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UNIVERSAL PROJECTIVE GEOMETRY VIA TOPOS THEORY

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The idea of this article is that linear algebra and projective geometry over a local commutative ring is equivalent to intuitionistic linear algebra and intuitionistic pure projective geometry over a field (at least in so far as *coherent* sentences are concerned; this term will be explained). By *pure* projective geometry is meant a formulation of synthetic geometry in terms of the predicates "incidence" and "equality" alone, in contrast to formulations which include a special apartness predicate. Heyting's intuitionistic projective geometry, for instance, has such an apartness predicate ω, and it is also necessary to have such ω if one wants to formulate a reasonable synthetic theory for the projective geometry over a local ring.

There are probably other and better reasons for looking for the "projective geometry over a local ring" than the one which gave rise to the present research: Study's transfer principle. It says that whatever is true in plane projective geometry over the ring of dual numbers $\mathbf{D} = \mathbf{R}[\varepsilon]$ (with $\varepsilon^2 = 0$), can be reinterpreted as a theorem about the set of lines in euclidean 3-space, see e.g. [6]. This set (which is a 4-dimensional manifold) is thereby made into a Hjelmslev *plane*; Hjelmslev used this fact. Of course, the value of the transfer principle is that one gets, or hopes to get, theorems for the projective plane over $\mathbf{D}$ by "analogy" with the well-known projective geometry over $\mathbf{R}$. We are studying the meta-mathematics of that "analogy".

Our approach is to construct a pure projective geometry $\mathcal{P}$, all of whose sentences on the one hand, written in "positive" form, hold for all local rings, and, on the other hand, because of their "purity" are immediately recognizable as familiar theorems of projective geometry.

The possibility of such a "universal" projective (pure) geometry is a consequence of the main theorem of the paper, Theorem 4.1, whose proof utilizes the theory of classifying toposes. Also, when we say "intuitionistically derivable", we actually mean "valid in any topos". These two notions are equivalent, see e.g. [3] or [12], but it is the latter which one can work with and is interested in.

Section 1 contains the logic by means of examples. It is roughly the content of a lecture given at the Aarhus Category Open House, May 1973 (except there we assumed that the site was a regular category with epimorphism topology). W.

Mitchell, Osius, Mulvey, Benabou and his students and the Montreal school have dealt more systematically and conceptually with language and interpretation for toposes. The conceptual way of formulating the above would be to pass to the topos sh($\boldsymbol{E}$) of sheaves on $\boldsymbol{E}$, where $\boldsymbol{E}$ is the site occurring in our approach. Instead of (informal) "statements" (as considered in section 1 here), one would construct conceptual objects in sh($\boldsymbol{E}$). For formal computations (e.g. matrix multiplication) I think, however, that it is an advantage not to conceptualize (or internalize) everything, but rather consider actual elements or matrices of elements in "external" rings, like hom(X, R) (where R is a ring object).

Section 2 contains a proof that the "generic local ring" is a field object, which is the main lemma for Theorem 4.1. It also contains some linear algebra for field objects in sites.

In section 3, we describe an intuitionistic 1st order theory $\mathcal{P}$ which we call universal projective geometry; it consists of ring theoretic sentences expressing pure incidence relations in the geometry over a field object in an arbitrary topos. As illustrated in section 4, $\mathcal{P}$ carries along with it a body of sentences of geometric type, valid for local rings; this is a corollary of the main Theorem 4.1. This body of sentences have the property that they admit transfer by Study's geometric transfer principle. So briefly, we end up by providing a logical transfer principle which supplements Study's geometrical transfer principle. In particular, we draw a conclusion of geometric nature, by transferring Pappos' Theorem (which is in $\mathcal{P}$) to line geometry.

A preliminary version [7] of this article was less meta-mathematical in the sense that it interpreted the theorems of $\mathcal{P}$ as theorems about specific objects ("projective plane") in a specific topos (the Zariski topos). That version has been presented at Giornate di Logica Categoriale, Firenze May 1974, and at Oberwolfach, August 1974. I wish to express my thanks to these institutions, as well as to Université de Montreal, where the present version was worked out.

1. Kripke–Joyal semantics

We do not attempt to describe a formal language $L(\boldsymbol{E})$ and a complete interpretation of it. Rather, we explain what are the principles behind the concrete interpretation of the concrete notions below (for more examples of this kind, see [8] and [9]).

Let $\boldsymbol{E}$ be a site, that is, a category equipped with a pretopology in the sense of [1] Exposé II; this means that for each $X \in |\boldsymbol{E}|$, we are given some families of maps ending in X

$$\{\beta_i : X_i \to X \mid i \in I\},$$

called the covering families. The axioms for this extra structure on the category $\boldsymbol{E}$ is given in loc. cit., Def. 1.3. Basically, there are two axioms: PT 1 which says that covering families are stable under pull-back, and PT 2 which says that covering families compose. (Also $\{\mathrm{id}_X : X \to X\}$ should be covering.)

Let B be an object in $\boldsymbol{E}$. Suppose we have already interpreted a certain statement φ about arrows ending in B; that is, for every $X \in |\boldsymbol{E}|$ and every $b : X \to B$ we assume we know what we mean by saying that φ holds for b, denoted $\varphi(b)$. Then the statement $\neg\varphi$ is interpreted by saying that for X and $b : X \to B$ arbitrary,

$(\neg\varphi)(b)$ iff for every arrow α which ends in X,
if $\varphi(\alpha . b)$ holds, then the domain of α
is covered by the empty family.

Interpreting $\alpha : Y \to X$ as passage from "time X" to the "later time Y", $\neg\varphi$ holds for b ("defined at time X", $b : X \to B$) iff for all later times Y, φ does not hold for b at time Y (unless Y is "an absurd time", that is, Y is covered by the empty family, which we shall denote $Y = \emptyset$).

This is (roughly) Kripke's semantics for negation. Implication and universal quantification are interpreted the analogous way, i.e. by introducing a *universally* quantified parametershift α (or "passage to later time"). (See illustration below in connection with linear independence, or [9].)

In Kripke's original semantics, validity of conjunctions, disjunctions and existential quantifications are decided "at the spot", that is, no α is required or allowed. This was pointed out to be inadequate (for disjunction and existential quantification) for many mathematical purposes by Joyal (private communication). For instance, in topology, *existence* of cross-sections (in fiber-bundles, or sheaves, say) is a rare thing compared to *existence-locally* of cross-sections. So the slogan was formed "existence means *local* existence". Similarly for disjunction. It turns out that the notion of pre-topology (or site) is an adequate framework for this notion of local. To be precise, suppose that we have already interpreted the statements φ_1 and φ_2, both being statements meaningful for arrows ending in B, as before. Then the statement $\varphi_1 \vee \varphi_2$ is interpreted by saying that for X and $b : X \to B$ arbitrary,

$(\varphi_1 \vee \varphi_2)(b)$ iff there exists a covering family
$\{\beta_i : X_i \to X \mid i \in I\}$, such that
for each $i \in I$, either $\varphi_1(\beta_i . b)$
or $\varphi_2(\beta_i . b)$ holds.

Similarly for n-fold disjunctions. For existentially quantified statements $\exists a \varphi(a, b)$, suppose that for each $Z \in |\boldsymbol{E}|$, we know what we mean by saying that φ holds for $\langle a, b\rangle$ (where $\langle a, b\rangle : Z \to A \times B$), denoted $\varphi(a, b)$. If now $b : X \to B$ is an arbitrary arrow ending in B, we say that b satisfies $\exists a \varphi(a, -)$ (and we write that $\exists a \varphi(a, b)$) if there exists a covering family $\{\beta_i : X_i \to X \mid i \in I\}$ and a family of elements $a_i : X_i \to A$ such that for each $i \in I$, $\varphi(a_i, \beta_i . b)$ holds.

(In both cases ($\vee$ and $\exists$) we call such a covering $\{\beta_i : X_i \to X\}$ a "witnessing cover" for the validity of the disjunctive, respectively the existential, statement in question.)

Such interpretations of statements φ are not very useful, unless they "define subfunctors of representable functors", or "are stable under base change": this means that if we have

$$Z \xrightarrow{\gamma} X \xrightarrow{b} B$$

and φ holds for b, then φ also holds for $\gamma . b$. So our interpretations of composite statements should define subfunctors provided the constituent statements do. This is automatically so for conjunction, implication, and universal quantification. For existential quantification, disjunction and negation, this follows from the fact that pulling a covering family back gives again a covering family (Axiom PT 1 in [1]).

Also for the interpretations to be useful, they should be "local"; this means that if $b : X \to B$ and if $\{\beta_i : X_i \to X \mid i \in I\}$ is a covering family such that $\varphi(\beta_i . b)$ holds for all i, then $\varphi(b)$ holds. So our interpretations of composite statements should be local if the constituent statements are. For conjunctions, this is immediate. For existential quantified statements and disjunctions, it follows because covering families compose to give covering families (Axiom PT 2 in [1]). For universal quantified statements and implications it follows by the pull-back property PT 1 of covering families. Finally, for negated statements, one uses PT 1 as well as PT 2. We do this in detail. Suppose φ is a property of morphisms ending in B which is local (as well as stable under base change). We shall prove that $\neg\varphi$ is a local property. So let

$$b : X \to B$$

be arbitrary, and suppose $\{\beta_i : X_i \to X \mid i \in I\}$ is a covering family such that

$$\neg\varphi(\beta_i . b) \qquad \forall i \in I.$$

We must prove $\neg\varphi(b)$. So let $\alpha : Y \to X$ be arbitrary, and suppose $\varphi(\alpha . b)$. We must then prove $Y = \emptyset$. For each $i \in I$, let α_i and ρ_i be maps defined by the pull-back diagram

(1.1)
$$\begin{array}{ccc} Y_i & \xrightarrow{\rho_i} & Y \\ {\scriptstyle \alpha_i}\downarrow & & \downarrow{\scriptstyle \alpha} \\ X_i & \xrightarrow[\beta_i]{} & X \end{array}$$

By PT 1, the ρ_i's cover Y. If we can prove that each Y_i is $\emptyset$, we are through, by PT 2. But, by assumption

$$\varphi(\alpha . b) \quad \text{holds}$$

so that

$\varphi(\rho_i . \alpha . b)$ holds $\forall i$ (by stability under base change)

so that

$\varphi(\alpha_i . \beta_i . b)$ holds $\forall i$ (by commutativity of (1.1))

so that $Y_i = \emptyset$ since $\neg \varphi(\beta_i . b)$, by assumption.

A final remark concerning this method of interpretation. For this, we have to assume that the representable functors

$$\hom_E(-, B) : E^{\mathrm{op}} \to \text{Sets}$$

are sheaves. Corollary 2.4 in [1] Exposé II may be used as definition of the notion of sheaf on a site in the sense used here. If representable functors are sheaves, one says that the topology of E is *less fine than the canonical*. Under these conditions, we can augment the principle

Existence means local existence

by

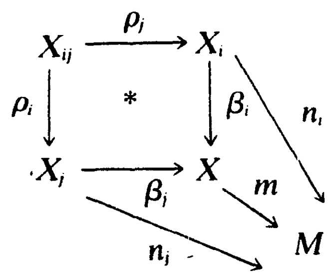
Unique existence implies global existence.

We illustrate this by an example.

Example. Let M be a ring-object in E (not necessarily commutative). Let $m : X \to M$ be a map in E. Then, by definition, m satisfies the existential statement "there exists a right inverse for m" if there is a covering family $\{\beta_i : X_i \to X \mid i \in I\}$ and for each $i \in I$, an element $n_i : X_i \to M$ such that in the ring $\hom_E(X_i, M)$

$$(\beta_i . m) \cdot n_i = e_i,$$

where e_i is the neutral element in the ring $\hom_E(X_i, M)$. From this local existence of a right inverse for m, it does not follow that m has a global right inverse, by which we mean a right inverse in the ring $\hom_E(X, M)$ itself. However, assume that M is a *commutative* ring object. In commutative rings, inverses are unique if they exist. For each $(i, j) \in I \times I$, let X_{ij} be defined by the pull-back diagram $*$ in the diagram



(the two triangles are not commutative). In the ring $\hom_E(X_{ij}, M)$, both $\rho_j . n_i$ and $\rho_i . n_j$ are inverses to the element $\rho_j . \beta_i . m = \rho_i . \beta_j . m$, and hence are equal. Thus the elements $(n_i)_{i\in I}$ in Π_I hom(X_i, M) are equalized by the two canonical maps from this product to $\Pi_{I\times I}$ hom(X_{ij}, M), and since the diagram

$$\text{(1.2)}\qquad \hom(X, M) \xrightarrow{\varepsilon} \Pi_I \hom(X_i, M) \rightrightarrows \Pi_{I\times I} \hom(X_{ij}, M)$$

is an equalizer diagram by the assumption that $\hom_E(-, M)$ is a sheaf, there exists a unique element $n \in \hom_E(X, M)$ with $\beta_i . n = n_i$, $\forall i \in I$. This element is an inverse for m; for, to see $m \cdot n = e$, it suffices to see $\beta_i . (m \cdot n) = \beta_i . e = e_i$ since the map ε in (1.2) is monic. So m has a "global" inverse n.

This means, in particular, that for a commutative ring object M in a site $\boldsymbol{E}$ where the topology is less fine than the canonical, one does not have to distinguish between the "Kripke–Joyal" interpretation of "m is invertible", and the "isolated" interpretation: "m is invertible in the ring hom(X, M)". ("Isolated" because we ignore the rest of the site $\boldsymbol{E}$ when deciding the invertibility of m in hom(X, M).)

The fact that ε in (1.2) is monic can also be expressed: the property of maps with codomain M being equal is a local property. Hence, in a site where the topology is less fine than the canonical, all properties expressed by using the logical operations $\wedge$, $\vee$, $\exists$ etc. to equations, are local properties.

Whenever $\boldsymbol{E}$ is a topos in the sense of [1], we may in a canonical way view $\boldsymbol{E}$ as a site, a family $\{X_i \to X \mid i \in I\}$ being covering if it is jointly epic (this is the same as the topology described in [1], II.2.5). Whenever we use the site-theoretic semantics introduced above for a topos, this canonical covering notion is to be understood.

If $\boldsymbol{E}$ is a small site with a topology less fine than the canonical, and $\mathscr{E}$ is the associated category of sheaves, we have a full and faithful functor ("Yoneda") $h : \boldsymbol{E} \to \mathscr{E}$. The following lemma will be useful.

Lemma 1.1. *The functor h preserves and reflects validity of statements.*

Proof. This is almost a rephrasing of the fact that our satisfaction-of-statements were constructed so that satisfaction is local and stable under base change (together with the fact that for each $X \in \mathscr{E}$, there is a covering family of the form $\{h(X_i) \to X \mid i \in I\}$). We omit details.

2. Linear algebra

In the following sections, $\boldsymbol{E}$ denotes a fixed site, whose topology is less fine than the canonical. We also assume that $\boldsymbol{E}$ has cartesian products. Let R be a commutative ring object in $\boldsymbol{E}$. We say that it is a *field object* (or just a *field*) if for each $n = 1, 2, \ldots$

$$\text{(2.1)}\qquad \neg\left(\bigwedge_{i=1}^{n} (a_i = 0)\right) \Rightarrow \bigvee_{i=1}^{n} (a_i \text{ is invertible}),$$

and

(2.2) $\neg(1 = 0)$.

Note that (2.1), according to section 1, means the following: *Assume* that $a_1, \ldots, a_n$ are maps $X \to R$ (with same domain X) having the property that if $\alpha : Y \to X$ is so that $\alpha . a_1 = \alpha . a_2 = \ldots = \alpha . a_n = 0$, then $Y = \emptyset$; *then* there exists a covering family $\{\beta_i : X_i \to X \mid i \in I\}$ such that for each $i \in I$, at least one of the $\beta_i . a_j$ ($j = 1, \ldots, n$) is an invertible element in the ring hom(X_i, R).

Similarly, (2.2) means that if Y is an object so that hom(Y, R) is the zero ring, then $Y = \emptyset$.

Note that (2.1) and (2.2) do not imply that each hom(X, R) is a field.

Again, if R is a commutative ring object in $\boldsymbol{E}$, we say (following Hakim [4]) that it is a *local* ring (object) if it satisfies (2.2) as well as the following, for each $n = 1, 2, \ldots$

$$(2.3) \qquad \Big(\sum_{i=1}^{n} a_i\Big) \text{ invertible } \Rightarrow \bigvee_{i=1}^{n} (a_i \text{ is invertible}).$$

(Actually, it suffices to have (2.3) for $n = 2$.)

The axiom (2.1) for $n = 1$ is what Mulvey [11] calls a "ring of fractions". The effect of having the axiom for all n is to build a little boolean logic (one of de Morgan's laws) into R.

Proposition 2.1. *If R is a field object, then it is a local ring object.*

Proof. Let $a_i : X \to R$ ($i = 1, \ldots, n$) be elements of R with $\Sigma a_i = 1$ in hom(X, R). Then these a_i satisfy

$$(2.4) \qquad \neg\Big(\bigwedge_{i=1}^{n} (a_i = 0)\Big);$$

for, suppose $\alpha : Y \to X$ is so that

$$\alpha . a_1 = \ldots = \alpha . a_n = 0;$$

then, in hom(Y, R)

$$0 = \Sigma \alpha . a_i = \alpha . \Sigma a_i = \alpha . 1 = 1.$$

Since (2.2) holds, we conclude from this $Y = \emptyset$. This proves (2.4). By (2.1), we conclude $\bigvee (a_i$ is invertible), whence (2.3) is proved.

Of course, the converse is false: not every local ring (object) is a field object. However, the generic local ring $\mathcal{O}_1$ is a field (Proposition 2.2 below).

Recall the construction of the Zariski topos $\mathcal{Z} \subseteq \mathcal{S}^{\mathcal{R}}$; $\mathcal{R}$ is the category of finitely presented commutative rings, and $\mathcal{S}^{\mathcal{R}}$ is the category of covariant functors from $\mathcal{R}$ to sets; $\mathcal{Z}$ consists of those functors $\mathcal{R} \to \mathcal{S}$ which are sheaves for the Zariski topology τ_1 (cf. Hakim [4], III.3, where $\mathcal{Z}$ is denoted $\mathcal{S}_1$). Among these is the underlying functor $\mathcal{O}_1 : \mathcal{R} \to \mathcal{S}$ which to a ring just associates the ring itself, considered as a set; $\mathcal{O}_1$ is a ring-object in $\mathcal{X}$, and Hakim proved that it is a local ring (loc. cit. Prop. 3.2). In particular, it satisfies (2.2).

Proposition 2.2. *The local ring object $\mathcal{O}_1 \in \mathcal{X}$ is a field object.*

Proof. By the lemma in the end of section 1, it suffices to prove that $\mathbf{Z}[T] \in \mathcal{R}^{\mathrm{op}}$ is a field object in the site $\mathcal{R}^{\mathrm{op}}$ with the Zariski topology (note that $h(\mathbf{Z}[T]) = \mathcal{O}_1$, since

$$\hom_{\mathcal{R}}(\mathbf{Z}[T], A) = A;$$

it is the coring structure $\mathbf{Z}[T] \to \mathbf{Z}[T] \otimes \mathbf{Z}[T]$ which makes $\mathbf{Z}[T]$ into a ring object in $\mathcal{R}^{\mathrm{op}}$). We write our diagrams in $\mathcal{R}$ instead of $\mathcal{R}^{\mathrm{op}}$. So an "element" of $B = \mathbf{Z}[T]$ is now rather a map in $\mathcal{R}$ with *domain B*. To prove (2.1), let

$$a_1, \ldots, a_n : \mathbf{Z}[T] \to X$$

be an arbitrary family of "elements" of $\mathbf{Z}[T]$ with same codomain $X \in \mathcal{R}$, and suppose they satisfy

$$\neg(\wedge\, a_i = 0).$$

This means that whenever $\alpha : X \to Y$ is a ring map with $a_i . \alpha = 0$ for $i = 1, \ldots, n$ (meaning $\alpha(a_i(T)) = 0$ for $i = 1, \ldots, n$), then Y is a ring covered by the empty family. The topology τ_1 is so that precisely the zero ring is covered by the empty family. Take in particular $Y = X/(a_1(T), \ldots, a_n(T))$ and let α be the canonical map $X \to Y$. Then $\alpha(a_i(T)) = 0$, $\forall i$, so Y is the zero ring. So

$$(a_1(T), \ldots, a_n(T)) = X,$$

that is, the $a_i(T)$'s generate the unit ideal of the ring X, and so by definition of the Zariski topology,

$$\{X \xrightarrow{\beta_i} X[a_i(T)^{-1}] \mid i = 1, \ldots, n\}$$

is a covering. Clearly $\beta_i(a_i(T))$ is invertible. So this covering witnesses the validity of

$$\bigvee_{i=1}^{n} (a_i \text{ is invertible}).$$

Propositions 2.1 and 2.2 together indicate that linear algebra over fields and over local rings in sites are closely related. Roughly speaking, the local ring theory is better. We developed some of it in [8]. However, the field theory allows for good natural notions of "linear independence" and "non-equality", a feature we are utilizing in section 3. In the rest of section 2, we discuss the basic linear algebra over fields which is needed to convert the natural (but negative) notions into notions which work for local rings.

Let $M \in |E|$ be a module object over a commutative ring object R. An n-tuple of elements of M

(2.5) $v_i : X \to M, \quad i = 1, \dots, n$

(all having the same domain X) is said to form a (*linearly*) *Independent* set (note the capital *I*), provided they satisfy (in the sense of section 1)

$$\forall t_1, \dots, t_n : (\Sigma t_i \cdot v_i = 0 \Rightarrow t_1 = \dots = t_n = 0).$$

In elementary terms this means: given arbitrary $\alpha : Y \to X$ and $t_i : Y \to R$ ($i = 1, \dots, n$), such that, in hom(Y, M),

$$\Sigma t_i \cdot (\alpha . v_i) = 0;$$

then $t_1 = \dots = t_n = 0$. This condition is stronger than: "$v_1, \dots, v_n$ form an independent set in the hom(X, R)-module hom(X, M)"; the extra strength is that for every $\alpha : Y \to X$, the $\alpha . v_i$ are independent in hom(Y, M). (In suggestive terms: "the v_i are not only independent now (at time X), but also at all later times Y".)

Let r be an integer $0 \leq r \leq n$: We say that the n-tuple (2.5) has Rank $\geq r$ if there exists a covering family $\{\beta_i : X_i \to X \mid i \in I\}$ such that for each $i \in I$, at least one sub-r-tuple of $\beta_i . v_1, \dots, \beta_i . v_n$ is Independent.

We do not talk about the Rank of $v_1, \dots, v_n$ being *equal* to r unless $r = n$.

We are mainly going to be concerned with the R-modules $M = R^k$ (k an integer). An element $v : X \to R^k$ can be identified with a k-tuple $v_i : X \to R$ ($i = 1, \dots, k$) of elements of R. An element $\mathbf{A} : X \to R^{nm}$ may be identified with an $m \times n$ matrix of elements $a_{ij} : X \to R$ (a matrix over the ring hom(X, R)), and this in turn may be identified with an m-tuple $r_i : X \to R^n$ ("the m-tuple of rows") or with an n-tuple $s_j : X \to R^m$ ("the n-tuple of columns"). Consider such a matrix $\mathbf{A} : X \to R^{nm}$. As in [8], we say that the (determinantal) Rank of $\mathbf{A}$ is $\geq r$ if there is a covering $\{\beta_i : X_i \to X \mid i \in I\}$ such that for each $i \in I$, the matrix $\beta_i . \mathbf{A}$ (over the ring hom(X_i, R)) has an invertible $r \times r$ minor. (This is the interpretation, according to section 1, of the formal statement "at least one of the $r \times r$ submatrices of $\mathbf{A}$ has invertible determinant", this being formally a disjunction with $\binom{m}{r} \times \binom{n}{r}$ terms. A similar remark holds for the other Rank-definition. In particular, these notions are local and stable under base change.)

We shall now see that the field axioms (2.1) and (2.2) are precisely what is needed to prove the implication $\Rightarrow$ in

Theorem 2.3. *Let R be a field object. Then for any matrix* $\mathbf{A} : X \to R^{nm}$, row-Rank ($\mathbf{A}$) $\geq r$ *if and only if* determinant-Rank ($\mathbf{A}$) $\geq r$. *Similarly for* column-Rank. (*In particular,* row-Rank ($\mathbf{A}$) $\geq r$ *if and only if* column-Rank ($\mathbf{A}$) $\geq r$.)

(Here, of course, row-Rank ($\mathbf{A}$) $\geq r$ means that the m-tuple of rows $r_i : X \to R^n$ ($i = 1, \dots, m$) has Rank $\geq r$ in the module R^n; similarly for column-Rank).

Proof. We start with the easy implication $\Leftarrow$, which does not use the field axioms.

By assumption, X can be covered by some $\{\beta_i : X_i \to X \mid i \in I\}$ such that for each $i \in I$, at least one $r \times r$ minor of $\beta_i . \mathbf{A}$ is invertible, $\beta_i . \mathbf{A}$ being a matrix over the ring $\hom(X_i, R)$. The rows of $\beta_i . \mathbf{A}$ which pass through that $r \times r$ minor are linearly independent, by standard linear algebra over any commutative ring. They are even Independent in the sense explained above, since for any $\alpha : Y \to X_i$, that $r \times r$ minor of $\alpha . \beta_i . \mathbf{A}$ which has same indices as the invertible minor of $\beta_i . \mathbf{A}$, remains invertible ($\hom(X_i, R) \to \hom(Y, R)$ being a ring-homomorphism, thus commuting with the formation of determinants). So for each $i \in I$, $\beta_i . \mathbf{A}$ has row-Rank $\geq r$, and therefore $\mathbf{A}$ has row-Rank $\geq r$, since having row-Rank $\geq r$ is a local property.

For the implication $\Rightarrow$, we list two simple lemmas.

Lemma A. *Any sub-tuple of a linearly Independent n-tuple* $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n : X \to M$ *is linearly Independent.*

This is true, and trivial, for any R. The next lemma holds for field objects.

Lemma B. *Let the* 1*-tuple in* R^k, $\boldsymbol{v} : X \to M = R^k$, *be linearly Independent. Then if* $\boldsymbol{v} = (a_1, \ldots, a_k)$, *the statement*

$$\bigvee_{i=1}^{k} (a_i \text{ is invertible})$$

holds.

Proof. By axiom (2.1) it suffices to prove $\neg(\wedge(a_i = 0))$. So let $\alpha : Y \to X$ be such that

$$\alpha . a_1 = \ldots = \alpha . a_n = 0$$

that is, $\alpha . \boldsymbol{v} = 0$. Let 1 denote the unit element in the ring $\hom(Y, M)$. Then $1 \cdot (\alpha . \boldsymbol{v}) = \alpha . \boldsymbol{v} = 0$, whence by linear Independence of $\{\boldsymbol{v}\}$, $1 = 0$ in $\hom(Y, R)$. By (2.2), therefore, $Y = \emptyset$, and this proves $\neg(\wedge(a_i = 0))$.

For the proof of $\Rightarrow$, assume that $\mathbf{A} = \{a_{ij}\}$ has row-Rank $\geq r$, $\mathbf{A} : X \to R^{nm}$. So there is some covering $\{\beta_i : X_i \to X \mid i \in I\}$ such that for each $i \in I$, r Independent rows of $\beta_i . \mathbf{A}$ can be picked. We now prove that each $\beta_i . \mathbf{A}$ has determinant-Rank $\geq r$; then also $\mathbf{A}$ will have this property, since it is a local property. For simplicity, consider an $i \in I$ where the *first* r rows of $\beta_i . \mathbf{A}$ are Independent. In particular, applying Lemmas A and B to the first row of $\beta_i . \mathbf{A}$, we get that the following statement holds:

$$\bigvee_{j=1}^{m} (\beta_i . a_{1j} \text{ is invertible}).$$

So there exists some covering

$$\{\beta'_k : X'_k \to X_i \mid k \in K\}$$

such that, for each $k \in K$, at least one of the n elements in $\hom(X_k, R)$ $\beta'_k . \beta_i . a_{1j}$ ($j = 1, \dots, n$) is invertible. It suffices to prove for each $k \in K$ that

$$\text{det.Rank}(\beta'_k . \beta_i . \mathbf{A}) \geq r.$$

For simplicity, let us consider a $k \in K$ where $\beta'_k . \beta_i . a_{11}$ is invertible. Clearly, we can perform row-operations (adding a multiple of one row to another) on any matrix over $\hom(X_k, R)$ without changing neither row-Rank nor determinant-Rank. We can now use the invertible $\beta'_k . \beta_i . a_{11}$ to sweep the rest of the first column by row operations. Having done that, we have a matrix $\mathbf{B}$ in $\hom(X_k, R)$ with same row- and determinant-Rank as $\beta_k . \beta_i . \mathbf{A}$, and which looks like

$$\mathbf{B} = \begin{Bmatrix} b_{11} & ? \\ 0 & \mathbf{B}' \end{Bmatrix}$$

with b_{11} invertible. Since row-Rank $(\mathbf{B}) \geqslant r$, we easily conclude that row-Rank $(\mathbf{B}') \geqslant r - 1$. By induction we may assume determinant-Rank $(\mathbf{B}') \geqslant r - 1$ (the start of the induction being again Lemma B). It is now clear that for any $\beta'' : X'' \to X'_k$ and any invertible $(r-1) \times (r-1)$ minor of $\beta'' . \mathbf{B}'$, we can construct an invertible $r \times r$ minor of $\beta'' . \mathbf{B}$, using the invertible element $\beta'' . b_{11}$. So $\mathbf{B}$ has determinant-Rank $\geqslant r$. This proves the theorem.

For field objects R, then, we do not have to distinguish between row-Rank or determinant-Rank, and shall sometimes just use the word Rank.

Consider again a commutative ring object R and a module object M over it. Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ be elements $X \to M$. An element $\boldsymbol{w} : X \to M$ is said to belong to Span $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m)$ if it satisfies (in the sense of section 1) the statement "there exist scalars $t_1, \dots, t_m$ such that $\boldsymbol{w} = \Sigma t_j . \boldsymbol{v}_j$", or in elementary terms that, locally, one can find such scalars t_j. (For a precise description, in elementary terms, of Span, see [8], section 2.) In general, Span$(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m)$ will be larger than the $\hom(X, R)$-submodule of $\hom(X, M)$ generated by the $\boldsymbol{v}_j$. (If the set $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ is Independent, it will not be larger, because of the principle "unique existence implies global existence".)

Now apply this for the case $M = R^n$. Let

$$\mathbf{A} : X \to (R^n)^p$$

and

$$\mathbf{B} : X \to (R^n)^q$$

be two matrices with same domain X, of size $p \times n$ and $q \times n$, respectively. We say $\mathbf{B} \leqslant \mathbf{A}$ if each row of $\mathbf{B}$ belongs to Span of the set of rows of $\mathbf{A}$. This defines a preorder-relation $\leqslant$ on the set of matrices with n columns having X as domain. Again $\leqslant$ is a local property, stable under base change. If $\mathbf{A}$ has a $p \times p$ submatrix $X \to R^{pp}$ which has invertible determinant, and $\mathbf{B} \leqslant \mathbf{A}$, then the coefficients t_{jk} used to display the rows of $\mathbf{B}$ as linear combinations of the rows of $\mathbf{A}$ are unique, hence globally defined, $t_{jk} : X \to R$, by "unique existence implies global existence". Using this principle once more, we see that it is even enough to assume that **A** has determinant-Rank p to get coefficients t_{jk} globally defined, $t_{jk}: X \to R$.

Using this, we can prove for an arbitrary commutative ring object R the following result, which for the case of matrices over ordinary commutative rings in the category of sets is well-known, [2], Exercise 20 p.A.III. 194.

Proposition 2.4. *Let* $\mathbf{A}: X \to (R^n)^p$ *be a* $p \times n$ *matrix of determinant-Rank* p. *If* $\mathbf{B}: X \to (R^n)^q$ *is a* $q \times n$ *matrix such that all* $(p+1)\times(p+1)$ *minors of the* $(p+q)\times n$ *matrix* $\begin{Bmatrix}\mathbf{A}\\ \mathbf{B}\end{Bmatrix}$ *are zero, then each of the rows of* **B** *is a linear combination of the rows in* **A** *(with coefficients from* $\hom(X, R)$*), and, in particular,* $\mathbf{B} \leqslant \mathbf{A}$.

Proof. We prove $\mathbf{B} \leqslant \mathbf{A}$. The fact that the coefficients can be taken globally (that is, in $\hom(X, R)$) then follows by the above remarks. By assumption, we can find a covering $\{\beta_i : X_i \to X \mid i \in I\}$ such that each matrix $\beta_i . \mathbf{A}$ actually has an invertible $p \times p$ minor, computed in the ring $\hom(X_i, R)$. The ordinary matrix theoretic result quoted from Bourbaki now gives, applied to this ring (in the category of sets) that each row of $\beta_i . \mathbf{B}$ can be written as a linear combination of the rows of $\beta_i . \mathbf{A}$, and the proposition follows.

If both **A** and **B** in the above proposition are $p \times n$ matrices of determinant-Rank p, then the coefficients used to write the rows of **B** as linear combinations of the rows of **A** form a $p \times p$ matrix **C** over the ring $\hom(X, R)$ with $\mathbf{B} = \mathbf{C} \cdot \mathbf{A}$. The determinant of **C**, by product rule for determinants, is necessarily invertible, so that **C** itself is invertible. So for such matrices **A** and **B**, $\mathbf{B} \leqslant \mathbf{A}$ is equivalent to **A** and **B** being congruent modulo the general linear group $\mathrm{GL}(p)$ over the ring $\hom(X, R)$. In particular, for $p \times n$ matrices **A**, **B** of determinant-Rank p, being congruent mod $\mathrm{GL}(p)$ in this standard way,

$$\mathbf{A} \equiv \mathbf{B} \bmod \mathrm{GL}(p),$$

is a local property (and stable under base change).

Proposition 2.5. *Let* R *be a field object in* **E**, *and let* **A** *and* **B** *be matrices* $X \to R^{np}$ *of size* $p \times n$, *both of Rank* p. *If*

$$\neg(\mathbf{A} \equiv \mathbf{B} \bmod \mathrm{GL}(p)),$$

then the $2p \times n$ *matrix* $\begin{Bmatrix}\mathbf{A}\\ \mathbf{B}\end{Bmatrix}$ *has Rank* $\geqslant p+1$.

(The converse implication also holds.)

Proof. The desired conclusion says that the statement

$$\bigvee_{\mathbf{H}} \{\mathbf{H} \text{ has invertible determinant}\}$$

holds, where **H** runs over the set of $(p+1)\times(p+1)$ submatrices of $\begin{Bmatrix}\mathbf{A}\\ \mathbf{B}\end{Bmatrix}$. To prove this conclusion, by (2.1) it is enough to see

$$(2.6)\qquad \neg\Big(\bigwedge_H \det(\mathbf{H})=0\Big),$$

with **H** as before. So let $\alpha : Y \to X$ be such that $\det(\alpha.\mathbf{H})=0$ for all **H**. This implies, by Proposition 2.4, that $\alpha.\mathbf{B}\leqslant\alpha.\mathbf{A}$, and, by the remarks after the proposition, it even implies

$$\alpha.\mathbf{A}\equiv\alpha.\mathbf{B} \bmod \mathrm{GL}(p)$$

in $\hom(Y,R)$. Since $\neg(\mathbf{A}\equiv\mathbf{B} \bmod \mathrm{GL}(p))$ was assumed, we conclude $Y=\emptyset$. This proves (2.6).

3. Universal projective geometry

Our aim here is to describe a subtheory of the theory of local rings, which might reasonably be called projective geometry over local rings. The idea is that this theory comes about in a canonical way from pure intuitionistic projective geometry over a field, and is thus a canonically algebraicized form of pure synthetic projective geometry, $\mathscr{P}$, or "universal projective geometry". By this we mean a certain theory formulated solely in terms of the primitive notions "point", "line", ..., and the predicates of incidence and equality.

We shall only be dealing with the plane case, $\mathscr{P}=\mathscr{P}_2$ (which is the case for which we have immediate geometric applications via Study's transfer). The algebra needed for the higher-dimensional cases has essentially been begun in [8].

We begin by describing the first order language $\mathscr{L}$ for $\mathscr{P}_2$. It contains variables of two sorts: $p,q,\ldots$ for points, and $l,m,\ldots$ for lines. The primitive predicates are

$p\equiv q$ (equality of points)

$l\equiv m$ (equality of lines)

$p\in l$ (incidence),

as well, for technical reasons, a unary predicate $\mathrm{P}(p)$, meaning "p is a point", and another one $\mathrm{L}(l)$, meaning "l is a line". Apart from that, it contains the logical symbols $\forall$, $\exists$, $\wedge$, $\vee$, $\Rightarrow$ and $\neg$. For a formula to be well-formed, we demand, for technical reasons that all quantifiers appear bound by the predicates P or L, that is, $\exists p(\varphi(p))$, say, should be of the form $\exists p(\mathrm{P}(p)\wedge\varphi'(p))$, and $\forall p(\varphi(p))$ should be of the form $\forall p(\mathrm{P}(p)\Rightarrow\varphi'(p))$; similarly for line-variables l. After having made the interpretation of this language $\mathscr{L}$ into the language $\mathscr{R}$ of the theory of commutative rings, we shall start writing more informally, and drop the P's and L's.

As an example of a sentence in $\mathscr{L}$, we mention

$$*\qquad \forall p,q([\mathrm{P}(p)\wedge\mathrm{P}(q)\wedge\neg(p\equiv q)]\Rightarrow\exists l[\mathrm{L}(l)\wedge p\in l\wedge q\in l]),$$

"through two different points passes a line". In an obvious way, $\mathscr{L}$ is a self-dual language, by interchange of "points" and "lines". The dual of * thus is

$$\forall l, m([\mathrm{L}(l) \wedge \mathrm{L}(m) \wedge \neg (l \equiv m)] \Rightarrow \exists p[\mathrm{P}(p) \wedge p \in l \wedge p \in m]).$$

("Two different lines intersect in a point".)

We now construct a map $\mathscr{L} \to \mathscr{R}$, which really just expresses that whenever one has a ring A, then one has a "projective plane" over it (a fairly bad one, though, unless A is a field). We start by dividing some of the variable symbols of $\mathscr{R}$ into disjoint, ordered 3-tuples, (1×3 matrices), and the rest of the variable symbols of $\mathscr{R}$ is divided into disjoint ordered 6-tuples (2×3-matrices); assume we have made a 1–1 correspondence between the variable symbols p of the point type, and the set of selected 1×3 matrices, and similarly a 1–1 correspondence between the set variable symbols l of $\mathscr{L}$ of the line type, and the set of selected 2×3 matrices. We shall write

$$p \sim (p_1\, p_2\, p_3)$$

as an abbreviation for "to the variable symbol p in $\mathscr{L}$ corresponds the 3-tuple (p_1, p_2, p_3) of the variable symbols of $\mathscr{R}$". Similarly

$$l \sim \begin{Bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \end{Bmatrix}. \tag{3.1}$$

Sometimes we write $\boldsymbol{p}$ for (p_1, p_2, p_3), and $(\boldsymbol{l}_1, \boldsymbol{l}_2)$ for the two rows in the matrix (3.1).

To the atomic formulae of $\mathscr{L}$ are now associated formulas in $\mathscr{R}$ in the following way:

To $\mathrm{P}(p)$, associate the formula in $\mathscr{R}$:

$$p_1 \text{ is invertible } \vee\ p_2 \text{ is invertible } \vee\ p_3 \text{ is invertible},$$

where $p \sim (p_1, p_2, p_3)$. To $\mathrm{L}(l)$ similarly associate the formula which says that at least one of the three 2×2 minors in the matrix (3.1) is invertible. To $p \equiv q$, associate the formula

$$\exists t : (t \text{ invertible}) \wedge t \cdot p_1 = q_1 \wedge t \cdot p_2 = q_2 \wedge t \cdot p_3 = q_3,$$

where $p \sim (p_1, p_2, p_3)$ and $q \sim (q_1, q_2, q_3)$. To $l \equiv m$, similarly associate the formula which says that there exists an invertible 2×2-matrix $\{t_{ik}\}$ such that

$$\{l_{ij}\} = \{t_{ik}\} \cdot \{m_{kj}\},$$

where $l \sim \{l_{ij}\}$ and $m \sim \{m_{kj}\}$. Finally, to the formula $p \in l$, associate the formula:

$$\text{"the determinant of } \begin{Bmatrix} p_1 & p_2 & p_3 \\ l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \end{Bmatrix} \text{ is zero"},$$

where $p \sim (p_1, p_2, p_3)$, $l \sim \{l_{ij}\}$.

To the composite formulas in $\mathscr{L}$ are now associated the corresponding composites of the above specific ring-theoretic formulas. We get in particular a map from the set $S(\mathscr{L})$ of sentences in $\mathscr{L}$ (formulas without free variables) to the set $S(\mathscr{R})$ of sentences in $\mathscr{R}$. The image of this map $S(\mathscr{L}) \to S(\mathscr{R})$ is the set of those ring-theoretic sentences which express pure incidence properties. For simplicity, let us view $S(\mathscr{L})$ as a subset of $S(\mathscr{R})$. If now $\mathscr{F} \subseteq S(\mathscr{R})$ denotes the set of sentences intuitionistically derivable from the axioms for commutative rings *plus* the field axioms (2.1) and (2.2), then we define $\mathscr{P} = \mathscr{F} \cap S(\mathscr{L})$. So $\mathscr{P}$ is the set of those pure incidence sentences, which are intuitionistically derivable from the notion of field. We call $\mathscr{P}$ *universal projective geometry*. The justification for this name is given in Corollary 4.2 below.

Note that, because from the field axioms (2.1) and (2.2) it follows that x invertible $\Leftrightarrow \neg(x = 0)$, we would have got the same $\mathscr{P}$, if the interpretation $\mathscr{L} \to \mathscr{R}$ had started off with $\mathrm{P}(p) \mapsto \neg(p_1 = 0) \vee \neg(p_2 = 0) \vee \neg(p_3 = 0)$. A similar remark can be made for the other primitive predicates. It will also be useful here to record a formula which is equivalent to the interpretation of $l \equiv m$.

Proposition 3.1. *Let R be a ring object in a site, and* $\mathbf{L}$ *and* $\mathbf{M}$ *two* 2×3 *matrices, that is, maps*

$$X \to (R^2)^3$$

of determinant-Rank 2. *Then* $\mathbf{L} \equiv \mathbf{M} \bmod \mathrm{GL}(2)$ *if and only if* $\mathbf{L}^\perp \equiv \mathbf{M}^\perp \bmod \mathrm{GL}(1)$ *(where* $\mathbf{L}^\perp$ *denotes that* 1×3 *matrix* $X \to R^3$ *which results by taking the usual vector product* $\boldsymbol{l}_1 \times \boldsymbol{l}_2$ *of the two rows of* $\mathbf{L}$*; similarly for* $\mathbf{M}^\perp$*).*

Proof. The implication $\Rightarrow$ follows from the product rule for determinants. Assume conversely $\mathbf{L}^\perp \equiv \mathbf{M}^\perp \bmod \mathrm{GL}(1)$. Then all 3×3 minors of the 4×3 matrix $\begin{Bmatrix} \mathbf{L} \\ \mathbf{M} \end{Bmatrix}$ are 0. By Proposition 2.4, $\mathbf{L} \leqslant \mathbf{M}$, and thus $\mathbf{L} \equiv \mathbf{M} \bmod \mathrm{GL}(2)$.

Remark 3.2. Since the coordinates of $\boldsymbol{l}_1 \times \boldsymbol{l}_2$ are the coordinates of $\boldsymbol{l}_1 \wedge \boldsymbol{l}_2$ in $\overset{2}{\wedge} R^3$ (with respect to the canonical basis there), the proposition just proved expresses that we might equivalently have interpreted lines as "proper" bivectors $\boldsymbol{l}$ in $\overset{2}{\wedge} R^3$, and equality $\equiv$ of lines as proportionality of proper bivectors (a bivector being *proper* if it has at least one coordinate invertible).

Remark 3.3. The statement "$p \in l$" also admits (under the assumption of the field axioms) an interpretation in $\wedge R^3$, namely as "$\boldsymbol{p}$ divides $\boldsymbol{l}$ in the algebra $\wedge R^3$". For, if $\boldsymbol{l} = \boldsymbol{l}_1 \wedge \boldsymbol{l}_2$ and $\boldsymbol{p} \in \boldsymbol{l}$, meaning that a certain 3×3 matrix has determinant 0, we conclude by Proposition 2.4 that $\boldsymbol{p} \in \mathrm{Span}(\boldsymbol{l}_1, \boldsymbol{l}_2)$. It is now easy to conclude that (locally) $\boldsymbol{p}$ divides $\boldsymbol{l}_1 \wedge \boldsymbol{l}_2$, and, in fact, by [8] Theorem 3.2 (Steinitz Exchange Theorem) that

$$\exists \boldsymbol{q} : \boldsymbol{p} \wedge \boldsymbol{q} = \boldsymbol{l}_1 \wedge \boldsymbol{l}_2$$

is valid.

For the sake of illustration we compile here a list of sentences in $\mathscr{P}$, that is, of theorems of pure intuitionistic projective geometry over a field. (It cannot be compared directly with Heyting's intuitionistic projective geomety (over a field), [5], since Heyting's geometry is not pure: it contains a separate apartness predicate "$p\omega q$".) The proofs, or references for proofs, will be given afterwards. We omit the P and L predicates. The duals of the theorems listed also hold.

Theorem P1. $\forall p, q : \neg(p \equiv q) \Rightarrow \exists ! l : p \in l \wedge q \in l$ (*"through two different points passes a unique line"*).

Theorem P2. $\forall p \forall l : p \in l \Rightarrow \exists q(\neg(p \equiv q) \wedge q \in l)$ (*"for any line and any point on it, we can always find one more point on it"*).

Theorem P3. $\forall p, q, r : \neg(p \equiv q) \Rightarrow (\neg(p \equiv r) \vee \neg(q \equiv r))$;

and finally Pappos' (or Pascal's) Theorem which we state by allowing ourselves to introduce the name $[pq]$ for the unique line (as asserted in Theorem P1) passing through p and q, and similarly using $l \cap m$ to denote the unique point simultaneously on two different lines l and m (as asserted in the dual of Theorem P1).

Theorem P4 (Pappos). *Let l and l' be lines such that $\neg(l \equiv l')$. Let p, q, r be points on l and p', q', r' points on l'; assume that all these six points are different. Then also*

$$\neg([pp'] \equiv [q'r]), \qquad \neg([p'q] \equiv [rr']) \quad \textit{and} \quad \neg([qq'] \equiv [r'p]),$$

and there is a unique line m such that

$$([pp'] \cap [q'r] \in m) \wedge ([p\ q] \cap [rr'] \in m) \wedge ([qq'] \cap [r'p] \in m).$$

Proof of Theorem P1. The sentence in Theorem P1 becomes under the map $\mathscr{L} \to \mathscr{R}$ the following sentence in ring theory

(3.2) $\forall p_1, p_2, p_3, q_1, q_2, q_3[\varphi(p_1, \ldots, q_3) \Rightarrow \psi(p_1, \ldots, q_3)]$,

where $\varphi(p_1, \ldots, q_3)$ is the statement

at least one of the p_i's is invertible, and at least one of the q_j's is invertible, and

$$\neg(\exists t : t \text{ invertible and } t \cdot (p_1, p_2, p_3) = (q_1, q_2, q_3)),$$

and where $\psi(p_1, \ldots, q_3)$ is the statement

$\exists$ 2×3 matrix $\{l_{rs}\}$ of Rank 2 such that the matrices

$$\begin{Bmatrix} p_1 & p_2 & p_3 \\ l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} q_1 & q_2 & q_3 \\ l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \end{Bmatrix}$$

both have determinant 0; and, if $\{l'_{rs}\}$ is another 2×3 matrix with this property, then $\{l_{rs}\}$ and $\{l'_{rs}\}$ are congruent mod GL(2).

To prove (3.2), assume that a topos $\boldsymbol{E}$ with a field object R is given, and that there are given six elements in R

$$p_1 : X \to R, \ p_2 : X \to R, \ldots, q_3 : X \to R,$$

all with same domain, and satisfying the statement φ. We must prove that they also satisfy ψ. Since φ is satisfied, we conclude, by Proposition 2.5 (with $p = 1$, $n = 3$) that the 2×3 matrix $\mathbf{B} : X \to R^6$, where

$$\mathbf{B} = \begin{Bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{Bmatrix},$$

has Rank 2. That matrix can clearly be used as $\{l_{rs}\}$, thus proving the existence statement in ψ (with $\mathrm{id}_X : X \to X$ as witnessing cover). To see the uniqueness part of ψ (which formally is a universally quantified statement: "for all $l'_{rs} \ldots$,") assume that $\alpha : X' \to X$ and $\mathbf{A} : X' \to R^6$ is given, where

$$\mathbf{A} = \{l'_{rs}\}$$

is a 2×3 matrix of Rank 2 which gives 0 determinant when taken together with the 1×3-matrix $\{\alpha . p_1, \alpha . p_2, \alpha . p_3\}$, and also when taken together with the 1×3-matrix $\{\alpha . q_1, \alpha . q_2, \alpha . q_3\}$. By Proposition 2.4 with $q = 1$,

$$\{\alpha . p_1, \alpha . p_2, \alpha . p_3\} \leq \mathbf{A}$$

and

$$\{\alpha . q_1, \alpha . q_2, \alpha . q_3\} \leq \mathbf{A}$$

whence $\alpha . \mathbf{B} \leqq \mathbf{A}$. Since both $\alpha . \mathbf{B}$ and $\mathbf{A}$ are Rank 2 matrices, it follows from the remarks just before Proposition 2.4 that $\alpha . \mathbf{B} \equiv \mathbf{A} \bmod \mathrm{GL}(2)$. This proves the uniqueness, and thus Theorem P1.

Remark 3.4. Note that the proof consisted in using the field assumptions (in the form of Proposition 2.5) to transform the negative statement $\varphi(p_1, \ldots, q_3)$ into a positive (in fact "coherent") statement, namely one about (determinant)-Rank of a certain 2×3-matrix. The implication also goes the other way. If the matrix $\mathbf{B}$ above has Rank 2, then validity of $\varphi(p_1, \ldots, q_3)$ follows from $\neg(1 \equiv 0)$.

Proof of Theorem P2. This is already almost proved, by Remark 3.3. We just have to see that the $\boldsymbol{q}$ asserted to exist in that remark satisfies $\neg(\boldsymbol{p} \equiv \boldsymbol{q})$. But $\mathrm{Rank}(\boldsymbol{p}, \boldsymbol{q}) = 2$ implies $\neg(\boldsymbol{p} \equiv \boldsymbol{q}) \bmod \mathrm{GL}(1)$.

Proof of Theorem P3. To prove that the ring theoretic translation (under $\mathcal{S} \to \mathcal{R}$) of the sentence in Theorem P3 holds for an arbitrary field object in a topos, we may replace the (ring theoretic translation of) $\neg(p \equiv q)$ by the equivalent statement:

Rank($\{p, q\}$) = 2, just as in the proof of Theorem P1 (see Remark 3.4). Thus the problem is reduced to proving that the followi.ıg sentence holds for an arbitrary field object R in an arbitrary topos:

For all 3×3-matrices

$$\begin{Bmatrix} p \\ q \\ r \end{Bmatrix} \tag{3.3}$$

where each row, as 1×3-matrix has (determinant-) Rank 1, and where $\begin{Bmatrix} p \\ q \end{Bmatrix}$ has (determinant-) Rank 2, we have

$$\left(\begin{Bmatrix} p \\ r \end{Bmatrix} \text{ has (determinant-) Rank } 2\right) \vee \left(\begin{Bmatrix} q \\ r \end{Bmatrix} \text{ has (determinant-) Rank } 2\right).$$

Since $\begin{Bmatrix} p \\ r \end{Bmatrix}$ is a 2×3-matrix, the statement that it has (determinant-) Rank $= 2$ is a three-fold disjunction "one of the three 2×2-minors of $\begin{Bmatrix} p \\ r \end{Bmatrix}$ is invertible". Similarly for $\begin{Bmatrix} q \\ r \end{Bmatrix}$, so that the desired conclusion is a six-fold disjunction

$$\left(\text{the first } 2 \times 2 \text{ minor of } \begin{Bmatrix} p \\ r \end{Bmatrix} \text{ is invertible}\right) \vee \ldots$$

$$\ldots \vee \left(\text{the third } 2 \times 2 \text{ minor of } \begin{Bmatrix} q \\ r \end{Bmatrix} \text{ is invertible}\right).$$

To prove this, it suffices, by the field axiom (2.1) (with $n = 6$) to prove

$$\begin{aligned} \neg\Big[&\left(\text{the first } 2 \times 2 \text{ minor of } \begin{Bmatrix} p \\ r \end{Bmatrix} \text{ is } 0\right) \wedge \ldots \\ &\ldots \wedge \left(\text{the third } 2 \times 2 \text{ minor of } \begin{Bmatrix} q \\ r \end{Bmatrix} \text{ is } 0\right)\Big]. \end{aligned} \tag{3.4}$$

So le $p : X \to R^3$, $q : X \to R^3$, $r : X \to R^3$ be elements of R^3 satisfying the assumptions in (3.3). We shall prove that also (3.4) is satisfied for these elements. Assume that $\alpha : Y \to X$ is so that the statement inside the square bracket holds, that is

$$\begin{aligned} \Big[&\left(\text{the first } 2 \times 2 \text{ minor of } \begin{Bmatrix} \alpha . p \\ \alpha . r \end{Bmatrix} \text{ is } 0\right) \wedge \ldots \\ &\ldots \wedge \left(\text{the third } 2 \times 2 \text{ minor of } \begin{Bmatrix} \alpha . q \\ \alpha . r \end{Bmatrix} \text{ is } 0\right)\Big]. \end{aligned}$$

So we have now an ordinary commutative ring hom(Y, R), and a 3×3 matrix $\mathbf{C}$ over it, with the property that those six 2×2 minors which contain elements from the last row, are all 0. Two applications of Proposition 2.4 then yields

$$\alpha . p \leqslant \alpha . r \quad \text{and} \quad \alpha . q \leqslant \alpha . r,$$

and then further, by the remark following the proof of that proposition

$$\alpha . p \equiv \alpha . r \mod \mathrm{GL}(1)$$

$$\alpha . q \equiv \alpha . r \mod \mathrm{GL}(1)$$

whence $\alpha . p \equiv \alpha . q \mod \mathrm{GL}(1)$. Since $\begin{Bmatrix} p \\ q \end{Bmatrix}$ has Rank 2, by assumption, we have $\neg (p \equiv q \mod \mathrm{GL}(1))$, whence we conclude $Y = \emptyset$. This proves (3.4).

Proof of Theorem P4. Again assume that R is a field object in a topos $\boldsymbol{E}$, and that

$$l : X \to R^6, \qquad l' : X \to R^6,$$

$$p : X \to R^3, \qquad q : X \to R^3, \qquad r : X \to R^3$$

$$p' : X \to R^3, \qquad q' : X \to R^3, \qquad r' : X \to R^3$$

are given matrices (respectively 2×3 matrices of Rank 2, and 1×3 matrices of Rank 1), satisfying the (ring theoretic translations of) the assumptions in Pappos' Theorem P4. Some preparations of logical nature are needed before the proof is reduced to standard linear algebra. Let $s : X \to R^3$ denote the point of intersection of l and l' (it can be computed explicitly as $(l_1 \times l_2) \times (l'_1 \times l'_2)$, which is why we may assume that s is globally defined, that is, has domain X; the uniqueness assertion in Theorem P1 (dualized) is not so strong that the principle "unique existence implies global existence" can be applied, since that theorem only asserts uniqueness modulo GL(1)). Now denote by P the statement $\neg (s \equiv p)$, by Q the statement $\neg (s \equiv q), \ldots,$ by R' the statement $\neg (s \equiv r')$. Since $\neg (p \equiv q)$, we have by Theorem P3 that $P \vee Q$ holds. Similarly, we get the other factors in the conjunction

$$(3.5) \qquad (P \vee Q) \wedge (P \vee R) \wedge (Q \vee R) \wedge (P' \vee Q') \wedge (P' \vee R') \wedge (Q' \vee R').$$

Now $\wedge$ distributes over $\vee$ in intuitionistic logic; so it is easy to see that (3.5) implies

$$(P \wedge P') \vee (Q \wedge Q') \vee (R \wedge R').$$

So the common domain X of our matrices $l, l', p, q, \ldots, r', s$ can be covered by $\{\beta_i : X_i \to X \mid i \in I\}$ such that, for each i, $P \wedge P'$ or $Q \wedge Q'$ or $R \wedge R'$ holds. For each $i \in I$, we can now prove the validity of the conclusion in Theorem P4, and this is enough since local validity implies validity. So consider for example an $i \in I$ where $P \wedge P'$ holds; that is

$$\neg (\beta_i . s \equiv \beta_i . p) \wedge \neg (\beta_i . s \equiv \beta_i . p').$$

The 3×3 matrix

$$\begin{Bmatrix} \beta_i . s \\ \beta_i . p \\ \beta_i . q \end{Bmatrix}$$

can now be seen to have Rank 3. For since $\neg(\beta_i . s \equiv \beta_i . p)$, the 2×3 matrix

$$\begin{Bmatrix} \beta_i . s \\ \beta_i . p \end{Bmatrix}$$

has Rank 2 by Proposition 2.5, and thus, by the uniqueness assertion in Theorem P1

$$\begin{Bmatrix} \beta_i . s \\ \beta_i . p \end{Bmatrix} \equiv \beta_i l \text{ mod GL}(2).$$

Similarly

$$\begin{Bmatrix} \beta_i . s \\ \beta_i . p' \end{Bmatrix} \equiv \beta_i . l' \text{ mod GL}(2).$$

Since $\neg(l \equiv l')$, we have $\neg(\beta_i . l \equiv \beta_i . l' \text{ mod GL}(2))$, and so

$$\neg\left(\begin{Bmatrix} \beta_i . s \\ \beta_i . p \end{Bmatrix} \equiv \begin{Bmatrix} \beta_i . s \\ \beta_i . p' \end{Bmatrix} \text{ mod GL}(2)\right),$$

and by Proposition 2.5 again the 4×3 matrix

$$\begin{Bmatrix} \beta_i . s \\ \beta_i . p \\ \beta_i . s \\ \beta_i . p' \end{Bmatrix}$$

has Rank 3. It has the row $\beta_i . s$ repeated, so also the 3×3 matrix obtained by removing the one occurrence of $\beta_i . s$ has Rank 3, thus is an invertible 3×3 matrix over the ring $\hom_E(X_i, R)$. We can now do computations in coordinates with respect to the basis $\mathbf{C} = \{\beta_i . s, \beta_i . p, \beta_i . p'\}$ for $\hom_E(X_i, R)$. From here on, the proof proceeds almost as the standard proof of Pappos' Theorem where one chooses the intersection of the lines plus two of the given points as "fundamental triangle". In coordinates with respect to the basis $\mathbf{C}$ (and omitting the β_i from notation)

$$s : (1, 0, 0)$$

$$p : (0, 1, 0)$$

$$p' : (0, 0, 1)$$

$$q : (b_1, b_2, 0).$$

Since $\neg(p \equiv q)$, the Rank of

$$\begin{Bmatrix} 0 & 1 & 0 \\ b_1 & b_2 & 0 \end{Bmatrix}$$

is 2, by Proposition 2.5, and therefore, b_1 is invertible. Thus we may replace the representative $(b_1, b_2, 0)$ for q by the simple $(1, b, 0)$. Similarly, for q', r, and r'

$$q : (1, b, 0)$$

$$q' : (1, 0, b')$$

$$r:(1,c,0)$$

$$r':(1,0,c').$$

To see that $\neg([pp']\equiv[q'r])$, it suffices to see that the matrix

$$\begin{Bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & b' \\ 1 & c & 0 \end{Bmatrix}$$

has Rank 3, which is obvious. The other non-equalities in the statement of the theorem are proved similarly. The three intersection points mentioned in the theorem can now be found explicitly. They are the rows of the matrix $\mathbf{D}$ =

$$\begin{Bmatrix} 0 & -c & b' \\ -c & -bc & bc'-cc' \\ b' & bb'-bc' & b'c' \end{Bmatrix}.$$

The determinant of this matrix is easily seen to be 0, and therefore the points are "colinear" in some weak sense. However, we are required to prove something stronger, namely the existence statement "there exists a line m such that ...". Applying Theorem P3 to $\neg(r\equiv q')$, we conclude

$$\neg(r\equiv s)\vee\neg(q'\equiv s),$$

so that the domain X_i of the matrix $\mathbf{D}$ can be covered, by $\{\delta_j : X'_j \to X_i \mid j\in J\}$, such that for each j, either $\neg(\delta_j . r\equiv\delta_j . s)$ or $\neg(\delta_j . q'\equiv\delta_j . s)$. This same covering will now be a witness of the existence of m; for a $j\in J$ where $\neg(\delta_j . r\equiv\delta_j . s)$, we have that $\delta_j . c$ is invertible, and the submatrix of $\delta_j . \mathbf{D}$ consisting of the two first rows then has Rank 2, and can be used as an $m : X'_j\to R^6$ satisfying the conclusion of the theorem. For a $j\in J$ where $\neg(\delta_j . q'\equiv\delta_j . s)$ we similarly have $\delta_j . b'$ invertible, and the submatrix of $\delta_j . \mathbf{D}$ obtained by omitting the middle row has Rank 2, and can be used as $m : X'_j\to R^6$ satisfying the conclusions of the theorem.

Uniqueness of m modulo GL(2) follows from the uniqueness statement in Theorem P1.

The proof of the dual of Theorem P1 is similar to the proof of Theorem P1; in fact, we already noted, in the Remark 3.2, that "equality of lines" was defined "correctly", namely as proportionality of elements in $\overset{2}{\wedge}R^3$. The self-duality of $\wedge R^3$ (see [8]) can now be used to deduce the dual of Theorem P1 from Theorem 1 itself. Similarly for the duals of the Theorems P2, P3 and P4.

4. A logical transfer principle, and the Study transfer

The ring theoretic translations of the sentences Theorem P1–Theorem P4 are all (under the assumption of the field axioms (2.1)–(2.2)) equivalent (in intuitionistic logic) to some ring theoretic sentences, not involving negation. We have in fact utilized this. For instance

$$\neg(p \equiv q) \quad \text{in} \quad \mathscr{L}$$

translates into

$$\text{Rank}(\boldsymbol{p}) = 1 \wedge \text{Rank}(\boldsymbol{q}) = 1 \wedge \neg(\boldsymbol{p} \equiv \boldsymbol{q} \mod \text{GL}(1))$$

which, however, is equivalent to

$$\text{Rank}\begin{Bmatrix}\boldsymbol{p}\\ \boldsymbol{q}\end{Bmatrix} = 2$$

by Proposition 2.5. For Theorem P3, we have even indicated the complete conversion of it into negation-free ring theoretic terms, namely in the sentence (3.3). Also the ring theoretic translation of

$$\neg(l \equiv m)$$

is equivalent (under the assumption of the field axioms (2.1), (2.2)) to a positive statement: for, the ring theoretic translation of $\boldsymbol{l} \equiv \boldsymbol{m}$ is by Proposition 3.1 equivalent to $\mathbf{L}^{\perp} \equiv \mathbf{M}^{\perp}$, thus $\neg(l \equiv m)$ is equivalent to $\neg(\mathbf{L}^{\perp} \equiv \mathbf{M}^{\perp})$ which in turn, by Proposition 2.5, is equivalent to the positive statement

$$\text{Rank}\begin{Bmatrix}\mathbf{L}^{\perp}\\ \mathbf{M}^{\perp}\end{Bmatrix} = 2,$$

(recalling that $\mathbf{L}^{\perp}$ and $\mathbf{M}^{\perp}$ are 1×3 matrices).

The idea is now that many theorems of universal projective geometry are "positive" except for occurrence of negated equality, so that their ring theoretic translations under the assumption of the field axiom are (intuitionistically) equivalent to certain *positive* ring theoretic sentences. More precisely, to *coherent* ring theoretic sentences. We recall the definition of this coherence notion:

A predicate in a 1[st] order language is said to be *coherent* [10] if it is built from atomic predicates by means of disjunction, conjunction, and existential quantification. A sentence of the form

$$\forall x_1, \ldots, x_n[\varphi(x_1, \ldots, x_n) \Rightarrow \psi(x_1, \ldots, x_n)],$$

where φ and ψ are coherent predicates, is called a coherent sentence. Likewise sentences

$$\forall x_1, \ldots, x_n[\varphi(x_1, \ldots, x_n)]$$

and

$$\forall x_1, \ldots, x_n[\neg\varphi(x_1, \ldots, x_n)]$$

are considered coherent. The interest in this notion lies in the fact that if $f : \boldsymbol{E} \to \boldsymbol{E}'$ is a geometric morphism of toposes, then $f^* : \boldsymbol{E}' \to \boldsymbol{E}$ preserves validity of coherent sentences.

For instance, "determinant-Rank(A) $\geq r$" is a coherent predicate, whereas "row-Rank(A) $\geq r$" is not a coherent predicate.

Our aim is to get information about geometry over local rings. This is achieved by considering ring-theoretic *coherent* sentences of universal projective geometry, together with

Theorem 4.1. *Every coherent sentence (in the language of the theory of rings) intuitionistically derivable from the field axioms* (2.1)–(2.2) *holds for any local ring (in fact any local ring object in any topos).*

Proof. This follows by combining two facts about the local ring object $\mathcal{O}_1$ in the Zariski topos $\mathcal{Z}$: 1) $\mathcal{O}_1$ is a field object (Proposition 2.2); 2) $\mathcal{O}_1$ is universal (or generic) among local ring objects in toposes, meaning that whenever $\mathcal{E}$ is a topos and $\mathcal{O}$ a local ring object in it, there exists a geometric morphism of toposes $f: \mathcal{E} \to \mathcal{Z}$ with $f^*(\mathcal{O}_1) \simeq \mathcal{O}$ (Hakim [4], Corollaire 3.10). Since f preserves validity of coherent sentences, we have that every coherent sentence which holds for $\mathcal{O}_1$, holds for $\mathcal{O}$. Now if a coherent sentence is derivable intuitionistically from the field axioms, it holds for $\mathcal{O}_1$, since $\mathcal{O}_1$ is a field object, and since "intuitionistically derivable" means "derivable in any topos", in particular in $\mathcal{Z}$. This proves the theorem.

Note that we cannot weaken the assumptions of the theorem: if we skipped the word "coherent", the theorem would allow us to conclude that the field axiom (2.1) itself also holds for any local ring. This is clearly false, since not all local rings are fields. If we skipped the word "intuitionistically", we would, from the field axiom (2.1) classically derive the coherent sentence

$$\forall x : (x = 0 \vee x \text{ is invertible})$$

which again is not true for all local rings.

Corollary 4.2. *Every ring theoretic coherent sentence intuitionistically equivalent (under the assumption of the field axioms) to a sentence of universal projective geometry $\mathcal{P}$, holds for arbitrary local rings.*

This "logical" transfer principle can be utilized together with Study's geometric transfer principle. We briefly discuss this. Let $\mathbf{D} = \mathbf{R}[\varepsilon]$ denote the ring of dual numbers ($\varepsilon^2 = 0$). If $p \in \mathbf{D}^3$ satisfies P(p), it is possible to associate to p a line in $\mathbf{R}^3$, and proportional p's define the same line. This line we call the Study-associate of p. Likewise if $l \in \mathbf{D}^{3\times 2}$ satisfies L(l), it is possible to associate to l a certain collection of lines in $\mathbf{R}^3$, or, alternatively, one single line in $\mathbf{R}^3$ called the Study associate of l: two l's which are proportional modulo GL(2, $\mathbf{D}$) define the same Study-associated line. Then $p \in l$ if and only if the line Study-associated to p intersects the line Study-associated to l at right angles. The property Rank$\begin{Bmatrix} p \\ q \end{Bmatrix} = 2$ (which under assumption of the field axioms is equivalent to $\neg(p \equiv q)$) then holds if and only if the lines in $\mathbf{R}^3$ associated to p and q are non-parallel. Similarly, that coherent property, which under the assumption of the field axioms is equivalent to $\neg(l \equiv m)$, holds if and only if the lines in $\mathbf{R}^3$ Study-associated to l and m respectively, are non-parallel.

We can now substitute all the negated constituents in the theorems P1–P4 by coherent formulas equivalent to them (under assumption of the field axioms); then P1–P4 become coherent ring theoretic sentences intuitionistically derivable from the field axioms and therefore valid for any local ring, by Corollary 4.2. In particular, they hold for $\mathbf{D}$, so by the above mentioned facts about the Study associates of points, lines, incidence and (the coherent equivalent of) non-equality, we get that P1–P4 have the following geometric meaning for lines in $\mathbf{R}^3$:

P1: Given two non-parallel lines p, q; then there is a unique line l intersecting both of them at right angles. (This l is called the *transversal* of p and q.)

P2: If a line p intersects a line l under right angles, we can find a line q which is not parallel to p and which also intersects l at right angles.

P3: If p and q are non-parallel lines, and r is an arbitrary line, then either r is not parallel to p, or r is not parallel to q; and finally, the transfer of P4, which gives a non-trivial geometric fact (which presumably is known by those geometers who invented the Study transfer):

Theorem 4.3. *Let l and l' be non-parallel lines in $\mathbf{R}^3$, and let a, b, c be lines intersecting l at right angles, and a', b', c' lines intersecting l' at right angles. The six lines a, b, c, a', b', c' are assumed pairwise non-parallel. Then the transversal of a, a' is not parallel to the transversal of b', c; let the transversal of these two transversals be denoted l_1. Similarly, l_2 is constructed out of $(a', b; c, c')$, and l_3 out of $(b, b'; c', a)$. Then there is a unique line m intersecting l_1, l_2 and l_3 at right angles.*

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