

Barycentric calculus, and the log-exp bijection

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Presented (online) at
“Tangent Categories and their Applications”
BIRS, Banff, June 2021

Thank you to the organizers !

Message:

Affine geometry on an infinitesimal scale
is possible in any good space

Need to explain the three key phrases. The first is standard: **affine geometry** is the geometry whose calculational aspect is barycentric calculus: barycentric calculus = the calculus of linear combinations where the sum of the coefficients is 1, e.g. formation of parallelograms.

What is a **good space**? Wrong question! It is an object in a **good category of spaces**.

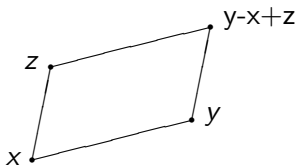
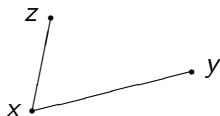
Examples (good in different ways - the word “good” has many meanings): e.g. a tangent category, or a topos model of SDG.

Also: the category \mathcal{A}^{op} of affine schemes

It is in the latter that I shall present a notion of **infinitesimal scale**, by defining a reflexive symmetric relation $f \sim_1 g$ on each of its hom sets.

The dual of the category of affine k -schemes is the category of commutative k -algebras, and most of the following deals with that category. For those of you who attended the talks of Finster, Joyal (and others), this is like the relationship between the category of toposes (the geometric aspect), and the category of logoses (the algebraic aspect).

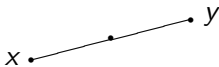
Barycentric calculus



$$1 \cdot y + (-1) \cdot x + 1 \cdot z$$

A **linear** combination where the sum of the coefficients is 1, also called an **affine** combination.

Similarly, given two points,



$$\text{midpoint} = \frac{1}{2}x + \frac{1}{2}y$$

More generally, can form

$$(1 - t) \cdot x + t \cdot y$$

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(\mathcal{A} = the category of commutative k -algebras)
2. Some affine combinations in $\text{hom}_{\mathcal{A}}(B, C)$
3. Geometric meaning in \mathcal{A}^{op} = the category of affine schemes over k

“Much like the theory of affine schemes and commutative rings, the theory of (higher) topoi leads a dual life: one algebraic and one geometric. . .” cf. Finster’s abstract.

4. log: from neighbour pairs to tangent vectors

1. (First order) neighbours

Neighbours in the category \mathcal{A} of commutative k -algebras:

For each B and C in \mathcal{A} , and f and g parallel maps:

$$B \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} C,$$

we say that they are (1st order) neighbours if for all $x \in B$:

$$f \sim_1 g :$$

$$(f(x) - g(x))^2 = 0$$

It is a reflexive and symmetric relation. It is not transitive.

Slightly stronger:

for all $x, y \in B$

$$(f(x) - g(x)) \cdot (f(y) - g(y)) = 0$$

So $f \sim_1 g$, is defined by validity, for all x and y in the domain B , of

$$(f(x) - g(x)) \cdot (f(y) - g(y)) = 0$$

This relation in $\text{hom}_{\mathcal{A}}(B, C)$ is equivalent to

$$f(x) \cdot f(y) + g(x) \cdot g(y) = f(x) \cdot g(y) + g(x) \cdot f(y)$$

No minus occurs !

(So it works for commutative **rigs** as well)

The value of f and g on a pair of elements in B gives a 2×2 matrix with entries from C

	x	y
f	$f(x)$	$f(y)$
g	$g(x)$	$g(y)$

It is stable under precomposition by any map $A \rightarrow B$ and under postcomposition by any map $C \rightarrow D$.

Can prove: $f \sim_1 g$ is equivalent to

$g - f$ is a derivation w.r.to f

In particular: a 1-neighbour of the identity map of B is of the form: identity map of B plus a derivation $B \rightarrow B$.

Or: first order deformation of the identity map of \overline{B} = vector field on \overline{B} = derivation on B .

We may consider not only neighbours *pairs* f, g of k -algebra maps $B \rightarrow C$, but for an n -tuple of mutual neighbouring k -algebra maps $B \rightarrow C$,

$$B \begin{array}{c} \xrightarrow{f_1} \\ \vdots \\ \xrightarrow{f_n} \end{array} C$$

For $x, y \in B$, we get similarly an $n \times 2$ matrix

	x	y
f_1	$f_1(x)$	$f_1(y)$
f_2	$f_2(x)$	$f_2(y)$
\cdot	\cdot	\cdot
\cdot	\cdot	\cdot
f_n	$f_n(x)$	$f_n(y)$

2. Affine combinations of n -tuples of mutual neighbours

Theorem.

Let f_1, \dots, f_n be an n -tuple of mutual neighbour k -algebra maps $B \rightarrow C$, and let t_1, \dots, t_n be elements of C with $t_1 + \dots + t_n = 1$. Then the affine combination

$$\sum_{i=1}^n t_i \cdot f_i : B \rightarrow C$$

is a k -algebra map. The construction is natural in B and in C .

Proof for $n = 2$. Let x and y in B . Multiplication in B denoted \bullet :

$$B \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} C$$

$$s + t = 1 \text{ in } C$$

$$(s \cdot f + t \cdot g)(x \bullet y) =$$

$$(s \cdot f)(x \bullet y) + (t \cdot g)(x \bullet y) =$$

$$s \cdot f(x) \cdot f(y) + t \cdot g(x) \cdot g(y)$$

since both f and g preserve multiplication.

Two terms!

$$(s \cdot f + t \cdot g)(x) \cdot (s \cdot f + t \cdot g)(y)$$

multiply out in C , get **four** terms

$$s^2 \cdot f(x) \cdot f(y) + s \cdot t \cdot f(x) \cdot g(y) + t \cdot s \cdot g(x) \cdot f(y) + t^2 \cdot g(x) \cdot g(y)$$

So compare with the previous poor little **two**-term expression:

$$s \cdot f(x) \cdot f(y) + t \cdot g(x) \cdot g(y)$$

Multiply it by $s + t = 1$! Get **four** terms

$$s^2 \cdot f(x) \cdot f(y) + t \cdot s \cdot f(x) \cdot f(y) + s \cdot t \cdot g(x) \cdot g(y) + t^2 \cdot g(x) \cdot g(y)$$

to be compared with

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$$s^2 \cdot f(x) \cdot f(y) + t \cdot s \cdot f(x) \cdot f(y) + s \cdot t \cdot g(x) \cdot g(y) + t^2 \cdot g(x) \cdot g(y)$$

to be compared with

$$s^2 \cdot f(x) \cdot f(y) + s \cdot t \cdot f(x) \cdot g(y) + t \cdot s \cdot g(x) \cdot f(y) + t^2 \cdot g(x) \cdot g(y)$$

The leftmost red term matches the leftmost blue term.

The rightmost red term matches the rightmost blue term.

The remaining (middle) terms are ($s \cdot t$ times) respectively

$$f(x) \cdot f(y) + g(x) \cdot g(y)$$

$$f(x) \cdot g(y) + g(x) \cdot f(y)$$

and they are equal by the “cross equation” defining $f \sim_1 g$. QED

The theorem may be augmented with the following, which are proved in a similar elementary way:

Let f_1, \dots, f_n be an n -tuple of mutual neighbour k -algebra maps $B \rightarrow C$.

Then any two affine combinations (with coefficients from C) of these maps are neighbours.

Or: in the C -module of k -linear maps $B \rightarrow C$, the affine span of the f_j s consists of k -algebra maps, and they are mutual neighbours.

In particular, the **barycenter** of a $(p + 1)$ tuple of mutual 1-neighbour maps $B \rightarrow C$ is a well defined map $B \rightarrow C$; e.g. the **mid-“point”** of two neighbour maps;

and **equations** between such affine combinations do hold, since they are derived from the C -module (“vector space”) $\text{hom}_k(B, C)$ (unlike infinitesimal constructions using the tangent spaces $T_p(M)$ which give a special status to the base “point” p).

Thus for $x \sim y$,

$$\text{midpoint of } x \text{ and } y = \text{midpoint of } y \text{ and } x$$

3. Geometric meaning

\mathcal{A} the category of k -algebras

\mathcal{A}^{op} the dual category.

so notation and terminology changes.

(similar to the logoi vs topos duality in the presentations by Fiers and Joyal)

$$B \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} C$$

becomes

$$\overline{C} \begin{array}{c} \xrightarrow{\overline{f}} \\ \xrightarrow{\overline{g}} \end{array} \overline{B}$$

" B a k -algebra" becomes " \overline{B} is an (affine) scheme (over k)"

The reflexive symmetric relation \sim_1 on the set $\text{hom}_{\mathcal{A}}(B, C)$ is also a reflexive symmetric relation \sim_1 on the set $\text{hom}_{\mathcal{A}^{op}}(\overline{C}, \overline{B})$, i.e. on points of \overline{B} (defined at same stage \overline{C})

The initial k -algebra k becomes the terminal affine scheme: $\bar{k} = 1$
Maps $\bar{C} \rightarrow \bar{B}$ (i.e. scheme maps) are called “*points* of \bar{B} (defined at *stage* \bar{C})”; Kripke-Joyal semantics applies in \mathcal{A}^{op} .
Thus points of \bar{B} at stage 1 (“global” points) are the same as k -algebra maps $B \rightarrow k$.

What about the “scalars” t_i , i.e. the elements of C that enter in the formulation of the Main Theorem ?

$k[X]$ = the free k -algebra in one generator X ; $R := \overline{k[X]}$

Elements t of C

correspond to

k -algebra maps $k[X] \rightarrow C$ (namely the one with $X \mapsto t$)
= maps in \mathcal{A}^{op} : $\overline{C} \rightarrow R$

points of R defined at stage \overline{C}

or: “scalars” $\in R$ defined at stage \overline{C} ,

$\text{hom}_k(B, C)$ is a C -module.

The Theorem can be expressed:

The subset $\text{hom}_{\mathcal{A}}(B, C) \subseteq \text{hom}_k(B, C)$ is stable under formation of affine combinations (in this C -module) of mutual neighbours (and the formation is natural w.r.to maps $C \rightarrow C'$)

The structure on $\text{hom}_{\mathcal{A}}(B, C)$ provided by the Theorem is a partially defined algebraic structure: (defined on mutually neighbouring maps): of such, affine combinations of such (over the k -module of linear maps $B \rightarrow C$) can be formed (natural w.r.to ...)

Expressed in \mathcal{A}^{op} :

$\text{hom}_{\mathcal{A}^{op}}(\overline{C}, \overline{B})$ carries structure of partially defined affine space over C (defined on mutually neighbouring maps $\overline{C} \rightarrow \overline{B}$ (natural w.r.to ...))

or

One may form affine combinations of mutual neighbouring points of \overline{B} (points at stage \overline{C}) with coefficients: scalars $\in R$ (at stage \overline{C} (natural w.r.to ...))

4. Logarithm

For $x \sim_1 y \in \overline{B}$ and $t \in R = \overline{k[X]}$, x, y neighbour points of \overline{B} , t a point of R , all three at stage \overline{C} , say, have

$$(1 - t) \cdot x + t \cdot y \in \overline{B}$$

(natural in \overline{C}) Synthetically: for $x \sim_1 y$, this defines a map $R \rightarrow \overline{B}$

$$t \mapsto (1 - t) \cdot x + t \cdot y$$

The 1-jet of this map at 0 is a tangent vector at x (in \overline{B}).

$$\log : \overline{B}_{(1)} \rightarrow T(\overline{B})$$

For good B , this is a bijection onto the subscheme of $(T(\overline{B}))_1$, the inverse is called \exp . For $\overline{B} = R^n$:

$(T(\overline{B}))_1 =$ tangent vectors whose principal part is $\sim_1 0$.

And \exp adds the principal part to the base point.

Filip Bar: Affine connections and second-order affine structures, to appear in Cahiers de Top. et Geom. Diff. Cat.

E. Dubuc and A. Kock.: On 1-form classifiers, Communications in Algebra 12 (1984), 1471-1531.

A. Kock: Synthetic Geometry of Manifolds, Cambridge Tracts in Mathematics 180, Cambridge University Press 2010.

A. Kock: Affine combinations in affine schemes, Cahiers de Top. et Geom. Diff. Cat., 58 (2017), 115-130.

A. Kock: Integration of 1-forms and connections, arXiv 1902.11003 .

<https://tildeweb.au.dk/au76680/>

Thank you !

$$B \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} C$$

Then $f \sim_1 g$ iff

$$(f, g) : B \otimes B \rightarrow C$$

factors through

$$B \otimes B \rightarrow B \otimes B / I^2$$

where I is the kernel of the multiplication map

$$B \otimes B \rightarrow B$$

In \mathcal{A}^{op} , this quotient map becomes a subscheme

$$\overline{B}_1 \subset \overline{B \otimes B}$$

the “first neighbourhood of the diagonal”.

The submodule $I/I^2 \subset B \otimes B / I^2$ is the Kaehler differentials of B .

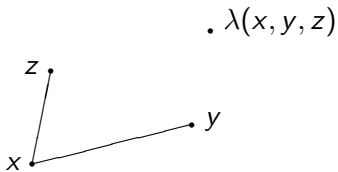
Interpreted in \mathcal{A}^{op} : the module of scalar functions $\omega : \overline{B}_1 \rightarrow R$

which vanish on the diagonal,

$\omega(x, y) \in R$ defined for $x \sim_1 y$, with $\omega(x, x) = 0$.

\mathcal{A} becomes a tangent category $B \mapsto B[\epsilon]$
 \mathcal{A}^{op} becomes a tangent category: B maps to the symmetric
 B -algebra on the Kaehler differentials of B .

If only $x \sim_1 y$, $x \sim_1 z$ is assumed, but not $y \sim_1 z$, the completion of the first figure into the second, is an added *structure*, namely an **affine connection** λ :



I have no license to write the affine combination $y - x + z$ unless $y \sim_1 z$.

In this case, $\lambda(x, y, z) = y - x + z$ iff λ is symmetric in y, z ,