.g. $\theta(\underline{x}, \underline{y}, \underline{x}) = 0$). In the context of SDG, it can be proved that such θ is alternating in the sense that the value changes sign when two of the input entries $\underline{x}, \underline{y}, \underline{z}$ are interchanged.

Any 1-form ω gives rise to a 2-form $d\omega$ ("coboundary" or "exterior derivative"), defined by the left hand side of (1); so a 1-form is closed iff $d\omega$ is the zero 2-form. Alternatively

$$d\omega(\underline{x}, \underline{y}, \underline{z}) = \omega(\underline{x}, \underline{y}) - \omega(\underline{x}, \underline{z}) + \omega(\underline{y}, \underline{z});$$

this formula is identical to the coboundary formula for cochains in simplicial algebraic topology.

If ω and α are two 1-forms, we may form a 2-form $\omega \wedge \alpha$ by the formula

$$(\omega \wedge \alpha)(\underline{x}, \underline{y}, \underline{z}) = \omega(\underline{x}, \underline{y}) \cdot \alpha(\underline{y}, \underline{z}); \tag{2}$$

again it is a "vanishing to the second order"-argument that this is indeed a 2-form. Note that this formula is identical to the formula for cup products of cochains in simplicial algebraic topology.

Proposition 3 *A sufficient condition for a distribution defined by a 1-form ω to be involutive is that $d\omega = \omega \wedge \alpha$ for some 1-form α.*

Proof. Let $\underline{x}, \underline{y}, \underline{z}$ be mutual neighbours in M, and assume $\underline{x} \approx \underline{y}$ and $\underline{x} \approx \underline{z}$. Then

$$\omega(\underline{y}, \underline{z}) = d\omega(\underline{x}, \underline{y}, \underline{z}) \pm \omega(\underline{x}, \underline{y}) \pm \omega(\underline{x}, \underline{z})$$

by the formula for coboundary. The two last terms vanish, since $\underline{x} \approx \underline{y}$ and $\underline{x} \approx \underline{z}$. So we are left with

$$d\omega(\underline{x}, \underline{y}, \underline{z}) = (\omega \wedge \alpha)(\underline{x}, \underline{y}, \underline{z}) = \omega(\underline{x}, \underline{y}) \cdot \alpha(\underline{y}, \underline{z}) = 0,$$

since $\underline{x} \approx \underline{y}$. So $\omega(\underline{y}, \underline{z}) = 0$, i.e. $\underline{y} \approx \underline{z}$), and this proves the Proposition.

In the context of classical differential geometry, involutivity of a (codimension 1) distribution is usually defined either in terms of a 1-form that represents it, or by an $(n-1)$-tuple of vector fields subordinate to it. In essence, the differential-form formulation is the sufficient condition in the Propsition above.

I would like to comment explicitly on a question raised during the presentation of my talk on June 7, namely why it is that the formula for wedge product of forms given here does not involve the standard *alternating sum* needed to to guarantee the alternating property. The point is the first order neighbour relation $\sim$ on R^n contains a little piece of nice "algebraic magic", which I shall illustrate in dimension 2. Consider namely a 3-tuple of mutual neighbours in R^2, and for simplicity, let one of them be the zero vector. The other two points then make up the columns of a 2×2-matrix $[x_{ij}]$. Since the first column is in $D(2)$ (being neighbour of 0) $x_{i1} \cdot x_{j1} = 0$ for all i, j, and similarly for the second column. But now the two columns are also neighbours of each other, and this implies

$$(x_{11} - x_{12}) \cdot (x_{21} - x_{22}) = 0.$$

Multiplying out, we get four terms, two of which vanish (those products where the second indices agree), and we are left with

$$-x_{11} \cdot x_{22} - x_{12} \cdot x_{21}$$

which therefore is 0. Equivalently,

$$-x_{12} \cdot x_{21} = x_{11} \cdot x_{22},$$

but this means that the two terms in the determinant of the matrix are equal, so that the determinant of the matrix (except for a factor 2!) equals $x_{11} \cdot x_{22}$; no alternating sum needed ! More generally, if one has an n-tuple of vectors in R^n, which are neighbours of 0, and are mutual neighbours, then the determinant of the $n \times n$-matrix formed by these vectors is $n!$ times the product of the diagonal entries (in more detail, all the terms in the sum of $n!$ terms that form the determinant, *agree* with this product).

There is a quite extensive literature on varius aspects of SDG; including in some of my papers. They can be downloaded from my home page,
http://home.imf.au.dk/kock/ .

There is an exposition of many geometric notions deriving from the neighbourhoods of the diagonal, and an extensive biblography, in my recent book

Anders Kock, *Synthetic Geometry of Manifolds*, Cambridge Tracts in Mathematics 180, Cambridge University Press 2010.