Revista Colombiana de Matemáticas
Vol. XX (1986) págs. 129-146

SYNTHETIC REASONING IN DIFFERENTIAL GEOMETRY

by

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All manifolds acquire in the context of synthetic differential geometry (SDG) a binary relation, the "neighbour" relation $\sim$, which is symmetric and reflexive, but not transitive. For instance, on the line R, the relation is given by

$$x \sim y \quad \text{iff} \quad x-y \in D \quad (= [\![d \in R \mid d^2 = 0]\!]).$$

It is *reflexive* ($x \sim x$) because $0 \in D$, and *symmetric* ($x \sim y \Rightarrow y \sim x$) because $d \in D \Rightarrow -d \in D$, but *not transitive*, since D is not stable under addition: if $d_1^2 = 0$ and $d_2^2 = 0$,

$$(d^1+d^2)^2 = d_1^2 + d_2^2 + 2d_1d_2 = 0+0+2d_1d_2 \ ,$$

but there is no reason for d_1d_2 to be zero.

Even in the category of sets, a reflexive symmetric relation $\sim$ on a set M gives rise to interesting combinatorial notions. In §1, we describe some of these. The logical character of this descriptions is evidently so that the notions described make sense in any topos, in particular in models of syntetic differential geometry where the notions acquire a geometric meaning. This meaning also motivates the terminology we shall use. It even *justifies* the terminology, a fact we shall, however, not prove.

§1. Combinatorial notions* derived from a reflexive symmetric relation ∿ on a set M.

Let such M, ∿ be given. If $x \sim y$, we say that x and y and *neighbour points*. The set $M_{(1)} \subseteq M \times M$ given by

$$M_{(1)} = [\![(x,y) \in (M \times M) \mid x \sim y]\!]$$

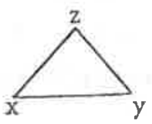
is called *the first neighbourhood of the diagonal*. It contains (the image of) the diagonal $\Delta : M \rightarrowtail M \times M$.

For any $x \in M$, the set $M_1(x) \subset M$ given by

$$M_1(x) = [\![y \in M \mid x \sim y]\!]$$

is called the 1-*monad* around x.

If $x \sim y$ and $y \sim z$, and also $x \sim z$, we say that the three elements x,y,z form an *infinitesimal triangle*:



the lines indicating the relation ∿. We say the triangle is *degenerate* if $x = y$ or if $x = z$ or if $y = z$.

A map $M_1(x) \to F$ (F any set) is called a 1-*jet* at x (with values in F).

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* Some of these notions were, in the present form, first considered by Joyal.

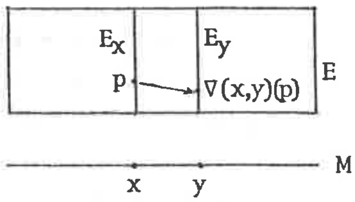
A map $\pi: E \to M$ (E any set) is called a *bundle* over M, and, for $x \in M$, the set $\pi^{-1}(x) \subseteq E$ is called the *fibre over* x, and may be denoted E_x.

A *connection* on a bundle $E \to M$ is a law ∇, which to any pair $x \sim y$ of neighbour points in M associates a map $\nabla(x,y)$: $E_x \to E_y$, such that

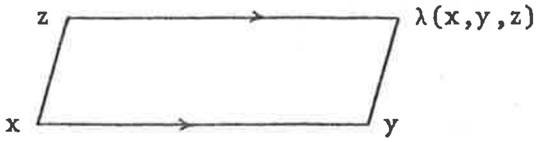
$$\nabla(x,x) = \nabla(y,x) \circ \nabla(x,y) = \text{identity map of } E_x.$$

(for many bundles in the context of synthetic differential geometry, $\nabla(x,x) = \text{id}$ will *imply* $\nabla(y,x) \circ \nabla(x,y) = \text{id}$ $\forall x \sim y$).

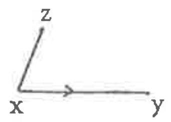
The effect of the connection ∇ may be *drawn:*



In particular, [3], consider the bundle $\text{proj}_1 : M_{(1)} \to M$ (which to $x \sim y$ associates x). The fibre over x is $M_1(x)$. A connection in this bundle is thus a law $\nabla(x,y)$ which to $z \sim x$ associates $\nabla(x,y)(z) \sim y$. If we write $\lambda(x,y,z)$ for $\nabla(x,y)(z)$, we see that a connection in the bundle $M_{(1)} \to M$ is the same as a partially defined ternary operation λ on M, with $\lambda(x,y,z)$ defined whenever $x \sim y$ and $x \sim z$, and with $\lambda(x,y,z) \sim y$. (In the context of SDG, it will then usually *follow* that also $\lambda(x,y,z) \sim z$). The effect of the operation λ may be drawn:



where the lines indicate the relation $\sim$. So it gives a way of completing a configuration



into a quadrangle, and quadrangles defined this way may be called the "infinitesimal parallellograms" of the connection λ. Note that we cannot conclude $\lambda(x,y,z) = \lambda(x,z,y)$ (if this holds, we say the connection is *torsion free*). Connections in the bundle $M_{(1)}$ correspond, in the context of SDG, to the classical notion of "connection on the *tangent* bundle $TM \to M$ of M".

Consider again the general case of an arbitrary bundle $E \to M$, and let ∇ be a connection. For an infinitesimal triangle x,y,z (see (1.1)), we may ask: do we have

$$\nabla(x,z) = \nabla(y,z)\circ\nabla(x,y) \tag{1.2}$$

Or equivalently

$$id_{E_x} = \nabla(z,x)\circ\nabla(y,z)\circ\nabla(x,y) \tag{1.3}$$

If this holds for all infinitesimal triangles, we say that the connection is *curvature-free.*

For $E \to M$, we can construct a groupoid FULL$(E \to M)$ with M as its set of objects, and where an arrow $x \rightarrowtail y$ is a bijective map $E_x \to E_y$.

Also, we can construct a groupoid $\Pi_o M$, with M as its set of objects, and for any x, y in M. there is exactly one arrow $x \rightarrowtail y$, denoted (x,y).

An *integral* for the connection ∇ is now defined as a *functor* $\tilde{\nabla} : \Pi_o M \to$ FULL$(E \to M)$ with $\tilde{\nabla}(x) = x$ for any object

$x \in M$, and with

$$\tilde{\nabla}(x,y) = \nabla(x,y) \quad \text{for} \quad x \sim y \tag{1.4}$$

Clearly, if the connection ∇ has an integral $\tilde{\nabla}$, it is curvature-free. For to say that $\tilde{\nabla}$ is a functor means that it preserves composition:

$$\tilde{\nabla}(x,z) = \tilde{\nabla}(y,z) \circ \tilde{\nabla}(x,y), \qquad \forall x,y,z \in M \tag{1.5}$$

But if x,y,z form an infinitesimal triangle, we may write ∇ instead of $\tilde{\nabla}$ in (1.5), because of (1.4). Then (1.5) becomes (1.2).

Many fundamental questions in differential geometry can be formulated as: for which bundles is it true that *every curvature-free connection has an integral* (in fact a unique one)?

Given a bundle $\pi : E \to M$, a *distribution on* E *transversal to the fibres of* π, is a law $\mathcal{D}$ which to each $p \in E$ associates a subset $\mathcal{D}(p)$ satistying

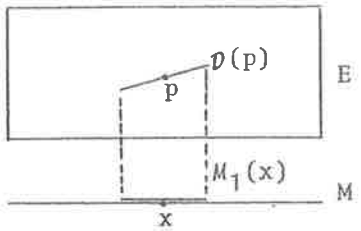
$$p \in \mathcal{D}(p) \tag{1.6}$$

$$q \in \mathcal{D}(p) \Leftrightarrow p \in \mathcal{D}(q) \tag{1.7}$$

and

$$\pi \text{ maps } \mathcal{D}(p) \text{ bijectively to } M_1(\pi(p)). \tag{1.8}$$

Such a distribution can be *drawn*:



(The drawing reflects (1.6) and (1.8), but not (1.7). However, for many bundles in the context of SDG, (1.7) will be implied by (1.6) and (1.8)).

There is a natural 1-1 correspondence between connections ∇ on $E \to M$, and such distributions: given ∇ define, for $p \in E_x$,

$$\mathcal{D}(p) := \{\nabla(x,y)(p) \mid y \in M_1(x)\},$$

and given $\mathcal{D}$, define ∇ by

$$\nabla(x,y)(p) := \text{unique element in } E_y \cap \mathcal{D}(p)$$

for $p \in E_x$.

To the notion of *integral* of connections corresponds a notion of *solution* of the distribution: a certain bijective map to E from a product-bundle, $M \times F \to E$, for a suitable F. We shall not go into it.

Let G be a group, written multiplicatively, and with neutral element e.

A *0-form* on M *with values in* G is a map $f: M \to G$.

A *1-form* on M *with values in* G is a map $\omega: M_{(1)} \to G$, with

$$\omega(x,x) = e \quad \text{and} \quad \omega(x,y) = \omega(y,x)^{-1} \quad \forall x \sim y.$$

(In the context of SDG, the second equation will usually follow from the first).

A 2-*form* on M, *with values in* G is a law Θ which to any infinitesimal triangle x,y,z associates an element $\Theta(x,y,z) \in G$, and associates e if the triangle is degenerate. The *zero* 2-form, denoted 0, takes value e on *all* infinitesimal triangles.

To a 0-form f we associate a 1-form df:

$$(df)(x,y) := f(y) \cdot f(x)^{-1},$$

and to a 1-form ω, we associate a 2-form $d\omega$:

$$(d\omega)(x,y,z) := \omega(z,x) \cdot \omega(y,z) \cdot \omega(x,y).$$

(Think of the right hand side here as the *curve integral* of ω around the *boundary* of the triangle x,y,z).

Clearly, d(df) = 0. A 1-form ω with $d\omega = 0$ is called *closed*. A 1-form ω which can be written $\omega = df$, for on $f: M \to G$ which is unique, modulo multiplication on the right by a fixed $a \in G$, is called *exact*. Because d(df) = 0, exact 1-forms are closed. If $\omega = df$, we say f is a *primitive* of ω.

Many fundamental question in differential geometry can be formulated as: for which $(M, \sim)$, G is it true that *every closed* G-*valued* 1-*form on* M *is exact* ?

Let N and M are sets, each equipped with a symmetric-reflexive relation $\sim$. Let ω be a G-valued (0,1, or 2-) form on M. Let $h: N \to M$ be a map preserving $\sim$. Then we get a (0,1, or 2-) form $h^*\omega$ on N by putting (for the 1-form case):

$$(h^*\omega)(n_1, n_2) := \omega(h(n_1), h(h_2))$$

($n_1 \sim n_2$ in N). Similarly for 0 or 2-forms. Clearly $d(h^*\omega) = h^*(d\omega)$. In particular

$$\omega \text{ closed} \Rightarrow h^*\omega \text{ closed.}$$

Let F be a fixed object. We denote by Diff(F) the group of all bijective maps $F \to F$.

Consider the product bundle $\pi: M\times F \to M$. Then there is a natural 1-1 correspondence between

$$\text{connections } \nabla \text{ on } M\times F \to M$$

and

$$\text{Diff}(F)\text{-valued 1-forms } \omega \text{ on } M.$$

For, given ω define ∇ by

$$\nabla(x,y)(x,u) := (y,\omega(x,y)(u)), \quad (u \in F) \tag{1.9}$$

and given ∇, define ω ("the *connection form*") by

$$\omega(x,y)(u) := \text{proj}_2(\nabla(x,y)(x,u)). \tag{1.10}$$

It is easy to see that ∇ is curvature-free if and only if ω is closed. Furthermore, if f is a primitive of ω, $df = \omega$, we can construct an integral $\tilde{\nabla}$ of ∇ by

$$\tilde{\nabla}(x,y)(x,u) := (y,f(y)(f(x)^{-1}(u))), \quad (u \in F) \tag{$\tilde{1.9}$}$$

and given an integral $\tilde{\nabla}$ of ∇, we can construct a primitive f of ω by *choosing* $x_0 \in M$, and putting

$$f(y)(u) := \text{proj}_2(\tilde{\nabla}(x_0,y)(x_0,u)). \tag{$\tilde{1.10}$}$$

If we had chosen x_1 instead of x_0, we would get another primitive, g, with $g\cdot a = f$ where $a \in \text{Diff}(F)$ is the constant element in Diff(F) given by

$$a(u) := \mathrm{proj}_2(\tilde{\nabla}(x_0,x_1)(x_0,u)) \quad (u \in F).$$

From this follows that (for product bundles) the question of integrals $\tilde{\nabla}$ for ∇ is equivalent to the question of primitives of the corresponding ω, and that the integral is *unique* iff the primitive is *unique modulo* right multiplication by a constant.

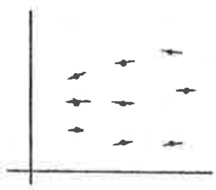
Many differential-geometric data present themselves naturally as connections (or distributions) on product bundles. The above considerations now prove that the question of the integration of the data reduces to the problems of finding primitives of closed G-values 1-forms, where $G = \mathrm{Diff}(F)$, i.e. to the problem of exactness of closed G-valued 1-forms. This question will be considered in the next numeral.

EXAMPLE. Consider an ordinary 1^{st} order differential equation of the form

$$y' = h(x,y).$$

The standard picture one draws of this has, in the present context, a mathematical status, namely as the picture of a distribution $\mathcal{D}$ on the bundle $R\times R \to R$

$$\mathcal{D}(x,y) = \{(x+d, y+h(x,y)\cdot d) \mid d \in D\}$$



We leave to the reader to write down the explicit formula for the connection ∇ and the connection form ω associated to this distribution. Also he many verify that a solution of $\mathcal{D}$ (respectively an integral for ∇, respectively a primitive for ω) "is" a solution of the differential equation.

Similarly, a partial differential equation of form

$$\begin{cases} D_1 z = h_1(x,y,z) \\ D_2 z = h_2(x,y,z) \end{cases}$$

presents itself as a distribution on the product bundle $R^2 \times R \to R^2$. Unlike in the 1-dimensional case above, it is easy to construct examples where such differential equation has no solution: 1-forms on R^2 need not be closed, whereas 1-forms on R^1, in the context of SDG, always turn out to be closed (for reasonable value-groups, like $(R,+)$ or Diff(R)).

§2. When are closed G-valued 1-forms on M exact?

In the context of SDG, we give a sufficient condition for this to be the case, which consists of two conditions:

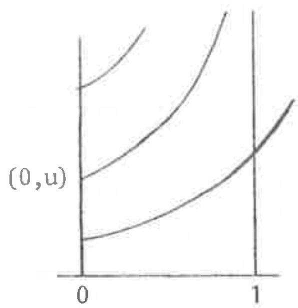
i) a condition on G ("G *admits integration*"), which does not depend on M,

ii) a condition on M ("M is *path connected* and *simply connected*"), which does not depend on G.

The context of SDG implies the existence of an ordering $\leqslant$ on "the line" R, so that it makes sense to talk about the "unit interval" $I = [0,1]$.

We say that a group G *admits integration* if any G-valued 1-form on I is exact. If $G = (R,+)$, this is the usual integration axiom. The group Diff(R) does not admit integration (see example below), but it has many large subgroups which do.

EXAMPLE. The differential equation $y' = y^2$ on I corresponds to a Diff(R)-valued 1-form which is not exact, since many of the solutions $y = (a-x)^{-1}$ do not extend over the whole of I. Expressed in terms of a connection and its integral, $\tilde{\nabla}$, $\tilde{\nabla}(0,1)$ can not be defined on (0,u) with u > 1:



(There should be a weaker sense in which Diff(R) admits integration, namely in a *local* sense: for any Diff(R)-valued 1-form ω on I, there exists a functor $\tilde{\omega}:\Pi_o I \to \mathrm{Diff}_{Loc}(R)$, extending ω, where $\mathrm{Diff}_{Loc}(R)$ is the groupoid of bijective maps from one open subset of R to another, for a suitable notion of open-ness).

We shall not discuss here the condition "G admits integration" any further. We shall sketch the proof of

THEOREM. *If* $G \subseteq \mathrm{Diff}(R^n)$ *is a subgroup which admits integration and* M *has connectedness properties* (ii) *above, then closed* G-*valued* 1-*forms on* M *are exact.*

Proof-sketch. Given a closed G-valued 1-form ω on M, to construct a primitive f of it, *choose* $x_o \in M$, and *choose* for each $x \in M$ a map $h_x : I \to M$ with $h_x(0) = x_o$, $h_x(1) = x$ ("h_x is a *path* from x_o to x"; the existence of such path is the pathwise connectedness-assumption on M). The crucial point of the proof is to prove that

$$f(x) := \int_{h_x} \omega \tag{2.1}$$

is *independent of the choice* of the path h_x. By the " curve integral" $\int_{h_x}\omega$, we mean $g(1)\cdot g(0)^{-1}$, where $g:I \to G$ is a *primitive* of the 1-form $h_x^*(\omega)$ on I. If we had chosen another such path, k_x, instead of h_x, the assumption of simply connectedness of M implies the existence of a map $H:I\times I \to M$ which restricts on the four sides of the square, to, respectively,

$$h_x, \text{ constant } x_o, \quad k_x, \text{ constant } x.$$

Since the two sides, where H is constant, do not contribute to the curve integral of $H^*\omega$ around the periphery, we get that $\int_{h_x}\omega = \int_{k_x}\omega$ iff the integral of $H^*\omega$ around the periphey yields $e \in G$. Now $H^*\omega$ is a *closed* 1-form on I×I which is the zero form on two opposite sides (where H is constant). So to prove $\int_{h_x}\omega = \int_{k_x}\omega$, it suffices to prove:

LEMMA. *Let* θ *be a closed* 1-*form on* I×I *with values in* G. *Then* $\int_{\partial(I\times I)}\theta = e$.

Proof. There are two steps in this proof, one conceptual, one arithmetic, but both typical for synthetic reasoning. The conceptual one reduces the question from the 'finite' rectangle I×I to 'infinitesimal' rectangles

$$[t,t+d] \times [s,s+\delta] \quad ((d,\delta) \in D\times D),$$

and is a two fold "infinitesimal induction": prove "by induction" in $t \in [0,1]$ that

$$\int_{\partial([0,t]\times I)} \theta = e$$

by proving that the integral around $\partial([0,t]\times I)$ equals the integral around $\partial([0,t+d]\times I)$ $\forall d \in D$, whence the derivative of this integral, as a function of t, has to be 0, so the integral itself has to be constant, in fact (take $t = 0$) the constant e. How are the two integrals in question proved to be equal? By proving that the integral around the strip $[t,t+d]\times I$ equals e, which in turn is achieved by a second infinitesimal induction in s for

$$\int_{\partial([t,t+d]\times[0,s])} \theta$$

which reduces the question to that of the infinitesimal rectangle $\partial([t,t+d]\times[s,s+\delta])$. The infinitesimal induction applied here reads:

$$k(0) = 0 \wedge (k(t) = k(t+d) \ \forall t,d) \Rightarrow k \equiv 0;$$

there is a stronger induction principle, which should hold for *analytic* functions:

$$k(0) = 0 \wedge (k(t) = 0 \Rightarrow k(t+d) = 0 \ \forall t,d) \Rightarrow k \equiv 0).$$

The proof of the lemma would now, of course, be finished if we had used the 'cubical' definition of forms (cf. the contribution of Moerdijk and Reyes), since, with that definition, *rectangular* infinitesimal Stokes theorem hold by definition. So

$$\int_{\partial([t,t+d]\times[s,s+\delta])} \theta = \int_{[t,t+d]\times[s,s+\delta]} d\theta = e$$

since $d\theta = 0$. But with the "combinatorial" definition we employ, it is the *triangular* infinitesimal Stokes theorem which holds by definition. So the arithmetical step of the proof is essentially to compare the "curve integral" of a 1-form θ around an infinitesimal triangle in $I\times I$, and an infinitesimal rectangle in $I\times I$:

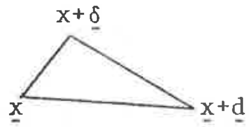
For ease of notation, we shall generalize and consider an arbitrary R^n instead of $I\times I$. The relation $x \sim y$ (for $\underline{x} = (x_1,\ldots,x_n)$, $\underline{y} = (y_1,\ldots,y_n)$) is defined by

$x \sim y$ iff $(\underline{x}-\underline{y}) \in D(n)$ $(= [\![(d_1,\ldots,d_n) \in R^n \quad d_i \cdot d_j = 0 \quad \forall i,j]\!])$

Also, we shall change the infinitesimal rectangle considered to an arbitrary parallellogram $P(\underline{x},\underline{d},\underline{\delta})$:

$(\underline{d},\underline{\delta}) \in D(n)\times D(n)$

Recall that the "curve integral" of θ around this is just the product (in G) of $\theta(\underline{x},\underline{x}+\underline{d})$ with three other similar factors, and this product must be proved = e. The closedness of θ tells us that we get e if we take the curve integral of θ around an arbitrary infinitesimal triangle $T(\underline{x},\underline{d},\underline{\delta})$:



$(\underline{d},\underline{\delta}) \in \tilde{D}(2,n) \subseteq D(n)\times D(n)$

where $\tilde{D}(2,n) \subset D(n)\times D(n)$ consists of those $(\underline{d},\underline{\delta}) \in D(n)\times D(n)$ with $\underline{d} \sim \underline{\delta}$. If $\underline{d} = (d_1,\ldots,d_n)$, $\underline{\delta} = (\delta_1,\ldots,\delta_n)$, this means

$$(d_i-\delta_i)\cdot(d_j-\delta_j) = 0 \quad \forall i,j,$$

or, in view of $d_i \cdot d_j = 0$ and $\delta_i \cdot \delta_j = 0$, that

$$d_i\delta_j + d_j\delta_i = 0 \quad \forall i,j. \tag{2.2}$$

The arithmetical part of the proof thus consists in proving for any G-valued 1-form θ, that

$$\int_{T(\underline{x},\underline{d},\underline{\delta})} \theta = e \quad \forall (\underline{d},\underline{\delta}) \in \tilde{D}(2,n)$$

implies

$$\int_{P(\underline{x},\underline{d},\underline{\delta})} \mathcal{O} = e \qquad \forall(\underline{d},\underline{\delta}) \in D(n)\times D(n) \qquad (*)$$

A proof of this for a general value group G may be found in [1]. Here, we shall just do the case G = (R,+). The point of the proof is that it utilizes some stronger versions of the KL-axiom:

Axiom 1. Any map $D(n) \to R$ extends uniquely to an *affine* map $R^n \to R$.

In particular, a map $D(n) \to R$ with $\underline{0} \to 0$ extends uniquely to a *linear* map $R^n \to R$. We express this by saying: "D(n) classifies linear maps". In particular

$$\mathcal{O}(\underline{x},\underline{x}+\underline{d}) = A(\underline{x};\underline{d}) \qquad \forall\underline{d} \in D(n)$$

where A(-,-) depends *linearly* in the second variable. Also,

$$A(\underline{x}+\underline{\delta};\underline{d}) = A(\underline{x};\underline{d}) + D_1A(\underline{x},\underline{d},\underline{\delta})$$

where $D_1A(-;-,-)$ depends *bilinearily* in the two last variables. It is now easy to calculate

$$\begin{aligned}\int_{T(\underline{x},\underline{d},\underline{\delta})}\mathcal{O} &= A(\underline{x};\underline{d}) + A(\underline{x}+\underline{d};\underline{\delta}-\underline{d}) + A(\underline{x}+\underline{\delta},-\underline{\delta}) \\ &= A(\underline{x};\underline{d}) + A(\underline{x};\underline{\delta}-\underline{d}) + D_1A(\underline{x};\underline{\delta}-\underline{d},\underline{d}) \\ &\quad + A(\underline{x};-\underline{\delta}) + D_1A(\underline{x},-\underline{\delta},\underline{\delta}).\end{aligned}$$

But if B(-,-) is any bilinear map then $B(\underline{d},\underline{d}) = 0 \quad \forall\underline{d} \in D(n)$, so, using this, and linearity in the variables after the semicolons, the above reduces to

$$\int_{T(\underline{x},\underline{d},\underline{\delta})} \mathcal{O} = D_1A(\underline{x};\underline{\delta},\underline{d}). \quad \forall(\underline{d},\underline{\delta}) \in \tilde{D}(2,n) \qquad (2.3)$$

A similar calculation gives

$$\int_{P(\underline{x},\underline{d},\underline{\delta})} \mathcal{O} = D_1A(\underline{x};\underline{\delta},\underline{d}) - D_1A(\underline{x};\underline{d},\underline{\delta})$$
$$\forall(\underline{d},\underline{\delta}) \in D(n)\times D(n).$$

We note that, because of (2.2), any *symmetric* bilinear map $R^n \times R^n \to R$ vanishes on $\tilde{D}(2,n)$. The bilinear map $D_1A(\underline{x};-,-)$ may be written uniquely as the sum of a a symmetric bilinear map and a skew bilinear map, call the latter $C(-,-)$, so $C(\underline{u},\underline{v}) = \frac{1}{2}(D_1A(\underline{x};\underline{u},\underline{v}) - D_1A(\underline{x};\underline{v},\underline{u}))$. Thus

$$\int_{T(\underline{x},\underline{d},\underline{\delta})} \mathcal{O} = C(\underline{x};\underline{\delta},\underline{d}) \qquad \forall(\underline{d},\underline{\delta}) \in \tilde{D}(n,n) \qquad (2.4)$$

and

$$\int_{P(\underline{x},\underline{d},\underline{\delta})} \mathcal{O} = 2C(\underline{x};\underline{\delta},\underline{d}) \qquad \forall(\underline{d},\underline{\delta}) \in D(n)\times D(n) \qquad (2.5)$$

We may pose still another version of KL:

<u>Axiom 1</u>": any map $\tilde{D}(2,n) \to R$ extends uniquely to a map $R^n \times R^n \to R$, which is the sum of an affine map and a *skew* bilinear map. Briefly "$\tilde{D}(2,n)$ classifies skew bilinear maps".

The proof of the implication (*) (for $G = (R,+)$) is now immediate: the assumption of closedness of $\mathcal{O}$ gives that (2.4) vanishes, thus the skew bilinear $C(-,-)$ map vanishes on $\tilde{D}(2,n)$. Since $\tilde{D}(2,n)$ classifies skew bilinear maps, it follow, that $C(-,-)$ is the zero map, in particular (2.5) vanishes.

The argument for the theorem can now easily be completed. Essentially, we have proved that the construction of f in (2.1) is independet of choice of the path h_x. It remains to be shown that $df = \omega$, and that f is unique modulo right multiplication by a constant from G. We leave this to the reader (for the case $M = R^m$, say), or refer to [1].

REMARK 1. Essentially the same calculations as those used in the arithmetic part of the proof may be used to establish a direct comparison between 2-forms in the "triangular" and "rectangular" senses. This is also possible for $n \geqslant 3$ (provided the values are R) but it is then more delicate, see [2] I.18.

REMARK 2. One might think that a "geometric" proof of the implication (*) could be achieved by "covering" or "paving" the infinitesimal parallellogram with infinitesimal rectangles. I have tried in vain to do it, and conjecture that it can be proved to be impossible.

REMARK 3. The reader may ask: how often do we want to pull another KL-axiom out of the hat? The answer is that they are all special cases of one 'uniform' axiom, called Axiom 1^W in [2]. It says that whenever an infinitesimal object $\hat{D} \subset R^k$ has been defined as the zero set of an ideal $I \subseteq C^\infty(\mathbb{R}^n)$ of finite codimension then:

Axiom $1^{\hat{D}}$: any function $\hat{D} \to R$ extends to a function $R^k \to R$ which is uniquely determined modulo the ideal I. (The finite codimension of I allows one to choose definite representatives, like when for $k = 1$, $I = (t^2)$, an equivalence class of maps mod I has a unique *affine* representative, a statement which, when internalized, is the appropriate axiom).

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