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Introduction to Functorial Semantics

by

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<u>Motivation.</u> As soon as one has formulated the notion of, say, 'semigroup' in its conceptual form

(0.1)

a map $A \times A \xrightarrow{\mu} A$

such that

$$\begin{tikzcd} A \times A \times A \arrow[r, "\mu \times 1"] \arrow[d, "1 \times \mu"'] & A \times A \arrow[d, "\mu"] \\ A \times A \arrow[r, "\mu"'] & A \end{tikzcd}$$

commutes

one can immediately ~~generalize,~~ interpret the theory of semigroups in any category with cartesian products; for instance, a <u>topological semigroup</u> is a diagrammatic situation like (0.1) <u>in the category of topological spaces</u>.

One can even construct a category with cartesian products, $\mathbb{T}$, "the theory of semigroups," such that there is a 1-1 correspondence between cartesian-product-preserving functors

$$\mathbb{T} \longrightarrow \mathcal{S}$$

($\mathcal{S}$ = the category of sets) on the one hand, and semigroups on the other. Then there will be a 1-1 correspondence between cartesian-product preserving functors

$$\mathbb{T} \longrightarrow (\text{Top-spaces})$$

on the one hand, and topological semigroups on the other. This point of view, functorial semantics of algebraic theories, (Lawvere 1963, [3]) has proved very useful in organizing concepts of universal algebra.

In the following, we shall carry out certain aspects of functorial semantics of 1. order theories (and to a certain extent also higher-order theories). A category, in which one can form models of 1. order theories should certainly have more structure than just cartesian products; it clearly should be quite rich in subobjects, so that a 1. order formula in n free variables $\varphi(x_1, \ldots, x_n)$ "carves out a unique sub "set" of A^n" (where A is "the base object").

To be more precise, we proceed to formulate what we mean by a category being "rich enough in subobjects", arriving at the notion of category with quantification (p. 11).

Again, the main example of a category with quantification will be $\mathbb{S}$, the category of sets. But also the functor-category $\mathbb{S}^{\Lambda}$, where Λ is a small category (for instance, a partially ordered set) is a quantification category. In fact, functorial semantics in $\mathbb{S}^{\Lambda}$ will be Kripke semantics. (We do not go into that here; see [1] or [5]).

The 1. order languages we shall meet here will not be formal, but conceptual, namely akin to polyadic algebras, just simpler; in fact, they will also be just quantification categories. We shall thus end up considering an "interpretation" of the "language" $\mathcal{S}$ in the "real world" $\mathbb{S}$,

$$\mathcal{S} \xrightarrow{\varphi} \mathbb{S}$$

(φ being quantification preserving), and relate this to ideas of non-standard analysis.

1. Categories with quantification

Let $\mathcal{C}$ be a category and $A \in |\mathcal{C}|$. A <u>subobject</u> of A is a monic (= right cancellable) map into A

(1.1) $$A' \xrightarrow{f} A .$$

The class of subobjects of A is denoted $\mathcal{P}(A)$.

A reflexive and transitive relation $\leq$ is put on $\mathcal{P}(A)$ by stipulating that

$$(A' \xrightarrow{f} A) \leq (A'' \xrightarrow{g} A)$$

if there is a map $A' \xrightarrow{h} A''$ making the diagram

$$\begin{tikzcd} A' \arrow[rr, "h"] \arrow[dr, "f"'] & & A'' \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

commutative. (If such h exists, it is unique, because g is monic; further, h is necessarily monic).

By abuse of notation, we write A' for the subobject given by (1.1). We define an equivalence relation $\sim$ on $\mathcal{P}(A)$ by ~~putting~~

$$A' \sim A'' \quad \text{if} \quad A' \leq A'' \text{ and } A'' \leq A' .$$

Let $\mathcal{P}_\sim(A)$ denote $\mathcal{P}(A)$ modulo this equivalence relation. The relation $\leq$ on $\mathcal{P}(A)$ defines a partial order relation (also denoted $\leq$) on $\mathcal{P}_\sim(A)$.

If $\mathcal{C} = \mathcal{S}$, the category of sets, $\mathcal{P}_\sim(A)$ is isomorphic to the boolean algebra of subsets of A. If $\mathcal{C}$ = category of groups, $\mathcal{P}_\sim(A)$ is isomorphic to the lattice of subgroups of A ; and so on, for virtually all "concrete" categories of structures. By abuse, we denote elements in $\underset{\sim}{\mathcal{P}}(A)$ by their representatives in $\mathcal{P}(A)$.

Suppose the category $\mathcal{C}$ has pull-backs. If $f : A \to B$ is a map in $\mathcal{C}$, we can construct a map

$$\underset{\sim}{\mathcal{P}}(f) : \underset{\sim}{\mathcal{P}}(B) \longrightarrow \underset{\sim}{\mathcal{P}}(A) ;$$

it associates to $(B' \overset{g}{\rightarrowtail} B) \in \underset{\sim}{\mathcal{P}}(B)$ the left hand column in the pull-back diagram

$$\begin{tikzcd}
A \underset{B}{\times} B' \arrow[r] \arrow[d, tail] & B' \arrow[d, tail] \\
A \arrow[r, "f"'] & B .
\end{tikzcd}$$

This is well defined modulo $\sim$, for trivial "composition-algebraic" reasons.

Since $\underset{\sim}{\mathcal{P}}$ not only is defined on objects now, but also on maps, we have in fact a contravariant ~~[illegible]~~ (= direction-inverting) functor

$$\underset{\sim}{\mathcal{P}} : \mathcal{C} \longrightarrow (\text{Po-sets})$$

where '(Po-sets)' denotes the category of partially ordered sets and monotone mappings.

Definition We shall say that the category $\mathcal{C}$ admits existential quantification if it has pull-backs, and, for all $f : A \longrightarrow B$ in $\mathcal{C}$, the monotone map $\underset{\sim}{\mathcal{P}}(f) : \underset{\sim}{\mathcal{P}}(B) \longrightarrow \underset{\sim}{\mathcal{P}}(A)$ has a left adjoint $\exists_f$.

Equivalently, ~~[illegible]~~ to every $f : A \to B$

there is a monotone map $\exists_f : \mathcal{P}_r(A) \longrightarrow \mathcal{P}_r(B)$ such that for all $A' \in \mathcal{P}_r(A)$, $B' \in \mathcal{P}_r(B)$:

$$\boxed{\exists_f(A') \leq B' \qquad \text{iff} \qquad A' \leq \mathcal{P}_r(f)(B')}$$

<u>Definition</u> We shall say that the category $\mathcal{C}$ admits <u>universal quantification</u> if it has pull-backs, and, for all $f : A \longrightarrow B$ in $\mathcal{C}$, the monotone map $\mathcal{P}_r(f) : \mathcal{P}_r(B) \longrightarrow \mathcal{P}_r(A)$ has a right adjoint $\forall_f$

Equivalently, for all $A' \in \mathcal{P}_r(A)$ and $B' \in \mathcal{P}_r(B)$:

$$\boxed{\mathcal{P}_r(f)(B') \leq A' \qquad \text{iff} \qquad B' \leq \forall_f(A').}$$

The category of sets, $\mathbb{S}$, has existential quantification as well as universal quantification. The category of groups has existential quantification, but <u>not</u> universal quantification. Existential quantification can in both cases be described as:

$\exists_f$ associates to $A' \leq A$ the direct image $f(A') \subseteq B$

Also, in both these cases, $\mathcal{P}_r(f)$ is just <u>forming inverse image along f</u>, whence we also denote $\mathcal{P}_r(f)$ by f^*.

It is clear that if $\mathcal{C}$ has pull backs, then each $\mathcal{P}_r(A)$ is a lower semilattice, i.e., has intersections. Thus, if $A' \xrightarrow{a} A$ and $B \longrightarrow A$ are in $\mathcal{P}_r(A)$, $A' \wedge B$ can be described as

$$\exists_a(a^*(B)) \in \mathcal{P}_r(A).$$

(note that $\exists_a$ always exists for <u>monic</u> a).

<u>Proposition</u> Let $\mathcal{C}$ be a category with universal quantification, and let $(A' \xrightarrow{a} A) \in \underset{\sim}{\mathcal{P}}(A)$. Then the monotone map

$$A' \wedge - \quad : \underset{\sim}{\mathcal{P}}(A) \longrightarrow \underset{\sim}{\mathcal{P}}(A)$$

has a right adjoint ~~equivalently~~ $A' \Rightarrow -$; equivalently, for all $B, C \in \underset{\sim}{\mathcal{P}}(A)$

$$\boxed{\begin{array}{lrcl} & A' \wedge B & \leq & C \\ \text{iff} & & & \\ & B & \leq & A' \Rightarrow C \end{array}}$$

<u>Proof</u>. We have

$$\begin{array}{llll} & A' \wedge B & \leq C & \\ \text{iff} & & & \\ & \exists_a(a^*(B)) & \leq C & \text{by construction of } A' \wedge B \\ \text{iff} & & & \\ & a^*(B) & \leq a^*(C) & \text{by } \exists_a \dashv a^* \\ \text{iff} & & & \\ & B & \leq \forall_a(a^*(C)) & \text{by } a^* \dashv \forall_a \end{array}$$

so that we may (and must) take $\forall_a a^*$ for $A' \Rightarrow$. (The proof given here really only spells out the fact that the adjoint of a composite functor is the composite of the adjoints).

Lower semilattices, where $A' \wedge -$ has a right adjoint for all A' are called <u>Brouwerian semilattices</u> So the Proposition says that if $\mathcal{C}$ has universal quantification, then each $\mathcal{P}_{\sim}(A)$ is a Brouwerian semilattice. It is not in general true, though, that $\mathcal{P}_{\sim}$ is a functor into the category of Brouwerian semilattices; for $\mathcal{P}_{\sim}(f) = f^*$ will not in general preserve the operation '$\Rightarrow$'; that is, one cannot just from existence of quantifications conclude that

$$(1.2) \qquad f^*(B' \Rightarrow B'') = f^*(B') \Rightarrow f^*(B'') .$$

Recall that a <u>coproduct</u> of objects A and B in a category $\mathcal{C}$ is a diagram of form

$$(1.3) \qquad A \xrightarrow{\mathrm{incl}_A} A+B \xleftarrow{\mathrm{incl}_B} B$$

with the universal property, that to each pair of maps $f : A \longrightarrow C$, $g : B \longrightarrow C$, there is a unique map $\left\{ \begin{smallmatrix} f \\ g \end{smallmatrix} \right\} : A+B \longrightarrow C$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "\mathrm{incl}_A"] \arrow[dr, "f"'] & A+B \arrow[d, "\left\{ \begin{smallmatrix} f \\ g \end{smallmatrix} \right\}"] & B \arrow[l, "\mathrm{incl}_B"'] \arrow[dl, "g"] \\
 & C &
\end{tikzcd}$$

In the category of sets, any <u>disjoint</u> union of A and B has this property.

A <u>nullary</u> coproduct is just an initial object $\emptyset$, 'initial' meaning that for every object C there is precisely one map $\emptyset \longrightarrow C$.

(The dual concepts are of course 'cartesian product $\times$' and 'terminal object 1'; we assume these known).

A category is said to <u>have finite coproducts</u> if it has an initial object, and a coproduct diagram (1.3) for each pair A, B of objects. A functor φ is said to <u>preserve</u> finite coproducts if φ applied to the initial object yields an initial object, and if φ applied to the coproduct diagram (1.3) yields a coproduct diagram. Similarly "to have finite products" and "to preserve finite products." Finally, a functor φ <u>preserves pullbacks</u> means that φ applied to a pull back diagram yields a pull back diagram. A functor φ which preserves pull-backs can quite easily be proved to carry monic maps to monic maps, thus it will, for every object A define a mapping (monotone, in fact), denoted $\overline{\varphi}_A$:

$$\underset{\sim}{\mathcal{P}}(A) \xrightarrow{\overline{\varphi}_A} \underset{\sim}{\mathcal{P}}(\varphi(A))$$

We can now describe a class of categories which allow interpretation of 1. order languages in them (just as categories with cartesian products allowed interpretation of ~~higher order~~ algebraic theories in them.)

<u>Definition</u> A category $\mathcal{C}$ is said to be a <u>category-with-quantification</u> if it has pull-backs, finite products, finite coproducts, and if it admits both kinds of quantification, (as defined p. 6 and 7.); further (for technical reasons, to get a good notion of equality predicate), we require (1.2) to hold whenever it makes sense – it will play no role here. (Consult [4]).

A functor $\mathcal{C} \xrightarrow{\varphi} \mathcal{C}'$ between categories with quantification is said to be <u>quantification preserving</u> if it preserves pull-backs, finite products, finite coproducts, <u>and</u> if, for every map $f : A \longrightarrow B$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} \underset{\sim}{P}(A) \arrow[r, "\exists_f"] \arrow[d, "\bar{\varphi}_A"'] & \underset{\sim}{P}(B) \arrow[d, "\bar{\varphi}_B"] \\ \underset{\sim}{P}(\varphi(A)) \arrow[r, "\exists_{\varphi(f)}"'] & \underset{\sim}{P}(\varphi(B)) \end{tikzcd} ,$$

as well as the similar diagram with $\exists$ replaced by $\forall$, commutes.

<u>Example</u> The category $\mathcal{S}$ of sets is a category-with-quantification; likewise $\mathcal{S}^{\wedge}$, as mentioned in the introduction. To give a non-trivial example of a quantification preserving functor, let I be an infinite set, and D a non-principal ultrafilter on I. Define $\varphi : \mathcal{S} \longrightarrow \mathcal{S}$ by associating to $X \in |\mathcal{S}|$ the ultrapower $\prod_I X / D$

[redacted].

The intended interpretation of quantification preserving functors is that a q.p. functor

$$q : \mathcal{C} \longrightarrow \mathcal{C}'$$

will be thought of as an <u>interpretation</u> of the <u>language</u> $\mathcal{C}$ into the <u>world</u> $\mathcal{C}'$. In fact, to any given ordinary 1. order theory $(\mathcal{L}, \Sigma)$, one may construct a quantification category $\mathbb{L}_1(\mathcal{L}, \Sigma)$ (the "Lindenbaum category" for $(\mathcal{L}, \Sigma)$), such that a model of Σ is the same as a q.p. functor

$$\mathbb{L}_1(\mathcal{L}, \Sigma) \longrightarrow \mathcal{S} .$$

(for details (and possibly some qualifications), the reader is referred to H. Volger: "<u>Logical Categories</u>" [6], (Thesis, Dalhousie U., 1971).

So the notion of quantification category unifies, in some sense, the notion of <u>language</u> and the notion of <u>world</u>. 'Language' understood as '1 order language'.

To have a similar unification for higher order language, we add two requirements to the notion of quantification category, and arrive at the notion of <u>elementary</u> <u>topos</u>

<u>Definition</u> A category $\mathcal{C}$ is called an <u>elementary</u> <u>topos</u> if it is a quantification category so that

1) "subobjects are representable"; this means that there exists an object Ω, such that

$$\mathcal{P}(A) \cong \hom_{\mathcal{C}}(A, \Omega)$$

naturally in A

2) $\mathcal{C}$ has exponentiation ; that is , the functor $- \times A : \mathcal{C} \to \mathcal{C}$ has a right adjoint functor $(-)^A : \mathcal{C} \longrightarrow \mathcal{C}$.

(That is, for all $B, C \in |\mathcal{C}|$

$$\hom_{\mathcal{C}}(B \times A, C) \cong \hom_{\mathcal{C}}(B, C^A)$$

naturally in B and C).
Also, $\mathcal{C}$ should have "pushouts" (the dual notion of pull-back).

The axioms for 'elementary topos' as stated here are not independent ; one can , in fact, prove that existence of $\exists_f$ and $\forall_f$ follow just from the remaining axioms , i.e. from

1. $\mathcal{C}$ has finite limits and finite colimits
2. subobjects are representable
3. $\mathcal{C}$ has exponentiation.

The category of sets, $\mathcal{S}$, is an elementary topos; for Ω, take any set with two elements; for exponentiation, take exponentiation C^A (= set of mappings from A to C).

A <u>logical morphism</u> from the elt. topos $\mathcal{C}$ to the elt. topos $\mathcal{C}'$ is a functor $\varphi : \mathcal{C} \longrightarrow \mathcal{C}'$ which preserves Ω and exponentiation, and is quantification preserving

It would be highly interesting to know whether there also exist Lindenbaum categories (which have now to be elementary toposes) for higher order theories. For instance, does there exist an elementary topos $\mathcal{T}$ so that a topological space is the same as a logical morphism $\mathcal{T} \xrightarrow{\varphi} \mathcal{S}$? $\mathcal{T}$ would have to have a specified object X and a specified object $\mathcal{O} \rightarrowtail \Omega^X$; $\varphi(X)$ would be the set of points of the space, and

$$\varphi(\mathcal{O}) \subseteq \varphi(\Omega^X) = \varphi(\Omega)^{\varphi(X)} = 2^{\varphi(X)}$$

would have to be the set of open subsets of the space. $\mathcal{O}$ would have to be "closed under arbitrary unions", the notion of "<u>arbitrary union of families of subsets of X</u>" being a map, denoted $\bigcup$,

$$2^{(2^X)} \xrightarrow{\ \bigcup\ } 2^X$$

which can be proved to exist in <u>any</u> elementary topos, and for any object X in it. (For details, see Kock and Wraith: <u>Elementary toposes</u>, Aarhus University Lecture Notes, Sept 1971, [2]; these notes elaborate on ideas due to Lawvere and Tierney, mainly; compare [5]).

In an elementary topos, Ω^X is an <u>intrinsic</u> object, having the nature of "power set of X", i.e., corresponding to the external $\mathcal{P}(X)$. One can prove that <u>internal</u> quantifications exist, i.e., for a map $f: X \to Y$, there exists a map $\Omega^X \xrightarrow{\exists_f} \Omega^Y$, being an internal version of $\mathcal{P}(X) \xrightarrow{\exists_f} \mathcal{P}(Y)$.

2. Interpretation of $\mathcal{S}$ in $\mathcal{S}$; non-standard notions.

Having said that an elementary topos at the same time can be viewed as a higher order language and as a world where to interpret such languages, we should now try to confront an elementary topos with itself. To be specific, we do it for the category of sets :

$$** \qquad \mathcal{S} \xrightarrow{\varphi} \mathcal{S} \; ;$$

the confrontation φ is only going to be "on the 1. order level", that is, we assume that φ is quantification preserving ; in particular, φ will preserve the subobject classifier Ω which here is just $2 = \{true, false\}$. The $\mathcal{S}$ on the left hand side of ** is not going to be a foreign tongue, nor is φ going to be a foreign way of using it for anybody who heard Machover's talks.

It should be remarked that, because of the good categorical properties of $\mathcal{S}$, a functor $\mathcal{S} \xrightarrow{\varphi} \mathcal{S}$ will be quantification preserving whenever it preserves finite products, pull-backs, finite coproducts, and 'onto' maps. (The last clause is superfluous if $\mathcal{S}$ satisfies the axiom of choice in the form : "onto maps have sections".) For instance, the fact that φ preserves finite coproducts implies that it preserves complementation in subobject lattices (which are here boolean algebras); $\exists$ is trivially preserved; and $\forall$ can be expressed in terms of $\exists$ and complementation.

We shall identify $\varphi(X \times Y)$ with $\varphi(X) \times \varphi(Y)$, and $\varphi(1)$ with 1, because φ preserves finite cartesian products. Also, we shall identify $\varphi(2)$ with 2, because φ preserves 1, preserves coproducts, and because $1+1 = 2$.

We proceed to prove elementary facts about φ

<u>Proposition 1</u> The functor φ is faithful (that is, if $a, b : X \longrightarrow Y$ are two different maps, then also $\varphi(a), \varphi(b)$ are different).

<u>Proof</u> Since 1 is a generator, it suffices to prove it for the case $X = 1$. Now $\langle a, b\rangle : 1 \longrightarrow Y \times Y$ factors through the complement of the diagonal $Y \longrightarrow Y \times Y$, essentially because 1 has only two subobjects: $\emptyset$ and 1. Since φ preserves diagonal and complementation, $\varphi\langle a, b\rangle$ factors through the complement of the diagonal $\varphi(Y) \to \varphi(Y) \times \varphi(Y)$.

The proposition implies the existence of a <u>monic</u> set theoretic mapping

$$*** \qquad \hom_{\mathcal{S}}(1, Y) \longrightarrow \hom_{\mathcal{S}}(1, \varphi(Y)).$$

From this, we cannot in general conclude the existence of a map $Y \xrightarrow{\eta_Y} \varphi(Y)$ inside $\mathcal{S}$, because $***$ is defined in terms which are not intrinsic to $\mathcal{S}$.

For any $A, B \in |\mathcal{S}|$, we have in $\mathcal{S}$ the map "evaluation"

$$B^A \times A \xrightarrow{ev} B$$

(the end adjunction for the adjointness $- \times A \dashv (\ )^A$).

If we apply φ to it, we get a map

$$\varphi(B^A) \times \varphi(A) = \varphi(B^A \times A) \xrightarrow{\varphi(ev)} \varphi(B).$$

Apply exponential adjointness to this map; then we get a map

$$\varphi(B^A) \xrightarrow{c_{A,B}} \varphi(B)^{\varphi(A)}.$$

It is natural in A and B; its crucial role in the following is to compare the "higher order logic" of the language $\mathcal{S}$ with the higher order structure of the world $\mathbb{S}$, relative to the "interpretation" $\varphi : \mathcal{S} \to \mathbb{S}$.
The following Proposition is therefore crucial

<u>Proposition 2</u>. For any A, B, $c_{A,B}$ is a monic (= injective) mapping.

DO NOT TRY TO READ THIS PROOF.

<u>Proof</u> For any pair of objects X, Y, there is a variant of the evaluation map

$$X^Y \times X^Y \times Y \xrightarrow{ev'_{Y,X}} X \times X$$

in terms of elements:

$$\langle f, g, y \rangle \rightsquigarrow \langle f(y), g(y) \rangle.$$

Extensionality for functions $Y \to X$ takes the form

$$(2.1) \qquad \Delta_{(X^Y)} = \forall_{proj_{1,2}}\left(ev'^{\,*}_{Y,X}(\Delta_X)\right),$$

where $\Delta_X : X \rightarrowtail X \times X \in \mathcal{P}(X \times X)$, and $proj_{12}$ is the projection onto the two first factors

$$X^Y \times X^Y \times Y \longrightarrow X^Y \times X^Y.$$

(2.1) is an esoteric way of writing $[f = g \text{ iff } \forall y\, [f(y) = g(y)]]$.

Apply φ to the equation (2.1) with $X=B$, $Y=A$; since φ preserves quantification, we get

$$\Delta_{\varphi(B^A)} = \forall_{proj_{12}} (\varphi(ev'_{A,B}))^* (\Delta_{\varphi(B)})$$

$$\overset{*}{=} \forall_{proj_{12}} (c\times c\times 1)^* \, ev'^{\,*}_{\varphi(A),\varphi(B)} (\Delta_{\varphi(B)})$$

$$\overset{**}{=} (c\times c)^* \forall_{proj_{12}} ev'_{\varphi(A),\varphi(B)} (\Delta_{\varphi(B)})$$

$$\overset{***}{=} (c\times c)^* \Delta_{(\varphi(B)^{\varphi(A)})}$$

$*$ being a rewriting of the map $\varphi(ev'_{A,B})$, $**$ by a so-called "Beck condition" [4], and $***$ being extensionality. Now it is easy to see that a map f such that $f\times f$ pulls the diagonal back to the diagonal is monic; hence c is monic.

The notions of <u>standard</u>, <u>internal</u>, and <u>external</u> maps $\phi(X)\longrightarrow\phi(Y)$.

A map $f:\phi(X)\longrightarrow\phi(Y)$ gives by exponential adjointness rise to a map

$$1 \xrightarrow{\ulcorner f \urcorner} \varphi(Y)^{\varphi(X)}$$

"the name of f". We shall say that f is an <u>internal</u> map (relative to φ) if $\ulcorner f \urcorner$ factors over

$$\varphi(Y^X) \xrightarrow{c_{X,Y}} \varphi(Y)^{\varphi(X)};$$

this factorization will then be unique, since $c_{X,Y}$ is monic, by Prop. 2.

If f is not internal, call it <u>external</u>. Finally if $\ulcorner f \urcorner$ is of the form $\varphi(\ulcorner g \urcorner)$ for some $1 \xrightarrow{\ulcorner g \urcorner} Y^X$, we call f <u>standard</u>. (This is equivalent to f being of the form $\varphi(g)$ for some $g : X \to Y$; such a g is unique, since φ is faithful, by Prop 1.).

A subobject $X' \rightarrowtail \varphi(X)$ of $\varphi(X)$ is said to be <u>internal</u>, provided its characteristic map

$$ch(X') : \quad \varphi(X) \longrightarrow 2 = \varphi(2)$$

is internal; else, X' is called external. $X' \rightarrowtail \varphi(X)$ is called a <u>standard</u> subobject if $ch(X')$ is a standard map; in this case, $X' \rightarrowtail \varphi(X)$ will be of the form $\varphi(X'') \xrightarrow{\varphi(i)} \varphi(X)$ for some monic $X'' \overset{i}{\rightarrowtail} X$.

It is a fairly simple minded matter to see that the a composite of two internal maps yields an internal map; a corollary of this is that the pull-back of an internal subobject along an internal map is an internal subobject. To get a similar statement for the "direct image ~~some~~ along f", i.e., for $\exists_f$, is slightly more difficult, but true. It depends on a property of the internal version of existential quantification, mentioned on p. 14.:

<u>Proposition 3</u> For any map $f : X \longrightarrow Y$, the following diagram commutes:

$$\begin{tikzcd}
\varphi(2^X) \arrow[r, "\varphi(\exists_f)"] \arrow[d, "c"'] & \varphi(2^Y) \arrow[d, "c"] \\
2^{\varphi(X)} \arrow[r, "\exists_{\varphi(f)}"'] & 2^{\varphi(Y)}
\end{tikzcd}$$

.

Proof omitted. It actually involves the use of Proposition 4 below.

For any object X, we denote by $\in_X$ the subobject of $2^X \times X$ classified by the evaluation map ev: $2^X \times X \longrightarrow 2$.

<u>Proposition 4</u> For any $X \in |\mathbb{S}|$, the diagram

$$\begin{tikzcd} \varphi(\in_X) \arrow[r, tail] \arrow[dd] & \varphi(2^X \times X) \arrow[d, equal] \\ & \varphi(2^X) \times \varphi(X) \arrow[d, tail, "c \times 1"] \\ \in_{\varphi(X)} \arrow[r, tail] & 2^{\varphi(X)} \times \varphi(X) \end{tikzcd}$$

is a pull-back diagram.

Proof omitted. — Any object of the form 2^X carries on it a canonical order-relation (expressing "inclusion among subobjects):

$$\leq_X \rightarrowtail 2^X \times 2^X$$

<u>Proposition 5</u> For any $X \in \mathbb{S}$, the diagram

$$\begin{tikzcd} \varphi(\leq_X) \arrow[r, tail] \arrow[dd, tail] & \varphi(2^X \times 2^X) \arrow[d, equal] \\ & \varphi(2^X) \times \varphi(2^X) \arrow[d, tail, "c \times c"] \\ \leq_{\varphi(X)} \arrow[r, tail] & 2^{\varphi(X)} \times 2^{\varphi(X)} \end{tikzcd}$$

is a pull-back diagram.

<u>Proof</u>. In principle by suitably quantifying universally on the diagram in Proposition 4.

By a <u>relation</u> from A to B, we understand a subobject $R \rightarrowtail A \times B$. The notion of <u>relational composition</u> of relations can in fact be defined in any category with existential quantification:

$$\mathcal{P}(A \times B) \times \mathcal{P}(B \times C) \longrightarrow \mathcal{P}(A \times C).$$

If R is a relation from $\varphi(A)$ to $\varphi(B)$, we say that is an <u>internal</u> relation if it is an internal subset of $\varphi(A) \times \varphi(B) = \varphi(A \times B)$.

<u>Proposition 6</u> Internal relations are closed under composition, i.e. the relational composition of two internal relations is again an internal relation.

By a <u>pseudo-set</u>, we understand a pair (a, A), where $A \in |\mathcal{S}|$ and a is a monic mapping into $\varphi(A)$; the domain of a is denoted $[a, A]$

$$[a, A] \xrightarrow{\ a\ } \varphi(A);$$

$[a, A]$ is required to be an <u>internal</u> subobject of $\varphi(A)$. (it does not make sense to ask that a be an internal <u>map</u>, since we have not assumed that $[a, A]$ is of form $\varphi(X)$).

By a <u>pseudo-map</u> from (a, A) to (b, B), we understand a map $[a,A] \xrightarrow{f} [b,B]$, such that the graph of f, viewed as a subobject of $\varphi(A)\times\varphi(B)$ is an internal subobjects.

Pseudomaps compose in an obvious way. We obtain a category $\mathcal{S}^*$ of pseudo-sets and pseudo-maps (relative to φ).

<u>Theorem</u> (hopefully). Let $\varphi : \mathcal{S} \longrightarrow \mathcal{S}$ be a quantification preserving functor, and let $\mathcal{S}^*$ be the category of pseudo-sets. Then there is a factorization of φ

$$\begin{tikzcd} \mathcal{S} \arrow[rr, "\varphi"] \arrow[dr, "\bar{\varphi}"'] & & \mathcal{S} \\ & \mathcal{S}^* \arrow[ur, "\psi"'] & \end{tikzcd}$$

Further, $\mathcal{S}^*$ is an elementary topos, and $\bar{\varphi}$ is a logical morphism.

<u>Proof</u>. The hard part is to prove that $\mathcal{S}^*$ is a topos. The hard part of that is to provide a construction of $(b,B)^{(a,A)}$ for two pseudo-sets (a,A) and (b,B): it is going to be a pair

$$(d, 2^{A\times B})$$

where $d : D \rightarrowtail \varphi(2^{A\times B})$, is a suitable subobject, to be constructed. Since $\varphi(2^{A\times B}) \overset{c}{\rightarrowtail} 2^{\varphi(A\times B)}$ is monic, we can describe D by describing a suitable subobject D' of $2^{\varphi(A\times B)}$, and putting

$$D = c^*(D')$$

i.e., the intersection of D' with $\varphi(2^{A\times B}) \overset{c}{\rightarrowtail} 2^{\varphi(A\times B)}$.

Now we describe various subobjects D_1', D_2', ... of $2^{\varphi(A\times B)}$ which express properties which the elements of $2^{\varphi(A)\times\varphi(B)}$ should have; for instance $D_1 \subseteq 2^{\varphi(A)\times\varphi(B)}$ is going to be the set of those elements in $2^{\varphi(A)\times\varphi(B)}$, which, viewed as relations R from $\varphi(A)$ to $\varphi(B)$ are partial graphs, which is equivalent to $R^{-1} \circ R \leq \Delta_B$ ('$\circ$' denoting relational composition); D_2' is going to express that the domain of that relation is equal to $[a, A] \rightarrowtail \varphi(A)$, etc.; we then have to argue that each of D_i' intersect $\varphi(2^{A\times B}) \rightarrowtail 2^{\varphi(A\times B)}$ in an <u>internal</u> subobject. We illustrate the proof technique by looking at D_1'. First, for any X, denote by $S_X \rightarrowtail 2^{X\times X}$ the subobject consisting of those relations from X to X which are contained in Δ_X. Because of Proposition 5, we can conclude that the diagram

$$\begin{tikzcd}
\varphi(2^{B\times B}) \arrow[r, tail, "c"] & 2^{\varphi(B)\times\varphi(B)} \\
\varphi(S_B) \arrow[u, tail] \arrow[r] & S_{\varphi(B)} \arrow[u, tail]
\end{tikzcd}$$

is a pull back. (which shows that $S_{\varphi(B)}$ intersects $\varphi(2^{B\times B})$ in an internal (even standard) subobject. D_1' is obtained by pulling back $S_{\varphi(B)}$ along a map "$\boxed{\text{composing with own inverse}}$": $2^{\varphi(A)\times\varphi(B)} \longrightarrow 2^{\varphi(B)\times\varphi(B)}$

We have, however, a commutative diagram

$$\begin{tikzcd}
2^{\varphi(A)\times\varphi(B)} \arrow[r, "\text{comp. with own inv.}"] & 2^{\varphi(B)\times\varphi(B)} \\
\varphi(2^{A\times B}) \arrow[u, tail, "c"] \arrow[r, "\varphi(\text{comp with own inv})"'] & \varphi(2^{B\times B}) \arrow[u, tail, "c"']
\end{tikzcd}$$

(essentially because of Prop. 3), which means that $D_1' \cap \varphi(2^{A\times B})$ will be

$$(\varphi(\text{comp with own inverse}))^{*}\,(\varphi(S^{B}))$$

which is an internal subobject of $\varphi(2^{A\times B})$, being the pull-back of the internal $\varphi(S^B)$ along an internal map. — And so on. (Some of the D_i' will involve a non-standard, but internal parameter, like $[a,A] \xrightarrow{a} \varphi(A)$, say.)

The point of the theorem is that it gives a systematic account of the interplay between the 1.st order, and the higher order structure ($\bar{\varphi}$ preserving the higher order structure).

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