

# Infinitesimal Deformations of Complete Vector Fields are Complete\*

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Recall that a vector field  $X$  on a smooth manifold  $M$  is *complete* if it generates a global flow  $\xi : M \times \mathbb{R} \rightarrow M$ . If  $X_\epsilon$  is a smoothly parametrized family of vector fields ( $\epsilon \in \mathbb{R}$ ), it may happen that  $X_0$  is complete but all  $X_\epsilon$  with  $\epsilon \neq 0$  are incomplete. This is the case for instance with the family of vector fields

$$\epsilon \cdot x^2 \cdot \frac{\partial}{\partial x}$$

on  $\mathbb{R}$ .

We shall prove, however, that in the context of synthetic differential geometry, if  $X_0$  is complete, then so is  $X_\epsilon$  for any  $\epsilon$  with  $\epsilon^2 = 0$ .

We presuppose some general notions from synthetic differential geometry (SDG), cf. [3], but also some more particular results [5]. The notion ‘connection in a groupoid’, ‘displacement’ and ‘displacement field’ that we use, are essentially due to Ehresmann, cf. [1].

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## 1 Generalities concerning vector fields and flows

Let  $D$  be an object equipped with a  $0 \in D$  and an inversion ‘minus’  $D \rightarrow D$ . An *action* of  $D$  on an object  $M$  is a map

$$X : M \times D \rightarrow M$$

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with

$$(1.1) \quad X(m, 0) = m = X(X(m, d), -d)$$

$\forall m \in M, \forall d \in D$ . We shall work in the context of SDG, with  $D = \{d \in D \mid d^2 = 0\}$ ; then  $X$  is a vector field on  $M$ , (this conception is due to Lawvere). If  $M$  is infinitesimally linear, the second equation in (1.1) follows from the first.

If  $(E, Y)$  and  $(M, X)$  are objects equipped with vector fields, we say that a map  $\pi : E \rightarrow M$  is a *morphism of vector fields* if  $\pi(Y(c, d)) = X(\pi(c), d) \forall c \in E \forall d \in D$ ; one also says that the ‘vector field  $Y$  is *projectable* to the vector field  $X$  via  $\pi$ ’.

For such  $\pi$ , we get for each  $m \in M$  and  $d \in D$  an invertible mapping (a ‘fibre infinitesimal transformation’) from  $E_m$ , the fibre over  $m$ , to  $E_{X(m, d)}$ , the fibre over  $X(m, d)$ , namely the restriction of  $Y(-, d) : E \rightarrow E$  to  $E_m$ . This map  $E_m \rightarrow E_{X(m, d)}$  may be viewed as an arrow  $m \rightarrow X(m, d)$  in the groupoid  $\text{Full}(\pi)$  consisting of invertible maps between the fibres of  $\pi$ . If  $\Psi \subset \text{Full}(\pi)$  is a subgroupoid, it makes sense to ask that all the maps

$$Y(-, d) : E_m \rightarrow E_{X(m, d)}$$

belong to the subgroupoid  $\Psi$ , in which case we shall say that the morphism  $\pi$  of vector fields is *subordinate* to  $\Psi$ . (In the terminology of §4 below, the fibre infinitesimal transformations form a ‘‘displacement field’’ in the groupoid  $\Psi$ .)

The line  $R$  in SDG carries a canonical vector field  $\frac{\partial}{\partial x}$ , namely  $(x, d) \mapsto x + d$  for  $x \in R, d \in D$ . We let  $I \subset R$  be any subset containing 0 and stable under the action of  $\frac{\partial}{\partial x}$ , e.g.  $I = R, I = [0, 1]$  or  $I = D_\infty$ . We shall assume that the vector fields  $(M, X)$  we consider have the property that for any  $m \in M$ , there is at most one morphism of vector fields

$$(I, \frac{\partial}{\partial x}) \rightarrow (M, X)$$

with  $0 \mapsto m$ . Such a morphism is adequately called a *streamline* for  $X$  through  $m$ .

For  $s, t \in I$ , we write  $s \sim t$  if  $s - t \in D$ . This is a reflexive symmetric relation.

Let  $I_{(1)} \subset I \times I$  be the set of pairs  $(s, t)$  with  $s \sim t$ . Let  $G$  be a group. A  $G$ -valued 1-form on  $I$  is a map

$$\omega : I_{(1)} \rightarrow G$$

with  $\omega(t, t) = e$  and  $\omega(s, t) = \omega(t, s)^{-1} \forall s \sim t$ . We say that the group  $G$  admits integration if for any such 1-form  $\omega$ , there exists a unique  $f : I \rightarrow G$  with  $f(0) = e$  and

$$\omega(s, t) = f(t) \cdot f(s)^{-1} \quad \forall s \sim t.$$

This holds in the models of SDG if  $G$  is a Lie group and  $I$  an interval on  $R$ , cf. [6] and [4], but fails for large groups like  $\text{Diff}(R)$  (= the object of bijective maps  $R \rightarrow R$ , cf. [4]), a fact closely related to the existence of incomplete vector fields.

Recall (from [5], say) that a groupoid  $\Psi$  over  $M$  is called *transitive* if for any  $m, n \in M$ , there is at least one arrow  $m \rightarrow n$  in  $\Psi$ . In this case, all the isotropy groups  $\Psi(m, m)$  are isomorphic.

**Theorem 1.1** *Let  $\pi : (E, Y) \rightarrow (M, X)$  be a morphism of vector fields, subordinate to a groupoid  $\Psi \subset \text{Full}(\pi)$ , and assume  $\Psi$  is transitive and that its isotropy groups admit integration. Let  $c \in E$ . Then any streamline  $\xi : I \rightarrow M$  for  $X$  through  $\pi(c)$  lifts to a streamline  $\eta$  for  $Y$  through  $c$ .*

**Proof.** Let  $\xi^*(\Psi)$  be the full image of the groupoid  $\Psi$  along  $\xi$  (i.e. the hom set  $\xi^*(\Psi)(s, t)$  can be identified with the hom set  $\Psi(\xi(s), \xi(t))$  of  $\Psi$ ). We get a connection  $\nabla$  ([5]) on the groupoid  $\xi^*(\Psi)$ , namely

$$\nabla(t, t + d) := Y(-, d) |_{E_{\xi(t)}} : E_{\xi(t)} \rightarrow E_{\xi(t+d)}$$

which by assumption is an arrow  $\xi(t) \rightarrow \xi(t + d)$  in  $\Psi$ , hence an arrow in  $\xi^*(\Psi)$  from  $t$  to  $t + d$ .

We now apply Theorem 4.1 of [5], with the identity map  $I \rightarrow I$  as  $f$ , to get a functor

$$\overline{\nabla} : I \times I \rightarrow \xi^*(\Psi)$$

over  $I$  extending  $\nabla$  (where  $I \times I = \pi_0 I$  is the codiscrete groupoid over  $I$ );  $\overline{\nabla}$  satisfies

$$\nabla(t, t + d) \circ \overline{\nabla}(s, t) = \overline{\nabla}(s, t + d)$$

$\forall s, t \in I \forall d \in D$ .

Now define  $\eta : I \rightarrow E$  by

$$\eta(s) := \overline{\overline{\nabla}}(0, s)(c),$$

where  $\overline{\overline{\nabla}} : I \times I \rightarrow \Psi$  is the functor obtained from  $\overline{\nabla}$  by composing with the canonical functor  $\xi^* \Psi \rightarrow \Psi$ . Then  $\pi \circ \eta = \xi$ , for

$$\pi(\eta(s)) = \partial_1(\overline{\overline{\nabla}}(0, s)) = \xi(s)$$

since the codomain of  $\overline{\nabla}(0, s)$  is  $s$ . So  $\eta$  lifts  $\xi$ . Also clearly  $\eta(0) = c$ . To see that  $\eta$  is a streamline for  $Y$ , note that we have

$$\begin{aligned} \eta(s+d) &= \overline{\overline{\nabla}}(0, s+d)(c) \\ &= \overline{\overline{\nabla}}(s, s+d) \circ \overline{\overline{\nabla}}(0, s)(c) \quad (\overline{\overline{\nabla}} \text{ being a functor}) \\ &= \overline{\overline{\nabla}}(s, s+d)\eta(c) \\ &= Y(-, d)(\eta(c)), \end{aligned}$$

since

$$\overline{\overline{\nabla}}(s, s+d) = \overline{\nabla}(s, s+d) = \nabla(s, s+d),$$

which is the fibre infinitesimal transformation  $Y(-, d) : E_{\xi(s)} \rightarrow E_{\xi(s+d)}$ .

## 2 Reduction to the fibrewise affine groupoid

We consider as in §1 a vector field  $X : M \times D \rightarrow M$  on an object  $M$ , and a deformation  $H$  of it, with parameter space for the deformation a pointed object  $(D', 0)$  (which ultimately will be taken to be  $D$ ). Thus, the data we consider is a map

$$M \times D \times D' \xrightarrow{H} M$$

with

$$(2.1) \quad H(m, d, 0) = X(m, d) \quad \forall m \in M, d \in D$$

and

$$(2.2) \quad H(m, 0, \epsilon) = m \quad \forall m \in M, \epsilon \in D'.$$

We write  $X_\epsilon$  for  $H(-, -, \epsilon)$ . Thus  $X_0 = X$ , by (2.1), and  $X_\epsilon$  is a vector field by (2.2), provided  $M$  is infinitesimally linear, which we henceforth assume (in fact in the ‘strong’ sense, cf. [7]).

Let  $M^{D'}$  denote the set of maps  $D' \rightarrow M$ . We equip this set with a vector field

$$\widetilde{H} : M^{D'} \times D \rightarrow M^{D'}$$

given by

$$\widetilde{H}(\tau, d)(\epsilon) := X_\epsilon(\tau(\epsilon), d) = H(\tau(\epsilon), d, \epsilon)$$

for  $\tau \in M^{D'}$ ,  $d \in D$ ,  $\epsilon \in D'$ . (Note that by (2.2)

$$\widetilde{H}(\tau, 0)(\epsilon) = H(\tau(\epsilon), 0, \epsilon) = \tau(\epsilon),$$

so  $\widetilde{H}(\tau, 0) = \tau$ , so  $\widetilde{H}$  is indeed a vector field,  $M^{D'}$  being infinitesimally linear since  $M$  is).

Consider, for fixed  $\epsilon \in D'$ , the map ‘evaluation at  $\epsilon$ ’

$$(2.3) \quad M^{D'} \xrightarrow{\text{ev}_\epsilon} M.$$

We note that this map is a morphism of vector fields

$$(M^{D'}, \widetilde{H}) \rightarrow (M, X_\epsilon);$$

for

$$\begin{aligned} \text{ev}_\epsilon(\widetilde{H}(\tau, d)) &= \widetilde{H}(\tau, d)(\epsilon) \\ &= X_\epsilon(\tau(\epsilon), d) \\ &= X_\epsilon(\text{ev}_\epsilon(\tau), d). \end{aligned}$$

Let us denote the map  $\text{ev}_0 : M^{D'} \rightarrow M$  by  $\pi$ ; so in particular

$$(2.4) \quad (M^{D'}, \widetilde{H}) \xrightarrow{\pi} (M, X)$$

is a morphism of vector fields.

We now take  $D' = D$ , so  $M^{D'} = M^D \xrightarrow{\pi} M$  is the tangent bundle of  $M$ . Since  $M$  is infinitesimally linear, the fibres of this map are  $R$ -modules. So it makes sense to talk about  $R$ -linear and  $R$ -affine maps from one fibre to another (affine = linear + constant).

**Lemma 2.1** *The fibre infinitesimal transformations of the vector field morphism (2.4) (for  $D = D'$ ) are affine maps.*

**Proof.** Let  $m \in M$ ,  $d \in D$ , and consider the corresponding fibre infinitesimal transformation

$$(2.5) \quad (M^{D'})_m \xrightarrow{\phi} (M^{D'})_{X(m,d)}.$$

Let  $\gamma \in (M^{D'})_{X(m,d)}$  be given by

$$\gamma(\epsilon) = H(m, d, \epsilon).$$

We prove that  $\phi$  minus the constant  $\gamma$  is a linear map, i.e. that the map  $\phi_0$  defined by

$$\phi_0(\tau) := \phi(\tau) - \gamma$$

depends in a linear way on  $\tau$ , so that  $\phi$  itself is affine. Now since  $M$  is infinitesimally linear, the  $R$ -modules  $(M^D)_n$  satisfy the ‘vector form of the Kock-Lawvere axiom’, as observed by Lavendhomme [9], so that linearity of  $\phi_0$  is equivalent, by [3] Prop. I.10.2, to homogeneity,

$$\phi_0(\lambda \cdot \tau) = \lambda \cdot \phi_0(\tau) \quad \forall \lambda \in R.$$

So consider  $\phi(\tau) - \gamma \in (M^{D'})_{X(m,d)}$ . To form this difference, we use the standard recipe for adding tangent vectors on infinitesimally linear objects (cf. [3]), namely ‘construct a map  $D(2) \rightarrow M$  and restrict it along the diagonal’. In the present case, we construct a map  $l : D(2) \rightarrow M$  given by

$$l(\epsilon_1, \epsilon_2) := H(\tau(\epsilon_1), d, \epsilon_1 - \epsilon_2)$$

( $d \in D$  being fixed, and  $\epsilon_1 - \epsilon_2 \in D$  since  $(\epsilon_1, \epsilon_2) \in D(2)$ .) We clearly have

$$l(\epsilon, 0) = H(\tau(\epsilon), d, \epsilon) = \phi(\tau)(\epsilon)$$

and

$$l(0, \epsilon) = H(\tau(0), d, -\epsilon) = (-\gamma)(\epsilon)$$

so that

$$l(\epsilon, \epsilon) = (\phi(\tau) - \gamma)(\epsilon)$$

or

$$\begin{aligned} \phi_0(\tau)(\epsilon) &= (\phi(\tau) - \gamma)(\epsilon) \\ &= H(\tau(\epsilon), d, \epsilon - \epsilon) = H(\tau(\epsilon), d, 0). \end{aligned}$$

We thus have, for  $\lambda \in R$

$$\begin{aligned}\phi_0(\lambda\tau)(\epsilon) &= H((\lambda \cdot \tau)(\epsilon), d, 0) \\ &= H(\tau(\lambda \cdot \epsilon), d, 0) \\ &= \phi_0(\tau)(\lambda \cdot \epsilon) \\ &= (\lambda \cdot \phi_0(\tau))(\epsilon),\end{aligned}$$

proving homogeneity of  $\phi_0$ , thus proving  $\phi$  affine.

### 3 The theorem

We now assume that  $M$  is a manifold, of dimension  $n$ , say. This implies that the tangent bundle is a vector bundle, so that the groupoid of fibrewise linear maps is transitive. Then the groupoid  $\Psi$  of fibrewise affine maps is transitive as well. The isotropy-groups of  $\Psi$  are Lie groups, namely the  $n^2 + n$  dimensional group of affine maps  $R^n \rightarrow R^n$ . So they admit integration. If  $H : M \times D \times D' \rightarrow M$  is a deformation of a vector field, as in §1, we therefore get, by Lemma 2.1 and Theorem 1.1 that the morphism (2.4) of vector fields has the property that streamlines for  $X$  lift to streamlines for  $\widetilde{H}$ ; we can then prove:

**Theorem 3.3** *Let  $M$  be a manifold, and let  $\{X_\epsilon \mid \epsilon \in D\}$  a deformation of a complete vector field  $X = X_0$ . Then each  $X_\epsilon$  is complete.*

**Proof.** Let  $m \in M$ , and let  $\epsilon \in D$  be fixed. By assumption, there is a streamline  $\xi : I \rightarrow M$  for  $X$  through  $m$ . By the above remarks, it lifts to a streamline  $\eta$  for  $\widetilde{H}$  through the element  $\tau : D \rightarrow M$  with constant value  $m$ . Compose  $\eta$  with  $\text{ev}_\epsilon : M^D \rightarrow M$  which is a morphism of vector fields  $(M^D, \widetilde{H}) \rightarrow (M, X_\epsilon)$ ; so we get a morphism of vector fields

$$(I, \frac{\partial}{\partial x}) \xrightarrow{\eta} (M^D, \widetilde{H}) \xrightarrow{\text{ev}_\epsilon} (M, X_\epsilon)$$

through  $\text{ev}_\epsilon(\eta(0)) = \text{ev}_\epsilon(\tau) = m$ . Since thus through each  $m$ , there is a streamline for  $X_\epsilon$ ,  $X_\epsilon$  is a complete vector field.

## 4 Complete displacement fields

The argument of §1 could have been presented in greater generality by the introduction of the notion of ‘displacement’ and ‘displacement field’. We sketch how this would go.

Let  $\Psi$  be a groupoid over  $M$ . A *displacement at  $m \in M$*  is a map  $g : D \rightarrow \Psi$  satisfying

$$\begin{aligned} g(0) &= \text{id}_m \\ \partial_0(g(d)) &= m \quad \forall d \in D. \end{aligned}$$

A *displacement field  $G$*  on  $\Psi$  is a law which to each  $m \in M$  associates a displacement at  $m$ , so  $G : M \times D \rightarrow \Psi$  satisfies

$$\begin{aligned} G(m, 0) &= \text{id}_m \quad \forall m \in M \\ \partial_0(G(m, d)) &= m \quad \forall m \in M \quad \forall d \in D. \end{aligned}$$

If  $G$  is a displacement field on  $\Psi$ , we get a vector field  $\partial_1 G$  on  $M$  by

$$(\partial_1 G)(m, d) = \partial_1(G(m, d)).$$

Inspecting the proof of Theorem 1.1, we see that we essentially might as well have proved

**Theorem 4.1** *Assume that  $\Psi$  is transitive and that its isotropy groupoids admit integration. Then if  $\partial_1 G$  is a complete vector field on  $M$ ,  $G$  itself is a complete displacement field on  $\Psi$  in the sense that there exists a map*

$$\bar{G} : M \times I \rightarrow \Psi$$

*satisfying*

$$\begin{aligned} \bar{G}(m, 0) &= \text{id}_m \quad \forall m \in M \\ \partial_0(\bar{G}(m, d)) &= m \quad \forall m \in M \quad \forall d \in D \end{aligned}$$

*and*

$$\bar{G}(m, s + d) = G(\partial_1(\bar{G}(m, s), d)) \circ \bar{G}(m, s)$$

$$\forall m \in M \quad \forall s \in I \quad \forall d \in D.$$



**Proof** (sketch). Let  $\xi : I \rightarrow M$  be a streamline for  $\partial_1 G$  through  $m$ . As in the proof of Theorem 1.1, obtain a functor  $\overline{\nabla} : I \times I \rightarrow \Psi$  over  $\xi$ ; then

$$\overline{G}(m, s) := \overline{\nabla}(0, s)$$

will satisfy the conditions.

This result may be seen as a paraphrasing of the Corollary 1 of §33 in [8],

For the viewpoint of a displacement field as an action on  $\Psi$  by a small pointed groupoid  $\mathbb{D}$ , see [2].

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