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THE STACK QUOTIENT OF A GROUPOID

by Anders KOCK

Resumé. On décrit un sens 2-dimensionnel précis dans lequel le champ des G-fibrés principaux est un quotient du groupoïde G. L'outil clé à cette fin est une reformulation de la descente (ou données de coégalisation), en termes de relèvements simpliciaux de diagrammes simpliciaux.

It is a well known conception, see [1], Ex. 4.8, that the stack $B(G_\bullet)$ of principal $G_\bullet$ bundles is in some sense a quotient of $G_\bullet$. I intend here to make this into a more precise statement, and to prove it in a quite general context – essentially that of a category with pull-backs and equipped with a class $\mathcal{D}$ of descent epis (as in [6] or [5]).

For an equivalence relation in a category $\mathbf{B}$, it is unambiguous what a quotient of it should be: If

$$R \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0$$

is an equivalence relation in $\mathbf{B}$, a quotient for it is a map $q : G_0 \to G_{-1}$ mediating, for every object X, a bijection

$$Coeq(R, X) \cong hom(G_{-1}, X),$$

where $Coeq(R, X)$ is the set of X-valued "coequalizing data" for R, meaning maps $p : G_0 \to X$ with $p \circ d_0 = p \circ d_1$.

For $G_\bullet$ a groupoid in $\mathbf{B}$,

$$G_\bullet = \quad (\ldots \quad G_2 \begin{array}{c}\longrightarrow\\[-8pt]\longrightarrow\\[-8pt]\longrightarrow\end{array} G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0),$$

and $\mathbf{X}$ a stack (or just a fibered category) over $\mathbf{B}$, we describe a category (groupoid, in fact), $\underline{Coeq}(G_\bullet, \mathbf{X})$ of $\mathbf{X}$-valued *coequalizing data* (or *descent data*). We describe in which sense the stack $BG_\bullet$ of "principal $G_\bullet$-bundles" is a 2-dimensional quotient (expanding the base category $\mathbf{B}$ into the 2-category of stacks over $\mathbf{B}$). So we look for an equivalence

$$\underline{Coeq}(G_\bullet, \mathbf{X}) \simeq \underline{hom}(BG_\bullet, \mathbf{X}); \tag{1}$$

there will be such an equivalence, provided $\mathbf{X}$ has a *stack* property, in the sense we shall recall in Section 5 below. The equivalence is not itself explicit, but is expressed in terms of two explicit equivalence functors (cf. (22) below),

$$\underline{Coeq}(G_\bullet, \mathbf{X}) \to \underline{Grpd}(\mathbf{X})/G_\bullet \leftarrow \underline{hom}(BG_\bullet, \mathbf{X}),$$

whose quasi-inverses are not completely explicit, since they depend on chosing cleavages or solutions of descent problems.

The first equivalence, $\underline{Coeq}(G_\bullet, \mathbf{X}) \to \underline{Grpd}(\mathbf{X})/G_\bullet$, is dealt with in Sections 2 and 3. Thus, also, a reformulation of the notion of descent data is provided. - The universal coequalizing data, i.e. the coequalizer itself, can in terms of groupoids in $BG_\bullet$ be given completely explicitly: it is Illusie's $Dec^\bullet(G_\bullet)$.

Our formulations of fibration theory, and of descent, are free of cleavages.

The question of quotients of groupoids may be relevant to the formulations of intentional type theory of e.g. [3], [7], who approximate the notion of types-with-an-intentional-equality in terms of groupoids.

I want to thank the Mittag-Leffler Institute, where the main part of the present work was carried out in May 2001.

1 Basics on fibrations

This section is mainly to fix notation and terminology. Consider a fibration $\pi : \mathbf{X} \to \mathbf{B}$ (see e.g. [5]). For G an object in $\mathbf{B}$, $\mathbf{X}_G$ denotes the fibre. We consider only fibrations where all fibres are groupoids; this is well known to be equivalent to the assumptions that all arrows in $\mathbf{X}$ are cartesian. (Nevertheless, we sometimes use the word "cartesian", as a reminder of the universal property.) By 2-category, we understand here a 2-category where all 2-cells are invertible; equivalently, a category enriched in the category of groupoids. We consider the 2-category of fibrations over $\mathbf{B}$, denoted $\underline{\underline{Fib}}_{\mathbf{B}}$. Morphisms are functors over $\mathbf{B}$ (which preserve the property of being a cartesian arrow; this is here automatic, by our assumption). And 2-cells are natural transformations all of whose components are vertical (mapping to an identity arrow by π). For two objects $\mathbf{X}$ and $\mathbf{Y}$ in $\underline{\underline{Fib}}_{\mathbf{B}}$, $\underline{hom}(\mathbf{X}, \mathbf{Y})$ denotes the hom-category (a groupoid, in fact, by our assumptions).

For any $G \in \mathbf{B}$, the domain functor $\mathbf{B}/G \to \mathbf{B}$ is a fibration, denoted $y(G)$; such fibrations one calls *representable*; they have discrete categories as fibres: $(y(G))_H$ is the set of arrows from H to G. An arrow in $\mathbf{B}/G$ *over* the arrow $f : X_1 \to X_0$ is a commutative triangle $h = g \circ f$, where g and h have codomain G; such triangle, when viewed as an arrow in $\mathbf{B}/G$, is denoted $(g; f)$.

We shall be interested in morphisms in $\underline{\underline{Fib}}_{\mathbf{B}}$ whose domain are representable fibrations $y(G)$, $y(H)$, etc. We collect some basic formulas. Note that since no "cleavage" or other arbitrary things are mentioned, the principle "whatever is meaningful, is true" is likely to be applicable. (We refer to these assertions as "Basic Item 1.-4.".)

1. Let $D : y(G) \to \mathbf{X}$, and let $d : H \to G$. The composite $D \circ y(d)$,

$$y(H) \xrightarrow{y(d)} y(G) \xrightarrow{D} \mathbf{X}$$

is given on objects $e \in (y(H))_K$ by

$$(D \circ y(d))(e) = D(d \circ e) \tag{2}$$

and on morphisms $(e; f)$ in $y(H)$ by

$$(D \circ y(d))(e; f) = D(d \circ e; f); \tag{3}$$

it is an arrow in $\mathbf{X}$ over f.

2. Next, we consider a 2-cell

$$y(G) \underset{D'}{\overset{D}{\underset{\downarrow \xi}{\rightrightarrows}}} \mathbf{X}.$$

So for $d \in y(G)_H$, the component $\xi_d : D(d) \to D'(d)$ is an arrow in $\mathbf{X}$, vertical over H. For an arrow $(d; e) : f \to d$ in $y(G)$ (where $f = d \circ e$), the naturality square is

$$\begin{array}{ccc} D(d \circ e) & \xrightarrow{D(d;e)} & D(d) \\ {\scriptstyle \xi_{d \circ e}} \downarrow & & \downarrow {\scriptstyle \xi_d} \\ D'(d \circ e) & \xrightarrow[D'(d;e)]{} & D'(d) \end{array} \tag{4}$$

3. We next consider the composition ("whiskering") of the form

$$y(H) \xrightarrow{y(d)} y(G) \underset{D'}{\overset{D}{\underset{\downarrow \xi}{\rightrightarrows}}} \mathbf{X}$$

where $d : H \to G$ in $\mathbf{B}$. For an object $e \in_K y(H)$, the component of the whiskering $\xi \circ y(d)$ at e is given as follows:

$$(\xi \circ y(d))_e = \xi_{d \circ e}; \tag{5}$$

it is a an arrow in $\mathbf{X}$, vertical over K.

4. Let D and D' be as in item 2. above. From the naturality square exhibited in (4), it is easy to conclude that if the values of D' are (cartesian) arrows, and if two 2-cells ξ and $\eta : D \to D'$ agree on the object 1_G (identity map of G), then they agree everywhere. For, from the naturality squares (4) for ξ and η with respect to $(1; d) : d \to 1$, it follows that $D'(1; d) \circ \xi_d = D'(1; d) \circ \eta_d$, but two parallel vertical arrows which postcompose with some cartesian arrow to give the same, are equal.

Remark. ("Yoneda Lemma") There is an explicit functor ev_1 (=evaluation at the object 1_G in $y(G) = \mathbf{B}/G$),

$$ev_1 : \underline{hom}(y(G), \mathbf{X}) \to \mathbf{X}_G,$$

and it is an equivalence, but a quasi-inverse is not explicit; a quasi-inverse amounts to *choosing* for each $X \in \mathbf{X}_G$ and each $h : H \to G$ a (cartesian) arrow $h^*(H) \to X$ over h. So it amounts to a "partial cleavage" of $\mathbf{X} \to \mathbf{B}$.

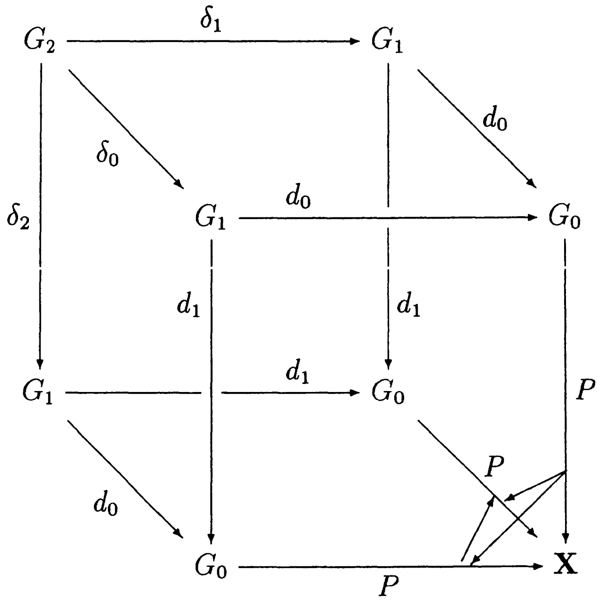
We have the full embedding y of the category $\mathbf{B}$ into the 2-category $\underline{\underline{Fib}}_{\mathbf{B}}$. It actually factors through the full subcategory of $\mathcal{D}$-stacks $\underline{\underline{S}}_{\mathbf{B}}$, to be described in Section 5 below. It is full and faithful on 1-cells as well as on 2-cells (viewing $\mathbf{B}$ as a locally discrete 2-category, in the sense that all 2-cells are identities). We sometimes omit the name of the embedding y from the notation. We shall not discuss coequalizers in $\underline{\underline{S}}_{\mathbf{B}}$ in general, but only coequalizers of groupoids $G_\bullet$ in $\mathbf{B} \subseteq \underline{\underline{S}}_{\mathbf{B}}$. In the first approximation, this means of course a diagram

$$G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0 \xrightarrow{\;P\;} \mathbf{X} \tag{6}$$

which commutes, and is universal (in a suitable sense) in $\underline{\underline{S}}_{\mathbf{B}}$ with this property. But since there are 2-cells available between parallel arrows to $\mathbf{X}$, two-dimensional wisdom says that the notion "the two composites are *equal*" should be replaced by "there is a *specified* 2-cell ψ comparing the two composites". But wisdom also says that specifications should come together with equations to be satisfied, and here it is a cocycle condition on ψ, which involves the three maps $G_2 \to G_1$. To make better room for the pasting geometry involved, we exhibit the fork (6) in terms of a square

$$\begin{array}{ccc} G_1 & \xrightarrow{\;d_0\;} & G_0 \\ {\scriptstyle d_1}\big\downarrow & \downarrow \psi & \big\downarrow{\scriptstyle P} \\ G_0 & \xrightarrow[\;P\;]{} & \mathbf{X} \end{array} \tag{7}$$

Then the equations to be satisfied are a cocycle condition, and a unit condition. The cocycle condition is expressed in terms of commutativity of the 2-cells in a cube,



(8)

The three faces adjacent to the vertex labelled $\mathbf{X}$ are equal, and are all filled with the (invertible) 2-cell ψ, and the three other faces, adjacent to the vertex labelled G_2, are strictly commutative, and express the three simplicial identities that obtains between the composite face operators $G_2 \to G_0$. As a pasting diagram, it makes sense, (ψ being an oriented 2-cell; there are in fact orientations on the three simplicial identities making this cube into a valid pasting scheme, namely $d_0\delta_0 \to d_0\delta_1$, $d_1\delta_0 \to d_0\delta_2$, and $d_1\delta_1 \to d_1\delta_2$).

The cocycle condition on ψ says that the pasting diagram commutes.

There is also a unit condition: it says that pasting the 2-cell ψ with $s : G_0 \to G_1$ yields an identity 2-cell,

$$\psi \circ s = 1_P.$$

If $\mathbf{X}$ is equipped with a cleavage, so that one has functors $d_0^* : \mathbf{X}_{G_0} \to \mathbf{X}_{G_1}$ etc., the cubic cocycle condition can be rendered in the usual form $\delta_2^*(\psi) \circ \delta_0^*(\psi) = \delta_1^*(\psi)$ for descent data.

The collection of such data form a groupoid $\underline{Coeq}(G_\bullet, \mathbf{X})$, whose arrows are 2-cells $P \to P'$ compatible with the ψ's. We may think of it as an alternative way of describing the category of descent data for descent along e, if d_0, d_1 happen to be the kernel pair of some map e.)

2 From coequalizing data to groupoids

We consider a fibration $\pi : \mathbf{X} \to \mathbf{B}$; we assume that all arrows in $\mathbf{X}$ are cartesian, so that the fibres $\mathbf{X}_G$ (for $G \in \mathbf{B}$) are groupoids. We also assume that $\mathbf{B}$ has pull-backs. Then it follows that $\mathbf{X}$ has pull-backs, and that π preserves them. Even more, π *reflects* pull-backs, in the sense that if a commutative square in $\mathbf{X}$ is mapped to a pull-back by π, then it is itself a pull-back.

A groupoid object in $\mathbf{B}$ may be given in terms of its nerve $G_\bullet$; a more economic way of giving the data of a groupoid object $\underline{G}$ is the following standard one: it consists of *truncated simplicial data*,

$$G_2 \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; G_1 \; \substack{\xleftarrow{s} \\ \longrightarrow \\ \longrightarrow} \; G_0 \tag{9}$$

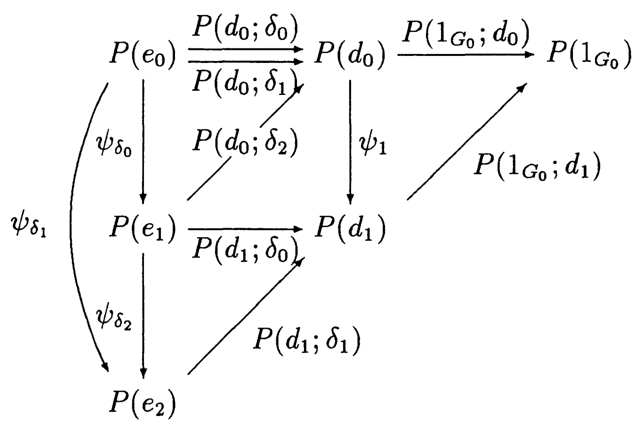
of face maps satisfying the simplicial identities, cf. Appendix, from where the notation is taken, plus a map $s : G_0 \to G_1$, splitting the two face maps $G_1 \to G_0$ (s "picks out identity arrows").

For such truncated data to be a groupoid, the three commutative squares that represent the three simplicial identities among face maps (see Appendix) should be pull-backs; also, with the middle of the three face maps $G_2 \to G_1$ as composition, this composition should be associative and have s as unit. If these conditions are satisfied, its "nerve" $G_\bullet$ may be formed. It is a full-fledged simplicial object, of which the given data then is a "truncation". The category of small groupoids becomes a full subcategory of the category of simplicial objects. – With the stated assumptions on $\pi : \mathbf{X} \to \mathbf{B}$, we then have

Proposition 1 *Let $\underline{X}$ be truncated simplicial data in $\mathbf{X}$ mapping to a groupoid $\underline{G}$ in $\mathbf{B}$. Then $\underline{X}$ is a groupoid.*

Proof. The associativity condition for the composition map $\delta_1 : X_2 \to X_1$ is expressed as an equality between two parallel maps $a_1, a_2 : X_3 \to X_1$ (where $X_3 = X_1 \times_{X_0} X_1 \times_{X_0} X_1$). Now since $X_\bullet$ maps to a groupoid $G_\bullet$, where the associativity condition holds, and since π preserves pull-backs, it follows that $\pi(a_1) = \pi(a_2)$. Since therefore a_1 and a_2 are parallel maps over the same map, it suffices to see that they become equal when post-composed with some (cartesian) map. But clearly for instance $d_0 : X_1 \to X_0$ will do this job.

So to construct a groupoid in $\mathbf{X}$ out of coequalizing data $P : G_0 \to \mathbf{X}$, ψ, as in (7), it suffices to construct truncated data X_2, X_1, X_0, with the relevant six maps in between. This is completely explicitly done, and exhibited in the diagram (10) below (as far as the five face maps are concerned). Namely, we take $X_0 := P(1_{G_0})$, $X_1 := P(d_0)$, $X_2 := P(e_0)$; they are the objects of the upper row in (10). The five face maps are also present in the diagram. We use notation for face maps as in the Appendix, and decorate the face maps in the $X_\bullet$ under construction by $\tilde{\delta}_i$, $\tilde{d}_j$, etc. We put $\tilde{d}_0 := P(1_{G_0}; d_0)$, and $\tilde{d}_1 := P(1_{G_0}; d_1) \circ \psi_{1_{G_1}}$. We put $\tilde{\delta}_0 := P(d_0; \delta_0)$, (note that by $d_0 \circ \delta_0 = e_0$, $(d_0; \delta_0) : e_0 \to d_0$ in $y(G_0)$, and similar for the other "semicolon" expressions). Similarly, we put $\tilde{\delta}_1 := P(d_0; \delta_1)$; for $\tilde{\delta}_2$, we need again to involve ψ: $\tilde{\delta}_2 := P(d_0; \delta_2) \circ \psi_{\delta_0}$; Finally, the construction is completed by putting $\tilde{s} : X_0 \to X_1$ equal to $P(d_0; s)$ (note that since $d_0 \circ s = 1$, $(d_0; s)$ is a morphism in $y(G_0)$ from 1 to d_0). The reader will find some of this data exhibited in the diagram



(10)

(the 1 on ψ refers to 1_{G_1}).

To prove the simplicial identities among the $\tilde{\delta}_i$, $\tilde{d}_j$ and $\tilde{s}$ is easier the fewer ψ's are involved, i.e. the smaller the indices i and j are. The method is in any case the same, so we are only going to present one of them, the "worst" one, – the only one involving the cocycle condition,

$$\tilde{d}_1 \circ \tilde{\delta}_1 = \tilde{d}_1 \circ \tilde{\delta}_2,$$

as well as the identity involving the unit condition,

$$\tilde{d}_1 \circ \tilde{s} = 1.$$

So we calculate

$$\tilde{d}_1 \circ \tilde{\delta}_1 = P(1; d_1) \circ \psi_1 \circ P(d_0; \delta_1) = P(1; d_1) \circ P(d_1; \delta_1) \circ \psi_{\delta_1}$$

(using naturality of ψ with respect to $(1; \delta_1) : \delta_1 \to 1_{G_1}$)

$$= P(1; e_2) \circ \psi_{\delta_1},$$

using functorality of P on the composite $(1; d_1) \circ (d_1; \delta_1) = (1; e_2)$. On the other hand,

$$\tilde{d}_1 \circ \tilde{\delta}_2 = P(1; d_1) \circ \psi_1 \circ P(d_0; \delta_2) \circ \psi_{d_0}$$
$$= P(1; d_1) \circ P(d_1; \delta_2) \circ \psi_{\delta_2} \circ \psi_{\delta_0},$$

by naturality of ψ w.r.to $(1; \delta_2) : \delta_2 \to 1_{G_1}$. By functorality of P, this is $P(1; e_2) \circ \psi_{\delta_2} \circ \psi_{\delta_0}$, and by the cocycle condition, this equals $P(1; e_2) \circ \psi_{\delta_1}$ as desired. – To prove the unit condition: $\tilde{d}_0 \circ \tilde{s} = 1$ is trivial by functorality of P; $\tilde{d}_1 \circ \tilde{s}$ uses the unit condition for ψ, namely $\psi_s = 1$.

3 From groupoids to coequalizing data

We consider a groupoid $X_\bullet$ in $\mathbf{X}$, mapping by π to the fixed groupoid $G_\bullet$ in $\mathbf{B}$, and proceed to construct coequalizing data $(P : G_0 \to \mathbf{X}, \psi)$ out of this data. This is not a completely explicit construction; one piece of information is not completely explicit, namely a functor (partial cleavage) $P : y(G_0) \to \mathbf{X}$ with $P(1) = X_0$ (1 denoting the identity map of G_0). We assume such a P *chosen.* (For instance, if $\mathbf{X}$ is equipped with a cleavage, then we may for $\epsilon : H \to G_0$ in $\mathbf{B}$ take $P(\epsilon)$ to be the cleavage-chosen cartesian arrow $\epsilon^* X_0 \to X_0$ over ϵ.) Being a functor over $\mathbf{B}$, we have for each $\epsilon : H \to G_0$ in $\mathbf{B}$ a given (cartesian) arrow $p(\epsilon) : P(\epsilon) \to X_0$ over it.

We have to provide the natural transformation $\psi : P \circ y(d_0) \to P \circ y(d_1)$ between the indicated functors $y(G_1) \to \mathbf{X}$. (The simplicial operators on $G_\bullet$ consist of maps d_i, δ_j, and e_k, as before; the simplicial operators on $X_\bullet$ are denoted similarly, but with a tilde: $\tilde{d}_i$, etc.)

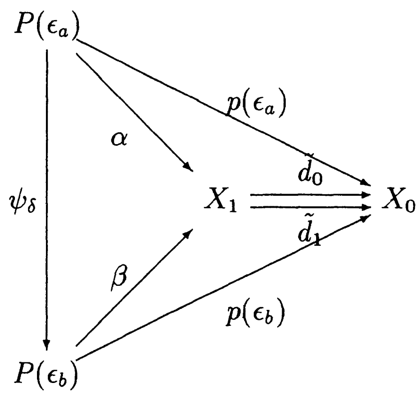
So consider an object $\delta : H \to G_1$ in $y(G_1)$, then ψ_δ should be a vertical arrow in $\mathbf{X}$ over H,

$$\psi_\delta : P(d_0 \circ \delta) \to P(d_1 \circ \delta);$$

denoting $d_0 \circ \delta$ by ϵ_a and $d_1 \circ \delta$ by ϵ_b, we then construct ψ_δ by the following recipe: Consider $p(\epsilon_a) : P(\epsilon_a) \to X_0$; then use that $\tilde{d}_0$ is cartesian, so that we may consider the comparison arrow $\alpha : P(\epsilon_a) \to X_1$ over δ, arising from the factorization $\epsilon_a = d_0 \circ \delta$; similarly, consider $p(\epsilon_b) : P(\epsilon_b) \to X_0$: then use that $\tilde{d}_1$ is cartesian, so that we may consider the comparison arrow $\beta : P(\epsilon_b) \to X_1$ over δ arising from the factorization $\epsilon_b = d_1 \circ \delta$. Since both α and β live over δ, and have common codomain X_1, we may use that β is cartesian, to get a unique vertical comparison from α to β, and this is to be our ψ_δ, so

$$\beta \circ \psi_\delta = \alpha. \tag{11}$$

For the convenience of the reader, we record the recipe in a diagram:



(12)

The unit condition $\psi \circ y(s) = 1$ follows by contemplating this diagram, with $\delta = s$, then the long sloping arrows will be 1_{X_0}; so $\alpha = \tilde{s}$ and $\beta = \tilde{s}$ by uniqueness of cartesian factorization, and so $\psi \circ y(s)$ is the identity 2-cell of 1_{X_0}.

To prove the cocycle condition (in the "cube" form, (8)), we need to calculate the whiskerings $\psi \circ y(\delta_i)$ for $i = 0, 1, 2$.

We claim that, for their components at the object 1_{G_2} (for brief denoted 1), we have, for certain canonical vertical arrows c_0, c_1 and c_2 to be given below,

$$(\psi \circ y(\delta_0))_1 = c_1 \circ c_0^{-1} \tag{13}$$

$$(\psi \circ y(\delta_1))_1 = c_2 \circ c_0^{-1} \tag{14}$$

$$(\psi \circ y(\delta_2))_1 = c_2 \circ c_1^{-1} \tag{15}$$

Since natural transformations in this case are determined by their component at the identity of the domain, these equations will establish the cocycle condition for ψ,

$$(\psi * y(\delta_2)) \circ (\psi * y(\delta_0)) = \psi * y(\delta_1),$$

(where we used $*$ rather than $\circ$ to denote horizontal composition (whiskering). The three calculations proceed in the same way, so we shall give only the one for (14). We use the cartesian property of $\tilde{d}_0$ to lift the factorization $d_0 \circ \delta_1 = e_0$ to a factorization of $p(e_0)$ through $\tilde{d}_0$, say

$$p(e_0) = \tilde{d}_0 \circ \alpha, \tag{16}$$

with $\pi(\alpha) = \delta_1$, and similarly, the factorization $d_1 \circ \delta_1 = e_2$ lifts to a factorization of $p(e_2)$ over $\tilde{d}_1$

$$p(e_2) = \tilde{d}_1 \circ \beta. \tag{17}$$

with $\pi(\beta) = \delta_1$. Also, by the definition of ψ_{δ_1},

$$\beta \circ \psi_{\delta_1} = \alpha \tag{18}$$

with ψ_{δ_1} vertical, for the α and β of (16) and (17). Let c_i denote the unique vertical comparison $X_2 \to P(e_i)$ with

$$p(e_i) \circ c_i = \tilde{e}_i \tag{19}$$

Then we claim

$$\alpha \circ c_0 = \tilde{\delta}_1. \tag{20}$$

These are parallel arrows over the same arrow δ_1 in $\mathbf{B}$, so it suffices to prove that they become equal by post-composition with some (cartesian) arrow; here, $\tilde{d}_0$ will do the job, since, by (16) $\tilde{d}_0 \circ \alpha \circ c_0 = p(e_0) \circ c_0 = \tilde{e}_0 = \tilde{d}_0 \circ \tilde{\delta}_1$. We can now prove

$$\psi_{\delta_1} \circ c_0 = c_2.$$

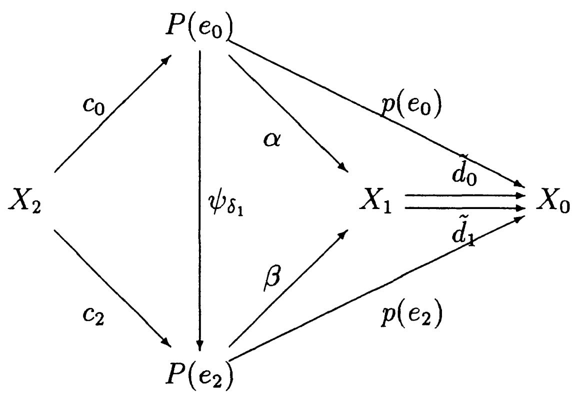
Since both sides of this equation are vertical, it suffices to prove that post-composing them with some (cartesian) arrow give same result; we shall utilize $p(e_2)$, so we intend to prove

$$p(e_2) \circ \psi_{\delta_1} \circ c_0 = p(e_2) \circ c_2.$$

We calculate

$$\begin{aligned} p(e_2) \circ \psi_{\delta_1} \circ c_0 &= \tilde{d}_1 \circ \beta \circ \psi_{\delta_1} \circ c_0 \text{ by } (17) \\ &= \tilde{d}_1 \circ \alpha \circ c_0 \text{ by } (18) \\ &= \tilde{d}_1 \circ \tilde{\delta}_1 \text{ by } (20) \\ &= \tilde{e}_2 = p(e_2) \circ c_2. \end{aligned}$$

This finishes the proof of (14). For the convenience of the reader, we compile the data of the proof of (14) into a diagram. Note that the corresponding diagrams for (13) and (15) will look similar, but that the α and β will denote something different (whereas the c_i's remain the same).



We now prove that the two processes (of Section 2 and the current part of the present Section) are inverse of each other, up to canonical isomorphism. If we start with coequalizing data (P, ψ), $P : y(G_0) \to \mathbf{X}$ in particular is a partial cleavage of $\mathbf{X}$ with codomain X_0 (so $P(1) = X_0$, 1 denoting 1_{G_0}); the groupoid constructed gives rise to, possibly new, coequalizing data (P', ψ'), whose construction starts out with choosing a partial cleavage $\overline{P}$ with $\overline{P}(1) = X_0 = P(1)$. Hence there is a unique isomorphism between them, and the compatibility with ψ means an assertion of equality of two natural transformations with domain $y(G_1)$. From Basic Item 4, it suffices to see agreement on 1_{G_1}, which is easy.

Conversely, if we have a groupoid $X_\bullet$ in $\mathbf{X}$ over $G_\bullet$ in $\mathbf{B}$, and produce coequalizing data, by some partial cleavage P, (with $P(1) = X_0$) then we have the vertical comparisons $c_0 : X_1 \to P(d_0)$ and $c_2 : X_2 \to P(e_2)$; and by the construction, these comparisons are immediately compatible with the face maps, except possibly with the last ones $\tilde{d}_1$ and $\tilde{\delta}_2$, whose definition involved ψ, cf. the display in (10). But contemplate the construction of ψ_1 in terms of the groupoid, cf. (12): in that diagram, the comparison α is just the inverse of the comparison c_0, and β similarly for c_1, the unique comparison for $\tilde{d}_1$, so $\psi_1 \circ c_0 = c_1$, and then the desired compatibility is clear. For the compatibility of the δ's, one can utilize that we are dealing with groupoids over the same groupoid $G_\bullet$, and prove the desired equality by post-composition with some suitable (cartesian) arrow $X_1 \to X_0$ (take $\tilde{d}_1$).

Summing up, we thus have our reformulation of coequalizing data (and hence of descent data):

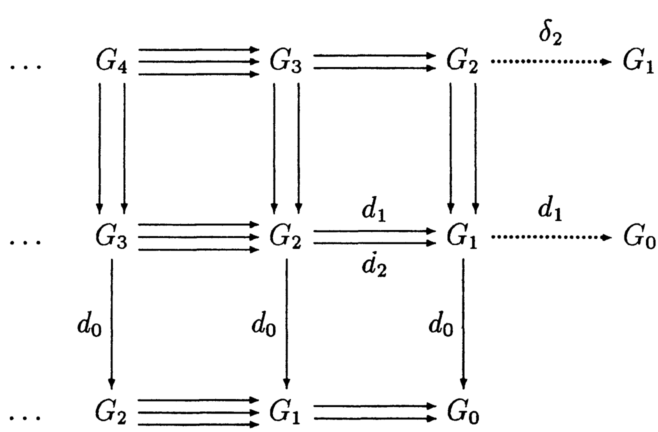
Theorem 1 *For any groupoid $G_\bullet$ in $\mathbf{B}$ and any fibration with groupoids as fibres $\mathbf{X} \to \mathbf{B}$, the explicit functor described in Section 2*

$$\underline{Coeq}(G_\bullet, \mathbf{X}) \to \underline{Grpd}(\mathbf{X})/G_\bullet$$

is an equivalence.

If $G_\bullet$ is a small groupoid (identified with its nerve, which is a simplicial set), a principal $G_\bullet$ bundle is a simplicial set over $p : G_\bullet$, $E_\bullet \to G_\bullet$ such that 1) all the squares, expressing that p commutes with the face-operators, are pull-backs, and 2) $E_\bullet$ is the (nerve of) an equivalence relation, with coequalizer $E_0 \to E_{-1}$, say, called the augmentation. We say that $E_\bullet$ is a principal $G_\bullet$-bundle *on* E_{-1}. The category of principal $G_\bullet$-bundles, with augmentation $E_\bullet \to E_{-1}$ as part of the data, form a fibered category over $\mathbf{B}$, $\pi : B(G_\bullet) \to \mathbf{B}$, where $\pi(E_\bullet \to E_{-1}) = E_{-1}$. All arrows in $B(G_\bullet)$ are cartesian (equivalently, the fibres are (large) groupoids.) (It is actually even a *stack* in the sense of Section 5, provided that the structural maps (face operators) of $G_\bullet$ are $\mathcal{D}$ -epis.)

A particular object in $B(G_\bullet)$ is Illusie's $Dec^1(G_\bullet)$, or Dec^1, for short, since $G_\bullet$ will be fixed; it is a principal bundle over G_0, and is given by $Dec^1_n = G_{n+1}$. It is depicted in row number two from below in the diagram



The row above that is called Dec^2, and above that (not depicted) Dec^3, etc. Although there are three maps from Dec^3 to Dec^2, and two maps from Dec^2 to Dec^1, they all compose to give, for each n, exactly one map from Dec^n to $G_\bullet$. In fact this map makes Dec^n into a principal $G_\bullet$-bundle over G_{n-1} for $n \geq 1$. Altogether, the various Dec^n's fit together into a simplicial object of principal bundles, augmented over the simplicial object $G_\bullet$ in the right hand column. Since all squares in sight are pull-backs, this means that the Dec^n's form a groupoid $Dec^\bullet(G_\bullet)$ in $B(G_\bullet)$, over the groupoid $G_\bullet$ in $\mathbf{B}$.

4 Stacks

The notion of stack that we shall use is relative to a class $\mathcal{D}$ of "descent epis" in the base category $\mathbf{B}$. If $\mathbf{B}$ is a topos, $\mathcal{D}$ could be taken to be the class of all epimorphisms. In the category of smooth manifolds, it could be taken to be the class of surjective submersions. An axiomatic treatment of the properties of such $\mathcal{D}$ has been given most succinctly in [6]; see also [5]. We shall not need to be specific here. Suffice it to say that any pull-back of a $\mathcal{D}$-epi is again a $\mathcal{D}$-epi, and all representable fibrations $y(B)$ are $\mathcal{D}$-stacks.

Let $q : G_0 \to G_{-1}$ be a map in $\mathbf{B}$, with simplicial kernel $G_\bullet =$

$$\dots \quad G_2 \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \; G_1 \; \substack{\longrightarrow\\ \longrightarrow} \; G_0.$$

Let $\mathbf{X} \to \mathbf{B}$ be a fibration. We then have an explicit functor

$$\underline{hom}(G_{-1}, \mathbf{X}) \xrightarrow{C} \underline{Grpd}(\mathbf{X})/G_\bullet. \qquad (21)$$

For, let $P \in \underline{hom}(G_{-1}, \mathbf{X})$, i.e. P is a functor $y(G_{-1}) = \mathbf{B}/G_{-1} \to \mathbf{X}$ above $\mathbf{B}$. Now $G_\bullet$ is a simplicial object in $\mathbf{B}/G_{-1}$ above the groupoid $G_\bullet$, so it goes by the functor $P : \mathbf{B}/G_{-1} \to \mathbf{X}$ to a simplicial object in $\mathbf{X}$, above $G_\bullet$; by Proposition 1, this simplicial object is a groupoid in $\mathbf{X}$.

Definition 1 *The fibered category* $\mathbf{X} \to \mathbf{B}$ *is a* stack *if for all* $q \in \mathcal{D}$, *the functor (21) is an equivalence.*

Proposition 2 *Let* $\pi : \mathbf{X} \to \mathbf{B}$ *be a stack, and let*

$$\delta : X_0 \to X_{-1}, \ \delta' : X_0 \to X'_{-1}$$

be arrows in $\mathbf{X}$ *with common domain, and with* $\pi(\delta) = \pi(\delta')$ *a* $\mathcal{D}$*-epi* $q : G_0 \to G_{-1}$. *Then there is a unique vertical isomorphism* $\xi : X_{-1} \to X'_{-1}$. *satisfying* $\xi \circ \delta = \delta'$.

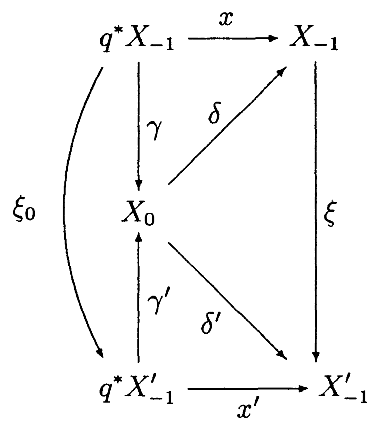
Proof. Choose a cleavage $(-)^*$ of $\mathbf{X} \to \mathbf{B}$. Then we get a functor $\overline{q}^* : \mathbf{X}_{G_{-1}} \to \underline{Grpd}(\mathbf{X})/G_\bullet$, namely the one which to an object $X \in \mathbf{X}_{G_{-1}}$ associates the (cartesian) lift with codomain X of the simplicial kernel $G_\bullet$ of $q : G_0 \to G_{-1}$. This functor makes

$$\begin{array}{ccc} \underline{hom}(G_{-1}, \mathbf{X}) & \xrightarrow{C} & \underline{Grpd}(\mathbf{X})/G_\bullet \\ {\scriptstyle ev_1} \downarrow & \nearrow {\scriptstyle \overline{q}^*} & \\ \mathbf{X}_{G_{-1}} & & \end{array}$$

commutative up to isomorphism. Since ev_1 is always an equivalence, and C is an equivalence by the assumed stack property of $\mathbf{X}$, we conclude that $\overline{q}^*$ is an equivalence. In particular, it is full and faithful. Now there is a unique vertical comparison $\gamma : q^*(X_{-1}) \to X_0$ with $\delta \circ \gamma$ equal to the cleavage chosen lift $q^*(X_{-1}) \to X_{-1}$ of q. Similarly, there is a comparison $\gamma' : q^*(X'_{-1}) \to X_0$. These two comparisons compose (inverting γ') to a comparison $\xi_0 : q^*(X_{-1}) \to q^*(X'_{-1})$. We may continue similarly for the lifts of $G_i \to G_{-1}$, and together, this provides an isomorphism

$$\xi_\bullet : \overline{q}^*(X_{-1}) \to \overline{q}^*(X'_{-1})$$

in $\underline{Grpd}(\mathbf{X})/G_\bullet$, and by fullness of $\overline{q}^*$, it comes from a vertical isomorphism $\xi : X_{-1} \to X'_{-1}$. The various arrows mentioned here sit in a diagram



in which the arrows x and x' are the cleavage-chosen lifts. It follows from commutativity of the remaining triangles in this diagram that also the triangle $\xi \circ \delta = \delta'$ commutes. – The uniqueness of such ξ follows similarly from the faithfulness of $\overline{q}^*$.

For a groupoid $X_\bullet$ over $G_\bullet$, any augmentation $X_0 \to X_{-1}$ over q deserves the name of *solution of the descent problem posed by* $X_\bullet$. By Proposition 2, such solution is essentially unique.

5 The coequalizer

We are now going to make precise in which sense and why $B(G_\bullet)$ is a coequalizer of the groupoid $G_\bullet$. This first of all means that one should specify the 2-category in which things take place; this is the 2-category $\underline{\underline{S}}_{\mathbf{B}}$, the full subcategory of stacks inside the 2-category $\underline{\underline{Fib}}_{\mathbf{B}}$. Secondly, one should specify the map $q : G_0 \to B(G_\bullet)$, which is to be the "coequalizing map", together with a 2-cell ϕ between $q \circ d_0$ and $q \circ d_1$. The map q is going to be $Dec^1(G_\bullet)$, more precisely, some partial cleavage of $B(G_\bullet)$ with codomain $Dec^1(G_\bullet)$. And q, ϕ is going to be "the" object in $\underline{Coeq}(G_\bullet, B(G_\bullet))$ which *corresponds* to the groupoid $Dec^\bullet(G_\bullet)$ over $G_\bullet$ in $B(G_\bullet)$, under the correspondence of Sections 2 and 3.

(One reason for reformulating coequalizing data/descent data in terms of groupoids is that $Dec^\bullet(G_\bullet)$ is a completely explicit piece of data, involving no choices, or quotation marks around definite articles.)

So consider, a fixed fibration-in-groupoids $\pi : \mathbf{X} \to \mathbf{B}$, and also a fixed ("small") groupoid $G_\bullet$ in $\mathbf{B}$. We have the following categories and functors

$$\underline{Coeq}(G_\bullet, \mathbf{X}) \rightleftarrows \underline{Grpd}(\mathbf{X})/G_\bullet \leftrightarrows \underline{hom}(B(G_\bullet), \mathbf{X}) \qquad (22)$$

The categories are, respectively, the category of coequalizing data $(p : G_0 \to \mathbf{X}, \psi)$, as explained in Section 2, the category of groupoid objects $X_\bullet$ in $\mathbf{X}$, over $G_\bullet$, and $\underline{hom}(B(G_\bullet), \mathbf{X})$ is the category of (cartesian) functors between fibrations-in-groupoids, over $\mathbf{B}$. All three categories are in fact (large) groupoids.

The functors displayed are all equivalences; the full arrows are explicit, the dotted ones are quasi-inverses, and depend on choice (say, of partial cleavages and solutions of descent problems); the functor

$$\underline{Grpd}(\mathbf{X})/G_\bullet \dashrightarrow \underline{hom}(B(G_\bullet), \mathbf{X}) \qquad (23)$$

requires for its construction that $\mathbf{X}$ is a stack. The two functors on the left in (22) are those that have been expounded in the previous sections. The functorality of the explicit functors in (22) is: pasting with $F : \mathbf{X} \to \mathbf{Y}$ on the left corresponds to applying F on groupoid objects in $\mathbf{X}$. The explicit functor on the right is just "evaluate at $Dec^\bullet$"; for, a functor over $\mathbf{B}$, say $\mathbf{Y} \to \mathbf{X}$, clearly takes groupoid objects over $G_\bullet$ in $\mathbf{Y}$ to groupoid objects over $G_\bullet$ in $\mathbf{X}$. This in particular applies to the groupoid object $Dec^\bullet(G_\bullet)$ in $B(G_\bullet)$.

So the remaining task is to provide the functor (23), provided that $\mathbf{X}$ is a stack, and prove it to be quasi inverse to the evaluation at $Dec^\bullet$.

When this has been carried out, we have the right to assert

Theorem 2 *Let $q : G_0 \to B(G_\bullet), \phi$ be the coequalizing data, corresponding under the left side equivalence of (22) to the groupoid object*

$Dec^\bullet(G_\bullet)$ in $B(G_\bullet)$. Then for any stack $\mathbf{X}$ over $\mathbf{B}$, pasting with q provides an equivalence

$$\underline{hom}(B(G_\bullet), \mathbf{X}) \longrightarrow \underline{Coeq}(G_\bullet, \mathbf{X}).$$

This is exactly to say that such q, ϕ is a coequalizer, in the 2-dimensional sense, of $G_\bullet$, recalling that universal properties 2-dimensionally should be expected to classify "up to equivalence", not "up to isomorphism".

So let us construct a functor (23). Let $X_\bullet$ be a groupoid over $G_\bullet$, in $\mathbf{X}$, assumed to be a stack. To construct its image under the functor (23) means to construct a functor over $\mathbf{B}$,

$$B(G_\bullet) \to \mathbf{X}. \tag{24}$$

The construction is going to involve some choosing of (cartesian) lifts; a partial cleavage $\mathbf{B}/G_0 \to \mathbf{X}$ will suffice. Also, it involves chosing solutions for descent problems in $\mathbf{X}$. So we assume given an object $E_\bullet \to E_{-1}$ on the left hand side, i.e., a principal $G_\bullet$-bundle with quotient E_{-1}; so there is in particular a simplicial map $a_\bullet : E_\bullet \to G_\bullet$. For each n, we take a cartesian lift over a_n with codomain X_n, say $\tilde{a}_n : X'_n \to X_n$. (Such lifts can be obtained canonically by comparison with the chosen lift of $d \circ a_n : E_n \to G_n \to G_0$, where $d : G_n \to G_0$ is the composite of a string of d_0's say.) Now by using the cartesian property of the $\tilde{a}_n$'s, and comparing with the simplicial map $E_\bullet \to G_\bullet$, one obtains a series of face operators between the X'_n's, making $X'_\bullet$ into a simplicial object in $\mathbf{X}$ above the groupoid $E_\bullet$. But such data is now precisely descent data for descent along the augmentation $E_0 \to E'_{-1}$, so since $\mathbf{X}$ is a stack, $X'_\bullet$ descends to an object X'_{-1} in $\mathbf{X}_{E_{-1}}$. The process $E_\bullet \mapsto X'_{-1}$ thus described is the requisite functor $B(G_\bullet) \to \mathbf{X}$.

We now prove that the two processes are inverse to each other, up to isomorphism. Let us start with a groupoid $X_\bullet$ over $G_\bullet$ in the stack $\mathbf{X}$; we want to evaluate the resulting functor $B(G_\bullet) \to \mathbf{X}$ on $Dec^\bullet(G_\bullet)$. But $Dec^\bullet(X_\bullet)$ sits in $\mathbf{X}$ above $Dec^\bullet(G_\bullet)$ in $\mathbf{B}$, so it follows from Proposition 2 that $Dec^1(X_\bullet) \to X_0$ is a solution of the descent problem posed by $Dec^1(X_\bullet)$, and similarly for $Dec^2(X_\bullet) \to X_1$, etc., so up to isomorphism, we recover the groupoid $X_\bullet$.

Conversely, let us start with a functor $P : B(G_\bullet) \to \mathbf{X}$, and evaluate it at $Dec^\bullet(G_\bullet)$, so as to get a groupoid $P(Dec^\bullet(G_\bullet))$ in $\mathbf{X}$; by the recipe provided, this groupoid gives rise to a functor $\overline{P} : B(G_\bullet) \to \mathbf{X}$, whose value at a principal $G_\bullet$-bundle $E_\bullet \to E_1$ may be desribed as follows: it amounts to use the stack property of $\mathbf{X}$ to descend a certain equivalence relation in $\mathbf{X}$ along $E_0 \to E_{-1}$ in $\mathbf{B}$, and this equivalence relation is described in terms of its nerve, which is simply

$$\overrightarrow{\overrightarrow{\longrightarrow}} P(Dec^2(E_\bullet)) \rightrightarrows P(Dec^1(E_\bullet)),$$

but since P preserves pull-backs and solutions of descent problems (this follows from Proposition 2), this solution is (isomorphic to) $P(E_\bullet)$.

Appendix. The faces of a triangle

For a simplicial object $X_\bullet$ in any category, we shall be interested in its lowest dimensional parts,

$$X_2 \overrightarrow{\overrightarrow{\longrightarrow}} X_1 \rightrightarrows X_0. \tag{25}$$

The three face operators $X_2 \to X_1$ we denote δ_0, δ_1 and δ_2, and the two face operators $X_1 \to X_0$, we denote d_0 and d_1. For the calculations, it is also convenient to have names for the three composites $X_2 \to X_0$, we call them e_0, e_1 and e_2, they are defined by the following basic equations

$$e_0 = d_0 \circ \delta_0 = d_0 \circ \delta_1$$
$$e_1 = d_0 \circ \delta_2 = d_1 \circ \delta_0$$
$$e_2 = d_1 \circ \delta_1 = d_1 \circ \delta_2$$

For the case where $X_\bullet$ is the nerve of a small category and we consider a 2-simplex x, i.e., a composable pair

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C,$$

$\delta_0(x) = f$, $\delta_1(x) = g \circ f$, $\delta_2(x) = g$, and for instance the middle equation can be rendered verbally: *the domain of the second arrow g is the codomain of the first arrow f* – and $e_0(x) = A$, $e_1(x) = B$, $e_2(x) = C$. The commutative square expressed by the middle equation is a pull-back, by definition of "composable pair"; the commutative squares expressed by the two other equations are pull-backs precisely when $X_\bullet$ is a groupoid.

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The diagrams were made using Paul Taylor's "Diagrams" package.

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