Aarhus Universitet
Matematisk Institut

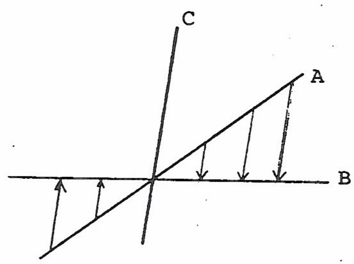
October 1974

Preprint Series
1974/75 No. 7.

THE CATEGORY ASPECT OF PROJECTIVE SPACE

Anders Kock

Let $\mathbb{P}(V)$ denote the set of 1-dimensional linear subspaces of the vector space V (over a fixed field k). If A and B are different elements in $\mathbb{P}(V)$, then a bijective linear map from A to B may be identified with a third element $C \in \mathbb{P}(V)$, since such a bijective linear map is "projection from A to B in a unique direction C":



Conversely, a C coplanar* with, but different from A and B, determines a bijective linear map from A to B in this way. (It can be described as a composite of two Noether isomorphisms:

$$A = A/_{A\cap C} \simeq A+C/_{C} = B+C/_{C} \simeq B/_{B\cap C} = B.)$$

This fact makes the full subcategory of $\text{Vect}(k)$ (= the category of vector spaces over k, and bijective linear maps) determined by the 1-dimensional subspaces of V into a category richer in structure than the equivalent category of <u>all</u> 1-dimensional vectorspaces over k. Let us call this category $\overline{\mathbb{P}}(V)$.

---

* By C <u>coplanar</u> with A and B, we mean $C \subseteq A+B$. Thinking of A, B, and C as points rather than as lines, the natural word would be <u>colinear</u>.

In this note, we explore this structure, and show how some geometric properties or structures of $\mathbb{P}(V)$ may be expressed in terms of the category (or groupoid) structure. In particular, "cross-ratio" appears as a special case of composition. Desargues' theorem appears as the statement that composition is associative.

The above fact about bijective linear maps $A \to B$ (for $A \neq B$) being determined by directions C gives rise to the following description of $\overline{\mathbb{P}}(V)$: the set of objects of $\overline{\mathbb{P}}(V)$ is $\mathbb{P}(V)$; for $A \neq B$ any two different objects,

$$\hom(A,B) = \mathbb{P}(A+B) \setminus \{A,B\};$$

also

$$\hom(A,A) = k \setminus \{0\}.$$

(This is a category with non-disjoint hom sets; see [2], I.8.)

By identifying a $C \in \mathbb{P}(A+B) \setminus \{A,B\}$ with a bijective linear map $A \to B$, as above, and identifying a scalar $t \in k \setminus \{0\}$ with the bijective linear map $A \to A$ consisting in multiplication by t, we get the compositions in the category $\overline{\mathbb{P}}(V)$ by transfer from composition of bijective linear maps, via the identification. Some properties of $\overline{\mathbb{P}}(V)$ are then immediate:

(1) any composite of form

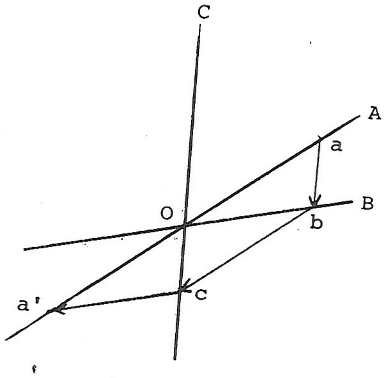
$$A \overset{C}{\to} B \overset{C}{\to} A$$

equals $1 \in \hom(A,A) = k \setminus \{0\}$. ($A$, B and C assumed mutually different, but coplanar.)

(2) any composite of form

$$A \overset{C}{\to} B \overset{A}{\to} C \overset{B}{\to} A$$

equals $-1 \in \hom(A,A)$ (A, B and C assumed mutually different but coplanar). To see this, consider the figure



where a is an arbitrary non-zero point on A. Then we let $b \in B$ be determined by $[ab]$ being parallel to C; we let $c \in C$ be determined by $[bc]$ being parallel to A; and let $a' \in C$ be determined by $[ca']$ being parallel to B. Then $Oabc$ is a parallellogram, and $Obca'$ is a parallellogram, hence

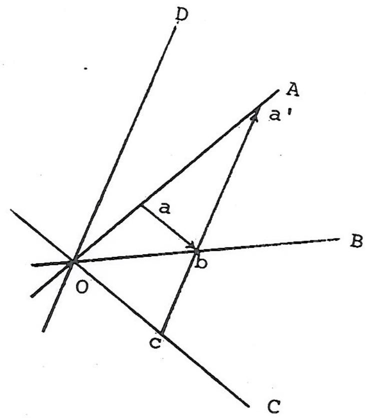
$$\vec{Oa} = \vec{cb} = \vec{a'O} ,$$

whence $a' = -a$.

(3) Let A, B, C, D be different, but coplanar. The composite

(*) $$A \overset{C}{\to} B \overset{D}{\to} A$$

equals the cross-ratio $(C,D;A,B)$. To see this, consider the figure



Let $a \in A$ be different from 0. Let $b \in B$ be determined by $[ab]$ being parallel to C and let $c \in C$ and $a' \in A$ be determined by $[bc]$ and $[ba']$ being parallel to D.

The cross-ratio $(C,D;A,B)$ is then given by $\frac{ca'}{cb'}$ because $[ca']$ is parallel to D. Since $[ab]$ is parallel to C, we get by similar triangles that

$$\frac{Oa'}{Oa} = \frac{ca'}{cb} = (C,D;A,B).$$

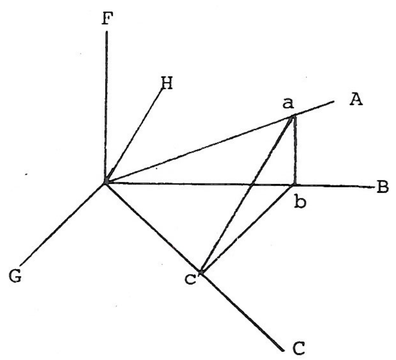
On the other hand, $Oa' : Oa$ is the scalar corresponding to that linear map which sends a to a', which is what the composite $*$ does by construction of the figure.

In both (1), (2), and (3) above, we have been considering sets of coplanar lines, which means that we have been considering the category structure of projective lines $\mathbb{P}(V)$ with V 2-dimensional. For a V of arbitrary dimension, the construction of a composite

$$** \qquad A \overset{F}{\to} B \overset{G}{\to} C$$

with A, B and C not coplanar can be carried out entirely in terms of the combinatorial structure (incidence relations) on $\mathbb{P}(V)$, without reference to V itself. For, if A, B and C are not coplanar, and the composite $**$ makes sense, then F is a 1-dimensional coplanar with A and B, and G a 1-dimensional subspace coplanar with B and C; consider

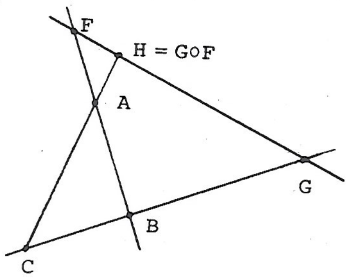
***



where $a \in A$ is non-zero and where $b \in B$ and $c \in C$ are constructed such that [ab] is parallel to F, [bc] parallel to G. Then clearly [ac] is parallel to the plane F+G. But also, [ac] is parallel to the plane A+C, so that the direction of [ac] (which equals the composite **) is given as the unique line of intersection of F+G with A+C.

Now A, B, C, F, G, and H all lie in a certain 3-dimensional subspace W of V. We get a plane picture of all the 1-dimensional linear subspaces of W by intersecting them with some affine plane A in W which does not go through 0; 1-dimensional linear subspaces give points in A, 2-dimensional linear subspaces give lines in A. This is the familiar affine picture of $\mathbb{P}(k^3)$. In this, *** becomes

****



and H is determined by pure incidence operations: the intersection of "line" FG with "line" AC.

Now, composition in a category is associative. Since we have been able to interpret a certain incidence-operational construction as composition in a category, we will by associativity know that two different incidence-operational constructions yield the same result; this will then give us a geometric theorem. We claim that it is the theorem of Desargues (dualized) which we get this way. To see this, consider the situation

(5) $$A \xrightarrow{F} B \xrightarrow{G} C \xrightarrow{H} D.$$

The two composites

$$A \overset{FG}{\rightarrow} C \overset{H}{\rightarrow} D$$

and

$$A \overset{F}{\rightarrow} B \overset{GH}{\rightarrow} D$$

agree; call it FGH. The triangles formed of

A, F, FG

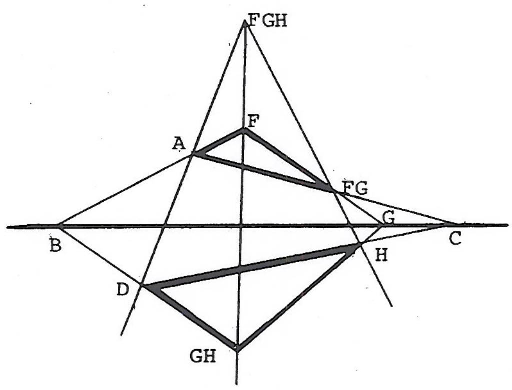
and

D, GH, H

respectively, are a pair of Desargues triangles (see figure below). For consider the line given by the colinear points G, B, C:

[A,F]	intersects	[D,GH]	in	B
[A,FG]	intersects	[D,H]	in	C
[F,FG]	intersects	[GH,H]	in	G,

by construction, and any pair of Desargues triangles can be viewed as a diagram (5). Associativity of composition gives that [AD], [F,GH], and [FG,H] are concurrent in FGH. This is the desired conclusion of the Desargues theorem.



One may in fact formulate this observation for a synthetically given projective plane as follows: The composition constructed by means of **** is associative if and only if the plane is Desarguesian.

The idea of points in a geometric structure playing the role of morphism between two other points (like F in the above picture **** playing the role of morphism from A to B) is due to Lawvere, who in [1] used it for points in a convex set rather than in a projective space. He also gave the construction of composition for the case where this can be carried out by pure incidence constructions. An analytic treatment of the case where the pure incidence constructions do not work (i.e. on the projective line), can be given as follows. We consider $\mathbb{P}(k^2)$.

If $(a_1,a_2) \neq (0,0) \in k^2$, we write $[a_1,a_2]$ for the 1-dimensional linear subspace determined by it. If all pairs in the diagram

$$(a_1,a_2) \xrightarrow{(f_1,f_2)} (b_1,b_2) \xrightarrow{(g_1,g_2)} (c_1,c_2) \tag{6}$$

are non-proportional, then the composite map is given by that 1-dimensional linear subspace of k^2 which contains the set of solutions (k_1,k_2) to the equation

$$\frac{\begin{vmatrix} k_1 & a_1 \\ k_2 & a_2 \end{vmatrix}}{\begin{vmatrix} k_1 & c_1 \\ k_2 & c_2 \end{vmatrix}} = \frac{\begin{vmatrix} f_1 & a_1 \\ f_2 & a_2 \end{vmatrix}}{\begin{vmatrix} f_1 & b_1 \\ f_2 & b_2 \end{vmatrix}} \cdot \frac{\begin{vmatrix} g_1 & b_1 \\ g_2 & b_2 \end{vmatrix}}{\begin{vmatrix} g_1 & c_1 \\ g_2 & c_2 \end{vmatrix}} .$$

If $(c_1,c_2) = (a_1,a_2)$, then the composite in (6) is given by the scalar on the right hand side in this equation. Also the other types of composites are given in this sort of way.

REFERENCES

1. F. W. Lawvere, Metric spaces, generalized logic, and closed categories. Perugia 1973.

2. S. Mac Lane, Categories for the working mathematician, Springer, 1972

/KG