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A SECONDARY PRODUCT STRUCTURE IN COHOMOLOGY THEORY

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1. Statement of results.

The cohomology $H^*(X; Z_2)$ with coefficients in Z_2 of a topological space X has the structure of a graded algebra over the Steenrod algebra $\mathscr{A} = \mathscr{A}(2)$. The link between the algebra structure and the $\mathscr{A}$-module structure is given by the equations

$$(1.1) \qquad Sq^i(\hat{x}) = 0, \qquad Sq^n(\hat{x}) = \hat{x}^2 ,$$

for all $\hat{x} \in H^n(X)$ and $i > n$. Still further structure on $H^*(X)$ is obtained by considering secondary cohomology operations. Given a relation in the Steenrod algebra of the form

$$R\colon \ \sum \hat{\alpha}_\nu \hat{a}_\nu + \hat{e} = 0, \quad \hat{\alpha}_\nu, \hat{a}_\nu, \hat{e} \in \mathscr{A} ,$$

where $\hat{e}$ is an unfactorized term of excess $\geqq N+1$, that is, $\hat{e}(\hat{x}) = 0$ for all cohomology classes $\hat{x}$ of dimension $\leqq N$; then there is a secondary operation Qu^R associated with R. Definitions can be found in [1] (for $\hat{e} = 0$) and in [3]: we indicate the definition from [3] in Section 2 below. The operation Qu^R is defined on cohomology classes $\hat{x}$ of dimension $n \leqq N$ satisfying $\hat{a}_\nu(\hat{x}) = 0$ for all ν, and in that case

$$Qu^R(\hat{x}) \in H^{n+i}(X) / \sum \hat{\alpha}_\nu H^{n+i-\deg \alpha_\nu}(X) ,$$

where $i+1 = \deg(R)$. In [3] we proved that Qu^R can be chosen such that

$$(1.2) \qquad Qu^R(\hat{x}) = 0 \quad \text{for} \quad \deg(\hat{x}) < j-1 ,$$

where j is such that the excess of $\hat{\alpha}_\nu \hat{a}_\nu$ is larger than or equal to j for all ν (and $j \leqq N$). Furthermore, if R is of the form

$$R\colon \ \hat{\alpha}\, Sq^j + Sq^{\frac{1}{2}(i+1-j)} \hat{\beta} + \sum \hat{\alpha}_\nu \hat{a}_\nu + \hat{e} = 0 ,$$

where the second term occurs only when $i+1-j$ is even, and where $\hat{\beta}$ has the property $\hat{\alpha}(\hat{x}^2) = \hat{\beta}(\hat{x})^2$, and the excess of $\hat{\alpha}_\nu \hat{a}_\nu$ is larger than j for all ν; then, if $\deg(\hat{x}) = j-1$,

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(1.3) $$Qu^R(\hat{x}) = \{\Sigma_s \hat{\alpha}_s'(\hat{x})\,\hat{\alpha}_s''(\hat{x})\}\,,$$

where $\hat{\alpha}_s', \hat{\alpha}_s''$ are determined by

$$\psi(\hat{\alpha}) = \Sigma\hat{\alpha}_s'\otimes\hat{\alpha}_s'' + \Sigma\hat{\alpha}_s''\otimes\hat{\alpha}_s' + \hat{\beta}\otimes\hat{\beta} \in \mathscr{A}\otimes\mathscr{A}\,.$$

Here ψ denotes the comultiplication in $\mathscr{A}$ arising from the Cartan formula.

The formulas (1.2) and (1.3) are in a sense analogous to (1.1). However, one could ask if there exists a secondary multiplication of cohomology classes $\hat{x}\circ\hat{y}$ (not the notation used later in this paper) such that

(1.4) $$Qu^R(\hat{x}) = \hat{x}\circ\hat{x}\,,$$

when $\deg(\hat{x}) = j$ and both sides are defined. If this is the case, then (1.2), (1.3), and (1.4) together would be the secondary analogue to (1.1).

Let $\hat{a}$ be a primary cohomology operation of degree $i+1$, and let

$$A:\begin{cases}\hat{a} = \Sigma\,\hat{\alpha}_\nu\hat{a}_\nu + \hat{e} & (\text{excess } \hat{e} > N)\,,\\ \psi(\hat{a}) = \Sigma\,\hat{b}_i\otimes\hat{c}_i + \Sigma\,\hat{b}_j\otimes\hat{c}_j, & i\in I \text{ and } j\in J\,,\end{cases}$$

be a relation for $\hat{a}$; that is, a factorization of $\hat{a}$ together with a splitting of $\psi(\hat{a})$ into two sums. Then there is a secondary product operation $\overset{A}{\smile}$,

$$\hat{x}\overset{A}{\smile}\hat{y} \in H^{p+q+i}(X)/\mathrm{Ind}\,,$$

natural in $\hat{x}$ and $\hat{y}$, defined whenever $\hat{x}\in H^p(X)$ and $\hat{y}\in H^q(X)$, $p+q\leqq N$, satisfy

$$\hat{a}_\nu(\hat{x}\hat{y}) = 0,\qquad \hat{b}_i(\hat{x}) = 0,\qquad \hat{c}_j(\hat{y}) = 0$$

for all ν, i, and j. The indeterminacy is given as the $(p+q+i)$-th grading of

$$\mathrm{Ind} = \Sigma\,\hat{\alpha}_\nu(H^*(X)) + \Sigma\,\hat{b}_i(H^*(X))\,\hat{c}_i(\hat{y}) + \Sigma\,\hat{b}_j(\hat{x})\,\hat{c}_j(H^*(X))\,.$$

This is proved in Section 4, where we also show that $\overset{A}{\smile}$ is bilinear except in the top dimension. The deviation from bilinearity is given (Theorem 4.8). Further properties of $\overset{A}{\smile}$ are:

1. Two product operations associated with A differ by a „primary" operation in two variables

$$\hat{x}\underline{\overset{A}{\smile}}\hat{y} - \hat{x}\overset{A}{\smile}\hat{y} = \Sigma\,\hat{\lambda}_\nu'(\hat{x})\,\hat{\lambda}_\nu''(\hat{y}),\qquad \hat{\lambda}_\nu', \hat{\lambda}_\nu'' \in \mathscr{A}\,.$$

This is proved in Section 4.

2. Deviation from commutativity. There is a product operation $\overset{A}{\smile}$ associated with A such that for $\hat{x}\in H^s(X)$ and $\hat{y}\in H^t(X)$

$$\hat{x}\overset{A}{\smile}\hat{y} - \hat{y}\overset{A}{\smile}\hat{x} = \hat{\eta}(\hat{x})\,\hat{\eta}(\hat{y}) + st\,\varkappa(\hat{e})(\hat{x}\hat{y}),\qquad \hat{\eta}\in\mathscr{A}\,,$$

whenever $s+t<N$, and the left side is defined. The mapping $\varkappa\colon \mathscr{A}\to\mathscr{A}$ is the derivation of degree -1 determined by $\varkappa(Sq^n) = Sq^{n-1}$. Obviously,

$\hat{\eta}=0$ if $\deg(a)$ is even. If $\deg(\hat{a})=2k+1$, the term $\hat{\eta}$ is determined by the fact that

$$(p+1)\varkappa(\hat{a})\,Sq^p + \hat{a}\,Sq^{p-1} + Sq^{k+p}\,\hat{\eta} \tag{1.6}$$

is of excess larger than p for all p; that is, (1.6) vanishes on all classes $\hat{x}$ with $\deg(\hat{x}) \leqq p$. As an example we have

$$\hat{\eta} = (\deg(I)+i_0+1)\,\zeta(\hat{\alpha})\,Sq^{i_0}Sq^I \qquad \text{if} \qquad \hat{a} = \hat{\alpha}\,Sq^{2i_0+1}Sq^{2I},$$

where $I=(i_1,i_2,\ldots,i_k)$, $\hat{\alpha}\in\mathscr{A}$, and $\zeta\colon \mathscr{A}\to\mathscr{A}$ is the homomorphism halving degrees dual to the squaring homomorphism

$$\zeta^*\colon\ \mathscr{A}^*\to\mathscr{A}^*, \qquad \zeta^*(x) = x^2 .$$

This is proved in Section 5.

3. If A is a relation for $\hat{a}$ and if $\hat{x}$ is an $(n-1)$-dimensional class such that $\hat{x}\overset{A}{\smile}\hat{x}$ is defined, then there is a secondary operation and a $\overset{A}{\smile}$-product with

$$Qu^R(\hat{x}) = \hat{x}\overset{A}{\smile}\hat{x},$$

where R is a relation derived from A,

$$R\colon\ \sum\hat{\alpha}_\nu[\hat{a}_\nu Sq^{n-1}] + Sq^{m+n-1}\zeta(\hat{a}) + \hat{e}_0 \quad (\text{excess } \hat{e}_0 \geqq n).$$

This is proved in Section 6.

4. If X is a suspension, $X=SY$, then for any relation A, $\hat{x}\overset{A}{\smile}\hat{y}=0$ whenever this is defined. This is done in Section 7, where we also consider $\overset{A}{\smile}$-products in connection with acyclic fibrations.

5. In Section 8 we give some relations between various secondary products, and Section 9 contains two Peterson–Stein formulas.

It is well known that other properties of primary operations also carry over to secondary operations. For instance, there is a Cartan formula for secondary operations. Here, however, the situation is as yet not entirely satisfactory. As for additivity there is a difference from the primary case. If Qu^R is an operation associated with a relation

$$R\colon\ \sum\hat{\alpha}_\nu\hat{a}_\nu + \hat{e} = 0,$$

where $\hat{e}\in\mathscr{A}$ is of excess $\geqq N+1$, then Qu^R is only defined on classes $\hat{x}$ of dimension $n\leqq N$, and

$$Qu^R(\hat{x}+\hat{y}) = Qu^R(\hat{x}) + Qu^R(\hat{y}) \qquad \text{if } n<N,$$
$$Qu^R(\hat{x}+\hat{y}) = Qu^R(x) + Qu^R(y) + \{D(\hat{x},\hat{y})\} \qquad \text{if } n=N,$$

where the deviation from additivity $D(\hat{x},\hat{y})$ is given by

$$D(\hat{x},\hat{y}) = \sum Sq^{J(k)}\big(Sq^{I(k)}(\hat{x})\,Sq^{I(k)}(\hat{y})\big),$$

if $\hat{e}$ has the form

$$\hat{e} = \sum Sq^{J(k)} Sq^{j(k)} Sq^{I(k)} + \hat{c} ,$$

where $j(k) = \deg(I(k)) + N + 1$ and excess of $\hat{c} \in \mathscr{A}$ is larger than $N+1$. This was proved in [3, Theorem 4.3]. The case $\hat{e} = Sq^{N+1}$ has frequently been used by Brown and Peterson (e.g. in [2], and to define the Arf invariant).

Operations connected with $\underset{\smile}{A}$-products have been considered by P. A. Schweitzer [6]. His approach is different. This connection is examined in Section 4.

The study of cohomology operations in this paper is based on a study of cochain operations. The cohomology operation associated with a cochain operation a commuting with coboundary will be denoted by $\hat{a}$. Similarly, $\hat{x}$ will denote a cohomology class, and a representing cocycle is denoted by x.

2. Preliminaries.

In this section we shall review some results from earlier papers [3] and [4].

Let $\mathscr{K}$ denote the category of CSS-complexes. On $\mathscr{K}$ we shall consider the cochain functor $C^{\cdot}(-; V)$, where the coefficient group is a graded Z_2-module of finite dimension (as vector space over Z_2). For each CSS-complex K, $C^{\cdot}(K; V)$ is a graded differential Z_2-module. For $V_0 = Z_2$ and $V_i = 0$ for $i \neq 0$, $C^{\cdot}(K; V)$ is a graded differential algebra; this case is simply denoted $C^{\cdot}(K)$.

Let us consider the set of all natural transformations

$$\theta\colon\ C^{\cdot}(-; V) \to C^{\cdot}(-)$$

satisfying

$$\theta(0) = 0, \qquad \deg(\theta(x)) = \deg(x) + i$$

for some integer i, which we denote $\deg\theta$. This set is denoted $\mathcal{O}^V$. It has in a natural way the structure of a graded differential Z_2-module,

$$(2.1) \qquad \begin{aligned} (\theta+\psi)(x) &= \theta(x)+\psi(x), & \deg\theta = \deg\psi , \\ (\Delta\theta)(x) &= \delta\theta(x) - (-1)^i\theta(\delta x), & \deg\theta = i . \end{aligned}$$

Since we are working over Z_2, we shall in what follows usually not write signs as $(-1)^i$. Also, let us consider the set $Q^{U,V}$ of all natural transformations ψ in two variables satisfying

$$\begin{aligned} \psi(x,y) &\in C^{\cdot}(K) && \text{for } x \in C^{\cdot}(K; U),\ y \in C^{\cdot}(K; V) , \\ \psi(x,0) &= \psi(0,y) = 0 && \text{for all } x \text{ and } y , \end{aligned}$$

and

$$\deg(\psi(x,y)) = \deg(x) + \deg(y) + i$$

for some integer i which, as above, we denote $\deg\psi$. The set $Q^{U,V}$ has, in an obvious way, the structure of a graded Z_2-module. A boundary operator ∇ is defined by

$$(\nabla\psi)(x,y) = \delta\psi(x,y) + \psi(\delta x,y) + \psi(x,\delta y) . \tag{2.2}$$

Obviously, $\nabla\nabla = 0$.

Let $Z(\mathcal{O}^V) = \operatorname{Ker}\Delta$ and $Z(Q^{U,V}) = \operatorname{Ker}\nabla$. We can define mappings

$$\begin{aligned} \varepsilon&: \ Z(\mathcal{O}^V) \to \mathscr{A}^V , \\ \varepsilon&: \ Z(Q^{U,V}) \to \mathscr{A}^U \otimes \mathscr{A}^V , \end{aligned} \tag{2.3}$$

where

$$\mathscr{A}^V = \mathscr{A} \otimes V^* ,$$

and $\mathscr{A}$ denotes the mod 2 Steenrod algebra. The definition of the mappings (2.3) goes as follows: Let

$$K(V,n) = \textstyle\prod_i K(V_i, n+i) ,$$

where $K(V_i,n+i)$ denotes an Eilenberg–MacLane complex. Since V is finite dimensional, this product is finite. The basic cocycle in $K(V,n)$ is denoted by

$$z_n = z_n^V \in C^n(K(V,n);\, V) .$$

Let $\theta \in Z^i(\mathcal{O}^V)$; then

$$\theta(z_n) \in C^{n+i}(K(V,n);\, Z_2)$$

and $\delta\theta(z_n) = 0$. The homomorphism

$$(\mathscr{A}^V)_i \xrightarrow{\ e\ } H^{n+i}(K(V,n))$$

defined by $\hat{a} \to \hat{a}(\{z_n\})$ is an isomorphism for n large. We define

$$\varepsilon(\theta) = e^{-1}\{\theta(z_n)\} \in (\mathscr{A}^V)_i .$$

This is independent of n. Similarly for $\psi \in Z^i(Q^{U,V})$

$$\varepsilon(\psi) = v^{-1}(\psi(z_m^U, z_n^V)\} \in (\mathscr{A}^U \otimes \mathscr{A}^V)_i ,$$

where

$$v: \ \mathscr{A}^U \otimes \mathscr{A}^V \to H^*(K(U,m) \times K(V,n);\, Z_2)$$

is the evaluation map.

The main theorem is

THEOREM 2.1. *The two sequences*

$$\mathcal{O}^V \xrightarrow{\ \Delta\ } Z(\mathcal{O}^V) \xrightarrow{\ \varepsilon\ } \mathscr{A}^V \to 0 ,$$

$$Q^{U,V} \xrightarrow{\ \nabla\ } Z(Q^{U,V}) \xrightarrow{\ \varepsilon\ } \mathscr{A}^U \otimes \mathscr{A}^V \to 0$$

are exact.

For the first of these sequences this is proved in [3]. In case $U = V = Z_2$, the second is proved in [4]. The same proof works for arbitrary U and V.

We shall give some simple applications of the $\mathcal{O}$-sequence in case $V_i = 0$ for $i \neq 0$. If $\dim(V_0) = n$, we have

$$C^i(K; V) \cong C^i(K) \oplus C^i(K) \oplus \ldots \oplus C^i(K) \quad (n \text{ summands}),$$

and $\theta \in \mathcal{O}^V = \mathcal{O}^n$ is considered as a function in n-variables.

Let $a \in Z(\mathcal{O}^1)$; then $b \in \mathcal{O}^2$ is defined by

$$b(x,y) = a(x+y) - a(x) - a(y) .$$

Obviously, $b \in Z(\mathcal{O}^2)$ and $b \in \operatorname{Ker}(\varepsilon)$. Hence there is $d(a) \in \mathcal{O}^2$ with $\Delta d(a) = b$. We can normalize $d(a)$ in such a way that

$$d(a)(0,y) = d(a)(x,0) = 0 .$$

We need only to replace

$$d(a)(x,y) \quad \text{by} \quad d(a)(x,y) - d(a)(x,0) - d(a)(0,y) .$$

More general, we put

$$d(a; x_1, \ldots, x_n) = \sum_{i=1}^{n-1} d(a)(x_i, x_{i+1} + \ldots + x_n) . \tag{2.4}$$

Then

$$\delta d(a; x_1, \ldots, x_n) + d(a; \delta x_1, \ldots, \delta x_n) = a(\sum x_i) - \sum a(x_i) , \tag{2.5}$$

and for each j, $1 \leq j \leq n$,

$$d(a; x_1, \ldots, x_n) = 0 \quad \text{if} \quad x_i = 0 \text{ for } i \neq j . \tag{2.6}$$

Let $\theta \in \mathcal{O}^1$ and let

$$\psi(x,y) = \theta(x+y) - \theta(x) - \theta(y) .$$

Since $\Delta\theta \in Z(\mathcal{O}^1)$, we can consider $d(\Delta\theta)$. Obviously, $\psi - d(\Delta\theta) \in Z(\mathcal{O}^2)$. Let

$$\varepsilon(\psi - d(\Delta\theta)) = \hat{a} \oplus \hat{b} \in \mathscr{A} \oplus \mathscr{A}, \qquad \text{where} \quad a,b \in Z(\mathcal{O}^1) .$$

Then for all pairs x,y of cocycles

$$\theta(x+y) - \theta(x) - \theta(y) - d(\Delta\theta)(x,y) \sim a(x) + b(y) .$$

Putting first x and then y equal to zero, we see that $\hat{a} = \hat{b} = 0 \in \mathscr{A}$. The theorem now gives the existence of $d(\theta) \in \mathcal{O}^2$ with

$$\Delta d(\theta) = \psi - d(\Delta\theta) . \tag{2.7}$$

Again, we can assume that

$$d(\theta)(x,0) = d(\theta)(0,y) = 0 .$$

We write (2.7) in the form

$$\Delta d(\theta; x_1,\ldots,x_n) + d(\Delta\theta; x_1,\ldots,x_n) = \theta(\Sigma x_i) - \Sigma\,\theta(x_i)\,, \tag{2.8}$$

where $d(\theta; x_1 \ldots, x_n)$ is defined by a formula similar to (2.4).

The following formulas are easily obtained. Let $\theta,\psi \in \mathcal{O}^1$; then

$$\begin{aligned}\Delta(\theta\psi)(x) \\ = (\Delta\theta)\psi(x) + \theta(\Delta\psi)(x) + \big(\Delta d(\theta) + d(\Delta\theta)\big)\big(\delta\psi(x), \psi\delta(x)\big)\,.\end{aligned} \tag{2.9}$$

If $\theta \in \mathcal{O}^1$ and $\psi \in Q^{1,1}$ $(=Q^{U,V},\ U=V=Z_2$, notation similar to $\mathcal{O}^1)$, then

$$\begin{aligned}\nabla(\theta\psi)(x,y) = (\Delta\theta)\psi(x,y) + \theta(\nabla\psi)(x,y) + \\ + \big(\Delta d(\theta) + d(\Delta\theta)\big)\big(\psi(\delta x,y), \psi(x,\delta y), \nabla\psi(x,y)\big)\,.\end{aligned} \tag{2.10}$$

In case $n=2$, equation (2.8) gives

$$\Delta d(\theta)(x,x) + d(\Delta\theta)(x,x) = 0\,.$$

Putting

$$d(\theta)(x,x) = \varkappa(\theta)(x)\,, \tag{2.11}$$

we get

$$\Delta\varkappa(\theta) = \varkappa(\Delta\theta)$$

and

$$\theta \in Z(\mathcal{O}^1) \Rightarrow \varkappa(\theta) \in Z(\mathcal{O}^1)\,.$$

In [4, Section 5], it is shown that $\varkappa$ induces a derivation $\varkappa\colon \mathcal{A} \to \mathcal{A}$ of degree -1 in the mod 2 Steenrod algebra and with $\varkappa(Sq^n) = Sq^{n-1}$.

Let $a \in Z(\mathcal{O}^1)$ and $\Phi(x,y) = xy$ (cup-product); then (2.10) gives

$$\nabla(a\Phi)(x,y) = \Delta d(a)\big(\Phi(\delta x,y), \Phi(x,\delta y)\big)\,.$$

Let

$$A(x,y) = d(a; \delta xy, x\delta y) + g(x)\varkappa(a)(x\delta y)\,,$$

where $g(x)$ denotes the degree of x; then $\nabla(a\Phi + A) = 0$. By the Cartan formula $\psi(\hat{a}) = \Sigma\hat{a}'\otimes\hat{a}''$ we get

$$\varepsilon(a\Phi + A) = \Sigma\hat{a}'\otimes\hat{a}''\,, \qquad \hat{a}',\hat{a}'' \in \mathcal{A}\,.$$

Let a' and a'' be cochain operations representing $\hat{a}'$ and $\hat{a}''$ respectively. Now a straightforward application of Theorem 2.1 yields

LEMMA 2.2. *Let $a \in Z(\mathcal{O}^1)$ and let $\psi(\hat{a}) = \Sigma\hat{a}'\otimes\hat{a}''$, where ψ denotes the comultiplication in $\mathcal{A}$. Then there is an element $T = T_a \in Q^{1,1}$ with*

$$(\nabla T)(x,y) = a(xy) - \Sigma a'(x)\,a''(y) + d(a; \delta xy, x\delta y) + g(x)\varkappa(a)(x\delta y) \tag{2.12}$$

for each pair x,y of cochains.

Later in this paper we shall also use Steenrod's cup-i products. Such products can be derived from Theorem 2.1 as follows. Let

$$D_0(x,y) = xy + yx .$$

Then, obviously $\nabla D_0 = 0$ and $\varepsilon(D_0) = 0$. Then there is $P_1 \in Q^{1,1}$ with $\nabla P_1 = D_0$. Let $D_1(x,y) = P_1(x,y) + P_1(y,x)$. Then $\nabla D_1 = 0$ and $\varepsilon(D_1) = 0$ for dimensional reasons. From this we get the existence of an operation $P_2 \in Q^{1,1}$ with $\nabla P_2 = D_1$. A continuation of this procedure gives the existence of operations $P_i \in Q^{1,1}$ with

$$(\nabla P_i)(x,y) = P_{i-1}(x,y) + P_{i-1}(y,x) .$$

We put

(2.13) $$x \cup_i y = P_i(x,y) .$$

LEMMA 2.3. *There are cochain operations $P_i \in Q^{1,1}$ with*

(2.14) $$\nabla P_i = P_{i-1}(x,y) + P_{i-1}(y,x) ,$$

or with the notation (2.13)

(2.15) $$\delta(x \cup_i y) = x \cup_{i-1} y + y \cup_{i-1} x + \delta x \cup_i y + x \cup_i \delta y .$$

For any cochain $x \in C^n(x)$ we define

(2.16) $$sq^i(x) = x \cup_{n-i} x + x \cup_{n-i+1} \delta x .$$

Then $sq^i \in Z(\mathcal{O}^1)$ and

$$\varepsilon(sq^i) = Sq^i$$

is the Steenrod reduced square.

The definition of a secondary operation Qu^R associated with $R\colon \sum \hat{\alpha}_\nu \hat{a}_\nu + \hat{e} = 0$, excess $(\hat{e}) > N$, goes as follows. Let $\alpha_\nu, a_\nu \in Z(\mathcal{O}^1)$ be representatives for $\hat{\alpha}_\nu, \hat{a}_\nu$. Let $e \in Z(\mathcal{O}^1)$ be a representative for $\hat{e}$ of excess $> N$. We say that e is of excess $> N$ if $e(x) = 0$ for all cochains x of dimension $< N$ and on all N-cocycles x. The existence of such a representative is shown in [3, Lemma 2.2]. Sometimes it is convenient to choose a representative for e of the form $\sum sq^I$, where sq^I is an iteration of operations sq^i from (2.16).

Obviously,

$$\nabla(\sum \alpha_\nu a_\nu + e) = 0 ,$$
$$\varepsilon(\sum \alpha_\nu a_\nu + e) = 0 .$$

Hence by Theorem 2.1 there is a $\theta \in \mathcal{O}^1$ with

$$\Delta\theta = \sum \alpha_\nu a_\nu + e .$$

If $\hat{x}$ is a cohomology class of dimension less than $N+1$, with $\hat{a}_\nu(\hat{x})=0$ for all ν, then

$$(2.17) \qquad Qu^R(\hat{x}) = \{\theta(x)+\sum\alpha_\nu(w_\nu)\}\,,$$

where w_ν are arbitrary cochains with $\delta w_\nu = a_\nu(x)$.

3. Lemmas.

Many of the lemmas we use do not have any interest in themselves. We have collected them in this section which is therefore of interest only in connection with the later sections. Therefore, we propose that this section is used for reference only. The proofs either use the exact sequences of Theorem 2.1 (the proof of Lemma 1 is typical) or they are purely computational checking (namely for lemmas of the form „∇ of this is that"). So we omit many of them. The cochain operations introduced in the lemmas have other cochain operations as parameters. For instance, in Lemma 3.1 below, the parameters for R are $a \in Z(\mathcal{O}^1)$, $d(a)$ and $d(\varkappa(a))$ (see 2.5). We shall not burden the notation by the parameters $R = R_{a,d(a),d(\varkappa(a))}$. The letters a, a_i, b, b_i always denote elements in $Z(\mathcal{O}^1)$.

LEMMA 3.1. *There is an operation $R \in \mathcal{O}^4$ satisfying*

$$(3.1) \qquad \Delta R(u_1,u_2,u_3,u_4) = d(a;\, u_1+u_2,u_3+u_4) + d(a;\, u_1+u_3,u_2+u_4) + \\ + d(a;\, u_1,u_3) + d(a;\, u_2,u_4) + \\ + d(a;\, u_1,u_2) + d(a;\, u_3,u_4)$$

and

$$(3.2) \quad R(u_1,u_2,u_1,u_2) + R(u_1,u_2,0,0) + R(0,0,u_1,u_2) + d(\varkappa(a);\, u_1,u_2) = 0\,.$$

PROOF. For short, denote by P the operation in $\mathcal{O}^4$ given by the right hand side of (3.1). Then an easy check gives $\Delta P = 0$. Also, since P is 0 if three of the arguments are zero, it follows that $\varepsilon P = 0$. So we may use Theorem 2.1 to find an $R'' \in \mathcal{O}^4$ with $\Delta R'' = P$. Consider $R''(u_1,0,u_3,0)$ and $R''(0,u_2,0,u_4)$ as operations in $\mathcal{O}^2$. They are easily seen to be in $\operatorname{Ker}\Delta$. Now the operation $R' \in \mathcal{O}^4$ defined by

$$R'(u_1,u_2,u_3,u_4) = R''(u_1,u_2,u_3,u_4) + R''(u_1,0,u_3,0) + R''(0,u_2,0,u_4)\,,$$

has $\Delta R' = P$ just like R'' and the further property

$$(3.3) \qquad R'(u_1,0,u_3,0) = R'(0,u_2,0,u_4) = 0\,.$$

Now, consider the operation $S \in \mathcal{O}^2$ defined by

$$S(u_1,u_2) = R'(u_1,u_2,u_1,u_2) + R'(u_1,u_2,0,0) + R'(0,0,u_1,u_2) + d(\varkappa(a);\, u_1,u_2)\,.$$

Then $S(u_1,0)=S(0,u_2)=0$ because of (3.3) and

$$\begin{aligned}\Delta S(u_1,u_2) &= P(u_1,u_2,u_1,u_2)+P(u_1,u_2,0,0)+P(0,0,u_1,u_2)+\Delta d(\varkappa(a);u_1,u_2)\\ &= d(a;u_1+u_2,u_1+u_2)+d(a;u_1,u_1)+d(a;u_2,u_2)+\Delta d(\varkappa(a);u_1,u_2)\\ &= 0\,.\end{aligned}$$

The operation R defined by

$$R(u_1,u_2,u_3,u_4) = R'(u_1,u_2,u_3,u_4)+S(u_1,u_4)$$

then has the properties (3.1) and (3.2).

LEMMA 3.2. *There is an operation $V \in \mathcal{O}^5$ satisfying*

$$\begin{aligned}\Delta V(u_1,u_2,u_3,u_4,u_5) = {}& d(a;\, u_1+u_2,u_3+u_4,u_1+u_4+u_5,u_2+u_3+u_5)+\\ &+d(a;\,u_1,u_2)+d(a;\,u_3,u_4)+d(a;\,u_1,u_4,u_5)+\\ &+d(a;\,u_2,u_3,u_5)+\sum_{i=1}^{5}\varkappa(a)(u_i)\,.\end{aligned}$$

LEMMA 3.3. *With V as in the preceding lemma there is an operation $U \in Q^{1,1}$ satisfying*

$$\begin{aligned}\nabla U(u,v) = {}& V(\delta u v,0,0,v\delta u,0)+V(0,\delta u v,v\delta u,0,0)+\\ &+V(\delta u v,\delta u v,v\delta u,v\delta u,\delta u v+v\delta u)+\\ &+d(\varkappa(a);\,\delta u v,v\delta u)\,.\end{aligned}$$

LEMMA 3.4. *Let f be an arbitrary mapping from $Z_0\times Z_0$ into Z_2, where Z_0 denotes the nonnegative integers. Let $\beta \in Z(\mathcal{O}^1)$. Then there is an operation $G \in Q^{1,1}$ with the properties*

$$\begin{aligned}\nabla G(u,v) &= f(g(u),g(v))\,\beta(\delta u\delta v)\,,\\ G(u,v) &= 0 \quad \textit{if} \quad \delta v = 0\,.\end{aligned}\tag{3.4}$$

PROOF. For $s,t \in Z_0$ put

$$G^{s,t}(u,v) = \begin{cases} 0 & \text{for } g(v)\neq t\,,\\ 0 & \text{for } g(v)=t,\ g(u)\leqq s\,,\\ \beta(u\delta v) & \text{otherwise}\,.\end{cases}$$

Then $\nabla G^{s,t}(u,v)=\beta(\delta u\delta v)$ for $g(u)=s$, $g(v)=t$ and 0 otherwise. Obviously, $G^{s,t}(u,v)=0$ if v is a cocycle. Let L be the set of (s,t) with $f(s,t)\neq 0$. Then

$$G(u,v) = \textstyle\sum_L G^{s,t}(u,v)$$

has the desired properties.

We shall need a Cartan formula for $\varkappa(\hat{a})$,

$$\psi(\varkappa(\hat{a})) = \sum \hat{\xi}_i'\otimes\hat{\xi}_i'' + \hat{\eta}\otimes\hat{\eta} + \sum \hat{\xi}_i''\otimes\hat{\xi}_i'$$

which we shall for short denote $\sum\hat{\zeta}'\otimes\hat{\zeta}''$. Pick cochain operations ζ' with $\varepsilon(\zeta')=\hat{\zeta}'$ and ζ'' with $\varepsilon(\zeta'')=\hat{\zeta}''$.

LEMMA 3.5. *There is an operation $F\in Q^{1,1}$, with*

$$\begin{aligned}(3.5)\qquad &\nabla F(u,v) = \\ &d(a;\,\delta u v+u\delta v,\delta v u+v\delta u,\delta u\cup_1\delta v+\delta u v+v\delta u,\delta u\cup_1\delta v+u\delta v+\delta v u) \\ &+d(a;\,\delta u v,u\delta v)+d(a;\,\delta v u,v\delta u)+d(a;\,\delta u v,v\delta u,\delta u\cup_1\delta v)+ \\ &+d(a;\,u\delta v,\delta v u,\delta u\cup_1\delta v)+g(u)\varkappa(a)(u\delta v)+g(v)\varkappa(a)(v\delta u)\end{aligned}$$

such that if y is a cocycle

$$\begin{aligned}(3.6)\qquad F(u,y) = {}& V(\delta u y,0,0,y\delta u,0)+g(y)d(\varkappa(a);\,\delta u y,y\delta u)+ \\ &+g(y)\varkappa(a)(\delta u\cup_1 y)+T_{\varkappa(a)}(\delta u,y)+T_{\varkappa(a)}(y,\delta u)+ \\ &+\sum\zeta' y\cup_1\zeta''\delta u+g(u)g(y)\varkappa(a)(uy)\,,\end{aligned}$$

where $T_{\varkappa(a)}$ satisfies (2.12) *with respect to $\varkappa(a)$, ζ', and ζ'', and V is as in Lemma* 3.2.

PROOF. We define an operation $F'\in Q^{1,1}$ by the formula

$$\begin{aligned}(3.7)\qquad F'(u,v) = {}& V(\delta u v,u\delta v,\delta v u,v\delta u,\delta u\cup_1\delta v)+U(u,\delta v)+ \\ &+g(v)d(\varkappa(a);\,\delta u v,v\delta u)+ \\ &+g(v)\varkappa(a)(\delta u\cup_1 v)+g(v)d(\varkappa(a);\,\delta u\cup_1\delta v,\delta u v+v\delta u)+ \\ &+T_{\varkappa(a)}(u,\delta v)+T_{\varkappa(a)}(\delta u,v)+T_{\varkappa(a)}(v,\delta u)+T_{\varkappa(a)}(\delta v,u)+ \\ &+\sum\zeta'(u)\cup_1\zeta''(\delta v)+\sum\zeta'(v)\cup_1\zeta''(\delta u)+ \\ &+g(u)g(v)d(\varkappa(a);\,u\delta v,\delta u v)+g(u)g(v)\varkappa(a)(uv)\end{aligned}$$

with V as in Lemma 3.2 and U as in Lemma 3.3. A straightforward but lengthy computation gives that $\nabla F'$ is equal to the expression (3.5) except for

$$\begin{aligned}&g(v)\varkappa^2(a)(\delta u\delta v+\delta v\delta u)+\bigl(g(v)+1\bigr)\varkappa^2(a)(\delta v\delta u)+ \\ &\qquad+\bigl(g(u)+1\bigr)\varkappa^2(a)(\delta u\delta v)+g(u)g(v)\varkappa^2(a)(\delta u\delta v)\,.\end{aligned}$$

According to Lemma 3.4, this expression may be put in the form $\nabla G(u,v)$ with $G(u,v)=0$ if $\delta v=0$. The operation $F=F'+G$ has the required properties.

LEMMA 3.6. *There is an operation $C\in\mathcal{O}^3$ satisfying*

$$\Delta C(u,v,w) = d(a;\,v,u,w)+d(a;\,w,u,v)\,.$$

In particular,

$$\Delta C(u,v,0) = d(a;\,v,u)+d(a;\,u,v)\,.$$

The proof is omitted.

LEMMA 3.7. *An operation $d(ba)$ satisfying* (2.5) *with a replaced by ba is given by*

$$d(ba;u,v) = b\,d(a;u,v) + \varkappa(b)au + \varkappa(b)av + \\ + d\big(b;a(u+v),au,av\big) + d\big(b;\delta d(a;u,v),d(a;\delta u,\delta v)\big).$$

The proof is omitted.

Lemmas 3.8–3.15 below are all applied in Section 8 only. Some of them are trivial and put up only for notational convenience.

LEMMA 3.8. *There is an operation $C' \in \mathcal{O}^3$ satisfying*

$$\Delta C'(u,v,w) = d(b;u,v) + d(b;w,u) + d(b;w,v) + \\ + d(b;u+w,v+w) + \varkappa(b)(w)$$

and such that $C'(u,v,w)=0$ if two of the cochains are 0.

LEMMA 3.9. *There is an operation $D \in \mathcal{O}^{4+2N}$ satisfying*

$$\begin{aligned}&\Delta D(u,v,w_1',\ldots,w_i',\ldots,w_N',w_1'',\ldots,w_N'',s,t)\\ &\quad = d(b;au+\textstyle\sum w_i'+s,av+\sum w_i'') + \\ &\qquad + d\big(b;a(u+w),w_1'+w_1'',\ldots,w_N'+w_N'',a(u+v)+au+av+t,s+t\big) + \\ &\qquad + d(b;au,w_1',\ldots,w_N',s) + d(b;av,w_1'',\ldots,w_N'') + \\ &\qquad + \textstyle\sum d(b;w_i',w_i'') + d(b;s,t) + d\big(b;a(u+v),au,av\big) + \\ &\qquad + d\big(b;a(u+v)+au+av+t,t\big) + \varkappa(b)au + \varkappa(b)av\,.\end{aligned}$$

LEMMA 3.10. *Let D be as above and let $a,b,a_i',a_i'' \in Z(\mathcal{O}^1)$ with*

$$\deg a_i' + \deg a_i'' = \deg a, \qquad 1 \leqq i \leqq N\,.$$

Then there is an operation $S \in Q^{1,1}$ satisfying

$$\nabla S(u,\delta v) = d\big(\varkappa(b);a(\delta u\,\delta v),a_1'\delta u\,a_1''\delta v,\ldots,a_N'\delta u\;a_N''\delta v\big) + (\nabla+\Delta)D(\Xi)\,,$$

when Ξ denotes the $4+2N$ tuple

$$\big(\delta u\,v,u\,\delta v,\ldots,a_i'\delta u\;a_i''v,\ldots,a_i'u\;a_i''\delta v,\ldots,(g(u)+1)\varkappa(a)(\delta u\,\delta v),\varkappa(a)(\delta u\,\delta v)\big)\,.$$

Here $(\nabla+\Delta)D$ means the sum of $(\Delta D)(\Xi)$ and the result of applying ∇ to $D(\Xi)$ considered as a cochain operation in the two variables u and v.

LEMMA 3.11. *Notation as in the two preceding lemmas. The following operation XI in $Q^{1,1}$,*

$$XI(u,v) = d\big(b;a(uv),\ldots,a_i'u\;a_i''v,\ldots,d(a;\delta u\,v,u\,\delta v),g(u)\varkappa(a)(u\,\delta v)\big) + \\ + D(\Xi) + S(u,\delta v)$$

goes by ∇ to

$$\begin{aligned}&\Delta d\big(b;\, a(uv),\ldots,a_i'u\, a_i''v,\ldots,d(a;\,\delta uv,u\delta v),g(u)\varkappa(a)(u\delta v)\big)+\\&\quad+d\big(\varkappa(b);\, a(\delta u\delta v),\ldots,a_i'\delta u\, a_i''\delta v,\ldots\big)+\\&\quad+d\big(b;\, a(\delta uv)+\textstyle\sum_I a_i'\delta u\, a_i''v\, +\\&\qquad\qquad+\big(g(u)+1\big)\varkappa(a)(\delta u\delta v),a(u\delta v)+\textstyle\sum a_i'u\, a_i''\delta v\big)\,+\\&\quad+\textstyle\sum_I d\big(b;\, a_i'\delta u\, a_i''v,a_i'ua_i''\delta v\big)+\big(g(u)+1\big)\varkappa(b)\varkappa(a)(\delta u\delta v)\,+\\&\quad+d\big(b;\, a\big(\delta(uv)\big),a(\delta uv),a(u\delta v)\big)\,+\\&\quad+d\big(b;\,\delta d(a;\,\delta uv,u\delta v),\varkappa(a)(\delta u\delta v)\big)+\varkappa(b)a(\delta uv)+\varkappa(b)a(u\delta v)\,.\end{aligned}$$

LEMMA 3.12. *Let* $\psi(\hat{a})=\sum\hat{a}_i'\otimes\hat{a}_i''$, $1\leqq i\leqq N$, *and let* $T=T_a$ *be as in* (2.12). *The folloving operation XII in* $Q^{1,1}$,

$$\begin{aligned}XII(u,v) = g(u)\varkappa(b)T(u,\delta v)+g(u)d\big(\varkappa(b);\, a(u\delta v),\ldots,a_i'u\, a_i''\delta v,\ldots\big)+\\+g(u)d\big(\varkappa(b);\,\delta T(u,\delta v),T(\delta u,\delta v)\big)\end{aligned}$$

goes by ∇ *to*

$$\begin{aligned}\varkappa(b)T(\delta u,\delta v)+g(u)\textstyle\sum\varkappa(b)(a_i'u\, a_i''\delta v)+g(u)\varkappa(b)a(u\delta v)\,+\\+d\big(\varkappa(b);\, a(\delta u\delta v),\ldots,a_i'\delta u\, a_i''\delta v,\ldots\big)\,.\end{aligned}$$

LEMMA 3.13. *Let* C' *be as in Lemma* 3.8 *and let* $A\in Q^{1,1}$. *The following operation XIII in* $Q^{1,1}$,

$$\begin{aligned}XIII(u,v) = C'\big(\delta A(\delta u,v),\delta A(u,\delta v),A(\delta u,\delta v)\big)+\\+d\big(b;\, A(\delta u,v),A(u,\delta v),\delta A(u,v)\big)\end{aligned}$$

goes by ∇ *to*

$$\begin{aligned}&\Delta d\big(b;\, A(\delta u,v),A(u,\delta v),\delta A(u,v)\big)+\\&\quad+d\big(b;\nabla A(\delta u,v),\nabla A(u,\delta v)\big)+\varkappa(b)\big(A(\delta u,\delta v)\big)\,.\end{aligned}$$

LEMMA 3.14. *Let* $b,a_i',a_i''\in Z(\mathcal{O}^1)$ *with* $\deg a_i'+\deg a_i''$ *independent of* i, $1\leqq i\leqq N$. *The following operation XIV in* $Q^{1,1}$,

$$\begin{aligned}XIV(u,v) = \textstyle\sum(\deg a_i')[g(v)\varkappa(b)(a_i'u\, a_i''v)\,+\\+g(u)g(v)\varkappa^2(b)(a_i'u\, a_i''\delta v)\,+\\+g(v)d\big(\varkappa(b);\, a_i'\delta u\, a_i''v,a_i'u\, a_i''\delta v\big)]\end{aligned}$$

goes by ∇ *to*

$$\textstyle\sum(\deg a_i')\varkappa(b)(a_i'u\, a_i''\delta v)\,.$$

LEMMA 3.15. *There is an operation XV in* $Q^{1,1}$ *with* $XV(u,v)=0$ *if* $\delta v=0$, *and which by* ∇ *goes to*

$$\big(g(u)+1\big)\varkappa(b)\varkappa(a)(\delta u\delta v)\,.$$

PROOF. By Lemma 3.4

We shall finally prove a useful proposition on symmetric cochain operations.

PROPOSITION 3.16. *Let* $K \in Z(Q^{1,1})$ *satisfy*

$$K(x,y) \sim K(x,y)\,,$$

for all pairs x,y *of cocycles. Then* $\varepsilon K \in \mathscr{A} \otimes \mathscr{A}$ *has the form*

$$\Sigma \hat{\beta}_i \otimes \hat{\gamma}_i + \Sigma \hat{\gamma}_i \otimes \hat{\beta}_i + \hat{\eta} \otimes \hat{\eta}$$

for certain $\hat{\beta}_i, \hat{\gamma}_i, \hat{\eta} \in \mathscr{A}$.

PROOF. Let n be greater than the degree of K. Let $\hat{x}$ and $\hat{y}$ be the two n-dimensional basic classes of $H^*(K(Z_2,n) \times K(Z_2,n))$. Now, if $\varepsilon K = \Sigma \hat{\varrho}_r \otimes \hat{\varrho}_r'$ with $\hat{\varrho}_r$ and $\hat{\varrho}_r'$ admissible monomials in $\mathscr{A}$, then by assumption and commutativity of cup product

$$\Sigma\, \hat{\varrho}_r(\hat{x})\, \hat{\varrho}_r'(\hat{y}) = \Sigma \hat{\varrho}_r'(\hat{x})\, \hat{\varrho}_r(\hat{y})\,.$$

Since all terms in this equation belong to a vectorspace basis for the cohomology, the proposition easily follows.

4. Definition and elementary properties.

Let $\hat{a} \in \mathscr{A}$. A *relation* A for $\hat{a}$ is defined as a collection of the following items: a factorizatio of $\hat{a}$,

$$\hat{a} = \sum \hat{\alpha}_\nu \hat{a}_\nu + \hat{e} \tag{4.1}$$

and a splitting of the Cartan formula sum into two sums

$$\psi \hat{a} = \sum \hat{b}_i \otimes \hat{c}_i + \sum \hat{b}_j \otimes \hat{c}_j, \qquad i \in I,\ j \in J\,. \tag{4.2}$$

Sometimes we are interested in making use of the symmetric form of $\psi\hat{a}$ so that instead of (4.2) we shall consider

$$\psi \hat{a} = [\Sigma \hat{c}_i' \otimes \hat{c}_i'' + \hat{b} \otimes \hat{b}] + \Sigma \hat{c}_i'' \otimes \hat{c}_i'\,. \tag{4.3}$$

The square bracket indicates the splitting of the Cartan formula sum into two sums.

Note that a relation for $\hat{0}$ is just a factorized relation $0 = \Sigma \hat{\alpha}_\nu \hat{a}_\nu + \hat{e}$ of the sort used in [1] or [3] for defining secondary cohomology operations.

Throughout this section, A will denote a fixed relation for $\hat{a}$ of the form (4.1), (4.2). To A we shall assign a secondary cohomology operation in two variables. It will be denoted $\hat{x} \overset{A}{\smile} \hat{y}$. The excess of $\hat{e}$ will be denoted $N+1$.

There is a choice of cochain operations involved in the definition, namely a choice of $\alpha_\nu, a_\nu, b_k, c_k \in Z(\mathcal{O}^1)$, $k \in I \cup J$, with $\varepsilon(\alpha_\nu) = \hat{\alpha}_\nu$, etc. Also, choose $d(\alpha_\nu), d(a_\nu) \in \mathcal{O}^2$ with

$$\Delta d(\alpha_\nu; x,y) = \alpha_\nu(x+y)+\alpha_\nu(x)+\alpha_\nu(y) ,$$

etc. For convenience, there will be no choice of e with $\varepsilon(e)=\hat{e}$. We determine e in the following way: write $\hat{e}$ as a sum of admissible monomials $\hat{e}=\sum Sq^{I(r)}$ of excess $\geqq N+1$. Then put $e=\sum sq^{I(r)}$. Then e is of excess $\geqq N+1$ in the sense of Section 2. Also, let $d(e)$ denote the operation in $\mathcal{O}^1$ given in [3], (2.14)–(2.16) with

$$\Delta d(e; x,y) = e(x+y)+e(x)+e(y) .$$

Having chosen these α_ν, a_ν, b_k, and e, put

$$a = \sum \alpha_\nu a_\nu + e .$$

Then $\varepsilon(a)=\hat{a}$. Also, we use the already chosen operations to define $d(a)$ according to Lemma 3.7

$$\begin{aligned} d(a; u,v) = d(e) &+ \sum \alpha_\nu d(a_\nu; u,v) + \sum \varkappa(\alpha_\nu)a_\nu(u) + \\ &+ \sum \varkappa(\alpha_\nu)a_\nu(v) + \sum d\big(\alpha_\nu; a_\nu(u+v), a_\nu(u), a_\nu(v)\big) + \\ &+ \sum d\big(\alpha_\nu; \delta d(a_\nu; u,v), d(a_\nu; \delta u, \delta v)\big) . \end{aligned}$$

Finally, we have to choose an operation $T \in Q^{1,1}$ with ∇T as in (2.12) with respect to a, $d(a)$, b_k, and c_k, $k \in I \cup J$. Let β denote the total choice $(\alpha_\nu, a_\nu, b_k, c_k, d(\alpha_\nu), d(a_\nu), T)$.

DEFINITION 4.1. Let $\hat{x},\hat{y}$ be cohomology classes in $H^*(X)$ such that

$$(4.4) \qquad \begin{aligned} \hat{a}_\nu(\hat{x}\hat{y}) &= 0 , \\ \hat{b}_i(\hat{x}) &= 0 \quad \text{for } i \in I , \\ \hat{c}_j(\hat{y}) &= 0 \quad \text{for } j \in J , \\ \deg(\hat{x}\hat{y}) &\leqq N . \end{aligned}$$

Then $\hat{x}\overset{A}{\smile}\hat{y}$, relative to the choice β, is the set of cohomology classes in $H^*(X)$ represented by cocycles z of the form

$$(4.5) \qquad z = T(x,y) + \sum \alpha_\nu(w_\nu) + \sum r_i c_i(y) + \sum b_j(x) r_j ,$$

where x is a cocycle in $\hat{x}$, y a cocycle in $\hat{y}$ and w_ν, r_i, and r_j are cochains with the properties

$$(4.6) \qquad \begin{aligned} \delta w_\nu &= a_\nu(xy) , \\ \delta r_i &= b_i(x) \quad \text{for } i \in I , \\ \delta r_j &= c_j(y) \quad \text{for } j \in J . \end{aligned}$$

Since $e(xy)=0$ by the last condition in (4.4), (2.12) gives that z is, indeed, a cocycle. Cochains satisfying (4.6) exist by the assumptions (4.4).

If (X,Y) is a pair of spaces and y is a relative cocycle satisfying (4.6) with $w_\nu,r_j \in C^\cdot(X,Y)$, then z in (4.5) is a relative cocycle. Hence, in a similar way, we get a set of relative cohomology classes $\hat{x}\overset{A}{\smile}\hat{y} \subset H^*(X,Y)$. Similarly in case $\hat{x} \in H^*(X,Y)$. For simplicity we consider only the absolute case in this section

For the moment we shall denote the cup-A product $\hat{x}\overset{A}{\smile}\hat{y}$ relative to the choice β by $(\hat{x}\overset{A}{\smile}\hat{y})_\beta$. We shall investigate the effect of this choice. Let

$$\beta = \big(\alpha_\nu,a_\nu,b_k,c_k,d(\alpha_\nu),d(a_\nu),T\big)\,,$$
$$\beta' = \big(\alpha'_\nu,a'_\nu,b'_k,c'_k,d(\alpha'_\nu),d(a'_\nu),T'\big)$$

denote two choices of the sort considered. We shall prove

PROPOSITION 4.2. *With β and β' as above,*

$$(\hat{x}\overset{A}{\smile}\hat{y})_\beta = (\hat{x}\overset{A}{\smile}\hat{y})_{\beta'} + \sum \hat{\lambda}'_\nu(\hat{x})\, \hat{\lambda}''_\nu(\hat{y})$$

for certain cohomology operqtions $\hat{\lambda}'_\nu,\hat{\lambda}''_\nu \in \mathscr{A}$.

PROOF. Since $\varepsilon(\alpha_\nu)=\varepsilon(\alpha'_\nu)$, there is an $A_\nu \in \mathcal{O}^1$ with $\Delta A_\nu = \alpha_\nu + \alpha'_\nu$. Similarly, there are operations $\bar{A}_\nu$, $\bar{B}_k$, $\bar{C}_k$ in $\mathcal{O}^1$ with

$$\Delta \bar{A}_\nu = a_\nu + a'_\nu, \quad \Delta \bar{B}_k = b_k + b'_k, \quad \Delta \bar{C}_k = c_k + c'_k\,.$$

Now one may easily verify that the operation $A \in \mathcal{O}^1$ given by

$$A(x) = \sum A_\nu a'_\nu(x) + \sum \alpha_\nu \bar{A}_\nu(x) + \\ + d\big(\alpha_\nu;\, \bar{A}_\nu(\delta x),\delta\bar{A}_\nu(x)\big) + d(\alpha_\nu;\, a_\nu x, a'_\nu x) \tag{4.7}$$

satisfies

$$\Delta A = a + a'\,,$$

where $a=\sum\alpha_\nu a_\nu + e$, $a'=\sum\alpha'_\nu a'_\nu + e$. This immediately yields $\Delta\theta = 0$ for $\theta \in \mathcal{O}^2$ given by

$$\theta(u,v) = d(a;\,u,v) + d(a';\,u,v) + A(u+v) + A(u) + A(v)\,.$$

Putting u or v equal to zero, one sees that $\varepsilon\theta = 0$. By Theorem 2.1 there is a $D \in \mathcal{O}^2$ with $\Delta D = \theta$. Just as for the normalization property (2.6) of $d(a)$, one can prove that D can be chosen normalized:

$$D(u,v) = 0 \qquad \text{if} \quad u=0 \text{ or } v=0\,.$$

Let $G \in Q^{1,1}$ be defined by

$$G(u,v) = \sum \bar{B}_k(u)\, c_k(v) + \sum b'_k(u)\, \bar{C}_k(v), \quad k \in I \cup J\,. \tag{4.8}$$

Then

$$\nabla G(u,v) = \sum b_k(u)\, c_k(v) + \sum b'_k(u)\, c'_k(v)\,.$$

Now it is easy to see that

$$T(u,v) + T'(u,v) + A(uv) + G(u,v) + D(\delta uv, u\delta v) + g(u) D(u\delta v, u\delta v)$$

is in $Z(Q^{1,1})$. Applying ε to the expression, we get an element of the form $\sum \hat{\lambda}_\nu' \otimes \hat{\lambda}_\nu''$ with $\hat{\lambda}_\nu', \hat{\lambda}_\nu'' \in \mathscr{A}$. Let $\lambda_\nu', \lambda_\nu''$ be representing cochain operations. Using Theorem 2.1, we get an operation $H \in Q^{1,1}$ with

$$(4.9) \qquad \nabla H(u,v) = T(u,v) + T'(u,v) + A(uv) + G(u,v) + D(\delta uv, u\delta v) + \\ + g(u) D(u\delta v, u\delta v) + \sum \lambda_\nu'(u)\, \lambda_\nu''(v) .$$

Let z be a cocycle of the form (4.5) representing a cohomology class in $(\hat{x} \overset{A}{\circlearrowleft} \hat{y})_\beta$. We shall prove that this class also belongs to $(\hat{x} \overset{A}{\circlearrowleft} \hat{y})_{\beta'} + \sum \hat{\lambda}_\nu'(\hat{x})\, \hat{\lambda}_\nu''(\hat{y})$. We have the equations

$$\begin{aligned} \delta[w_\nu + \bar{A}_\nu(xy)] &= a_\nu'(xy) , \\ \delta[r_i + \bar{B}_i(x)] &= b_i'(x) \qquad \text{for } i \in I , \\ \delta[r_j + \bar{C}_j(y)] &= c_j'(y) \qquad \text{for } j \in J . \end{aligned}$$

Hence

$$(4.5') \qquad z' = T'(x,y) + \sum \alpha_\nu'\big(w_\nu + \bar{A}_\nu(xy)\big) + \\ + \sum \big(r_i + \bar{B}_i(x)\big) c_i'(y) + \sum b_j'(x) \big(r_j + \bar{C}_j(y)\big)$$

represents a cohomology class in $(\hat{x} \overset{A}{\circlearrowleft} \hat{y})_{\beta'}$. Using (4.7), (4.8), and (4.9), we get

$$z + z' + \sum \lambda_\nu'(x)\, \lambda_\nu''(y) = \delta[H(x,y) + \sum A_\nu(w_\nu) + d(\alpha_\nu'; w_\nu, \bar{A}_\nu(xy)) + \\ + \sum r_i \bar{C}_i(y) + \sum \bar{B}_j(x) r_j + \sum \bar{B}_i(x)\, \bar{C}_i(y)] .$$

A similar argument proves the opposite inclusion. This concludes the proof of the proposition.

Remark. Let $T^+(u,v) = T(u,v) + \sum \lambda_\nu'(u)\, \lambda_\nu''(v)$. Since $\nabla T = \nabla T^+$, it follows that we may change the T in β with $\sum \lambda_\nu'(u)\, \lambda_\nu''(v)$ without changing anything else. Hence to any product $(\overset{A}{\circlearrowleft})_\beta$ we may find another product, $(\overset{A}{\circlearrowleft})_{\beta+}$ such that

$$(\hat{x} \overset{A}{\circlearrowleft} \hat{y})_\beta = (\hat{x} \overset{A}{\circlearrowleft} \hat{y})_{\beta+} + \sum \hat{\lambda}'(\hat{x})\, \hat{\lambda}''(\hat{y}) .$$

For the remainder of this section we shall work with a fixed choice β. We shall prove two bilinearity lemmas. These will be used to determine the indeterminacy of $\overset{A}{\circlearrowleft}$, and to determine deviation from bilinearity of $\overset{A}{\circlearrowleft}$.

Lemma 4.3. *Let $R \in \mathcal{O}^4$ be as in Lemma 3.1. Then there are operations $d_1(T) \in Q^{2,1}$ and $d_2(T) \in Q^{1,2}$ with*

$$\nabla d_1(T; x_1, x_2; y) = T(x_1 + x_2, y) + T(x_1, y) + T(x_2, y) + \\ + d(a; x_1 y, x_2 y) + \sum_{I \cup J} d(b_k; x_1, x_2)\, c_k(y) + \\ + g(x_1) d\big(\varkappa(a); x_1 \delta y, x_2 \delta y\big) + \\ + R(\delta x_1 y, \delta x_2 y, x_1 \delta y, x_2 \delta y)$$

and

$$\begin{aligned}\nabla d_2(T;x;y_1,y_2) = {} & T(x,y_1+y_2) + T(x,y_1) + T(x,y_2) + \\ & + d(a;xy_1,xy_2) + \textstyle\sum_{I\cup J} b_k(x)\,d(c_k;y_1,y_2) + \\ & + g(x)d(\varkappa(a);x\delta y_1,x\delta y_2) + \\ & + R(\delta x y_1,\delta x y_2,x\delta y_1,x\delta y_2)\,.\end{aligned}$$

PROOF. Immediate from Theorem 2.1.

It will be convenient to introduce an auxiliary functor S and a functor transformation $c\colon S \to C^{\cdot}$.

DEFINITION 4.4. Let X be a CSS-complex. We denote by $S(X)$ the set of tuples s of cochains in $C^{\cdot}(X)$ of the form

$$s = [x,y \mid \{w_\nu\},\{r_i\},\{r_j\}]\,,$$

where x and y are cocycles, and where

$$\begin{aligned}\delta w_\nu &= a_\nu(xy)\,, \\ \delta r_i &= b_i(x) \qquad \text{for } i \in I\,, \\ \delta r_j &= c_j(y) \qquad \text{for } j \in J\,.\end{aligned}$$

We shall call s a *system* for x,y. We define $c = c(\beta)\colon S \to C^{\cdot}$ by

$$c(s) = T(x,y) + \sum \alpha_\nu(w_\nu) + \sum r_i c_i(y) + \sum b_j(x) r_j\,.$$

It is obvious that $c(s)$ is a cocycle if $\deg(xy) \leqq N$, and that $\hat{x}\overset{A}{\smile}\hat{y}$ is just the set of cohomology classes of the form $\{c(s)\}$, where s is a system for x,y and $x \in \hat{x}$, $y \in \hat{y}$.

LEMMA 4.5. *Let* $s = [x,y \mid \{w_\nu\},\{r_i\},\{r_j\}]$, $s' = [x',y \mid \{w'_\nu\},\{r'_i\},\{r_j\}]$ *and* $s'' = [x,y'' \mid \{w''_\nu\},\{r_i\},\{r''_j\}]$ *be systems for* (x,y), (x',y), *and* (x,y'), *respectively. Then systems for* $(x+x',y)$ *and* $(x,y+y')$ *are given by*

$$\begin{aligned}s_1^+ &= [x+x',y \mid \{w_\nu + w'_\nu + d(a_\nu;xy,x'y)\},\{r_i + r'_i + d(b_i;x,x')\},\{r_j\}]\,, \\ s_2^+ &= [x,y+y' \mid \{w_\nu + w'_\nu + d(a_\nu;xy,xy')\},\{r_i\},\{r_j + r''_j + d(c_j;y,y')\}]\,,\end{aligned}$$

respectively, and we have

$$\left.\begin{aligned} c(s) + c(s') &\sim c(s_1^+) \\ c(s) + c(s'') &\sim c(s_2^+)\end{aligned}\right\} \quad \textit{for} \quad \deg(xy) < N\,. \tag{4.10}$$

Furthermore, there exist operations $sq^{J(\lambda)}$, $\lambda \in \Lambda$, *such that for* $\deg(xy) = N$

$$\begin{aligned} c(s) + c(s') &\sim c(s_1^+) + \textstyle\sum_\Lambda sq^{J(\lambda)}(xy)\, sq^{J(\lambda)}(x'y)\,, \\ c(s) + c(s'') &\sim c(s_2^+) + \textstyle\sum_\Lambda sq^{J(\lambda)}(xy)\, sq^{J(\lambda)}(xy')\,.\end{aligned}$$

PROOF. Applying Lemma 4.3 and Lemma 3.7, we easily get

$$\begin{aligned} c(s)+c(s') = c(s_1^+) &+ d(e; xy,x'y) + \\ &+ \delta[d_1(T; x,x'; y) + \textstyle\sum d(b_j; x,x')r_j + \\ &+ d(\alpha_\nu; d(a_\nu; xy,x'y) + w_\nu + w'_\nu, w_\nu, w'_\nu) + \\ &+ \textstyle\sum \varkappa(\alpha_\nu)(w_\nu) + \sum \varkappa(\alpha_\nu)(w'_\nu)] . \end{aligned}$$

But

$$e = \textstyle\sum_R sq^{J(r)} = \sum_\Lambda sq^{N+1+\deg J(\lambda)} sq^{J(\lambda)} + \sum_{R-\Lambda} sq^{J(r)} ,$$

where we have collected the monomials of exact excess $N+1$ in the first sum. So

$$d(e) = \textstyle\sum_\Lambda d(sq^{N+1+\deg J(\lambda)} sq^{J(\lambda)}) + \sum_{R-\Lambda} d(sq^{J(r)}) .$$

Now [3], Corollary 3.6 gives

$$d(sq^{N+1+\deg J(\lambda)} sq^{J(\lambda)}; xy,x'y) = sq^{J(\lambda)}(xy)\, sq^{J(\lambda)}(x'y)$$

for $\deg(xy)=N$, and [3, Lemma 2.6], gives that

$$d(sq^{I(r)}; xy,x'y) = 0$$

if $\deg(xy)=N$ and $r \in R-\Lambda$, or if $\deg(xy)<N$. This proves the one half of the lemma. The other half is similar.

COROLLARY 4.6. *If x' and y' are coboundaries, then* (4.10) *holds for* $\deg(xy) \leqq N$.

PROPOSITION 4.7. *Let* $\deg(\hat{x}\hat{y})=P\leqq N$. *Then the indeterminacy for* $\hat{x} \overset{A}{\smile} \hat{y}$ *is the* $(P+\deg a-1)^{\text{th}}$ *grading of*

$$\operatorname{Ind}^*(\hat{x},\hat{y}) = \textstyle\sum \hat{\alpha}_\nu H^*(X) + \sum H^*(X)\, \hat{c}_i(\hat{y}) + \sum \hat{b}_j(\hat{x})\, H^*(X) .$$

PROOF. We first prove that if s_0 is a system for $(x,\delta v)$, then $\{c(s_0)\} \in \operatorname{Ind}^*(\hat{x},\hat{y})$. Let $s_0=[x,\delta v \mid \{w_\nu\},\{r_i\},\{r_j\}]$. Using (2.12), we easily get

$$\begin{aligned} c(s_0) = \textstyle\sum_J b_j(x)\big(c_j(v)+r_j\big) &+ \textstyle\sum \alpha_\nu\big(a_\nu(xv)+w_\nu\big) + e(xv) + \\ &+ \delta[T(x,v) + \textstyle\sum r_i c_i(v) + g(x)\varkappa(a)(xv) + \sum d\big(\alpha_\nu; a_\nu(xv)+w_\nu, w_\nu\big)] . \end{aligned}$$

Since $\deg(xv)<\deg\big((x\delta v)\leqq N\big)$, $e(xv)=0$. Obviously, $\big(c_j(v)+r_j\big)$ and $\big(a_\nu(xv)+w_\nu\big)$ are cocycles. Therefore

$$(4.11) \qquad \{c(s_0)\} \in \textstyle\sum \hat{\alpha}_\nu H^*(X) + \sum \hat{b}_j(\hat{x}) H^*(X) \subseteqq \operatorname{Ind}^*(\hat{x},\hat{y}) .$$

Similarly, one may prove $\{c(s_0)\} \in \operatorname{Ind}^*(\hat{x},\hat{y})$ if s_0 is a system for $(\delta u,y)$.

Now, let s and $\tilde{s}$ be arbitrary systems for (x,y), $(x+\delta u, y+\delta v)$, respectively. Obviously, there exists a system s_1 for $(x,\delta v)$ so that Lemma 4.5 gives a system s_2 for $(x,y+\delta v)$, and by Corollary (4.6) and (4.11),

$$\{c(s)+c(s_2)\} \equiv 0 \mod \operatorname{Ind}(\hat{x},\hat{y}) .$$

Also, there is a system s_3 for $(\delta u, y+\delta v)$, so that Lemma 4.5 gives a system s_4 for $(x+\delta u, y+\delta v)$, and, again,

$$\{c(s_2)+c(s_4)\} \equiv 0 \quad \bmod \operatorname{Ind}(\hat{x},\hat{y}) .$$

Since s_4 and $\tilde{s}$ both are systems for $(x+\delta u, y+\delta v)$, we obviously have

$$\{c(s_4)+c(\tilde{s})\} \equiv 0 \quad \bmod \operatorname{Ind}(\hat{x},\hat{y}) .$$

On the other hand, if $s=[x,y \mid \{w_\nu\},\{r_i\},\{r_j\}]$ is a system for x,y, then for arbitrary cocycles z_ν, z_i, and z_j,

$$s^+ = [x,y \mid \{w_\nu+z_\nu\},\{r_i+z_i\},\{r_j+z_j\}]$$

is a system for x,y, and

$$\{c(s)+c(s^+)\} = \sum \hat{\alpha}_\nu(\hat{z}_\nu) + \sum \hat{z}_i\, \hat{c}_i(\hat{y}) + \sum \hat{b}_j(x)\, \hat{z}_j .$$

This proves the proposition.

THEOREM 4.8. *Let* $\deg\hat{x}=\deg\hat{x}'$ *and* $\deg\hat{y}=\deg\hat{y}'$. *If* $\hat{x}\overset{A}{\smile}\hat{y}$ *and* $\hat{x}'\overset{A}{\smile}\hat{y}$ *are defined, then, so is* $(\hat{x}+\hat{x}')\overset{A}{\smile}y$, *and modulo indeterminacy we have*

$$\hat{x}\overset{A}{\smile}\hat{y} + \hat{x}'\overset{A}{\smile}\hat{y} = (\hat{x}+\hat{x}')\overset{A}{\smile}y$$

for $\deg(xy)<N$. *Furthermore, if*

$$e = \sum_R sq^{J(r)} = \sum_\Lambda sq^{N+1+\deg J(\lambda)} sq^{J(\lambda)} + \sum_{R-\Lambda} sq^{J(r)} ,$$

and

$$\text{excess } sq^{J(r)} > N+1 \quad \textit{for} \quad r \in R-\Lambda ,$$

then for $\deg(\hat{x}\hat{y})=N$, *the following equation holds modulo indeterminacy*

$$\hat{x}\overset{A}{\smile}\hat{y}+\hat{x}'\overset{A}{\smile}\hat{y} = (\hat{x}+\hat{x}')\overset{A}{\smile}\hat{y} + \sum Sq^{J(\lambda)}(\hat{x}\,\hat{y})\, Sq^{J(\lambda)}(\hat{x}'\,\hat{y}) .$$

Similarly, we have modulo indeterminacy

$$\hat{x}\overset{A}{\smile}\hat{y} + \hat{x}\overset{A}{\smile}\hat{y}' = x\overset{A}{\smile}(\hat{y}+\hat{y}') \qquad \textit{for } \deg(\hat{x}\hat{y})<N ,$$
$$\hat{x}\overset{A}{\smile}\hat{y} + \hat{x}\overset{A}{\smile}\hat{y}' = x\overset{A}{\smile}(\hat{y}+\hat{y}') + \sum Sq^{J(\lambda)}(\hat{x}\,\hat{y})\, Sq^{J(\lambda)}(\hat{x}\,\hat{y}') \qquad \textit{for } \deg(\hat{x}\hat{y})=N .$$

PROOF. This is an immediate consequence of Lemma 4.5 and Proposition 4.7.

Let A be the following relation for $\hat{a} \in \mathscr{A}$

$$(4.12) \qquad A: \begin{cases} \hat{a} = \hat{a}\cdot 1 + 0 \\ \psi\hat{a} = \sum \hat{b}_i\otimes\hat{c}_i + \sum \hat{b}_j\otimes\hat{c}_j, \quad i \in I,\ j \in J . \end{cases}$$

Then there is a close connection between the operation $\hat{a}_d$ defined in Schweitzer's paper [6] and the secondary product $\overset{A}{\smile}$. For any space X

and cohomology operation $\hat{a}: H^*(X) \to H^*(X)$, the operation $\hat{a}_d$ is an additive relation

$$H^*(X^n, \text{fat wedge}) \to H^*(X)\,.$$

The operation $\hat{a}_d$ is more general than $\overset{A}{\smile}$ in the following sense: It can be defined for any finite number of variables (we cannot do this until for any positive integer n we have established an exact sequence

$$Q \xrightarrow{\nabla} Z(Q) \xrightarrow{\varepsilon} \mathscr{A}\otimes\ldots\otimes\mathscr{A} \to 0 \quad (n \text{ copies of } \mathscr{A})\,,$$

where Q denotes cochain operations in n variables). Secondly, we have restricted ourselves solely to Z_2-coefficients. So we shall compare only a special case of the Schweitzer operation with $\overset{A}{\smile}$, A as in (4.12). Suppose $\hat{a} \in \mathscr{A}$ is of degree $s-r$. In this case the definition of $\hat{a}_d$ is given by means of the diagram below. In this diagram $X \vee X$ is a subspace of $X \times X$ in the usual way, X_d is the diagonal subspace of $X \times X$. The rows are exact sequences for the injection

$$(X_d,*) \xrightarrow{d} (X \times X, X \vee X)\,.$$

Commutativity holds because of the stability of $\hat{a}$.

$$\begin{array}{ccccccccc}
 & & H^{r-1}(X_d,*) & \xrightarrow{\delta^*} & H^r(X\times X, X_d \cup (X\vee X)) & \xrightarrow{j^*} & H^r(X\times X, X\vee X) & \xrightarrow{d^*} & H^r(X_d,*) \\
 & & \downarrow{\scriptstyle\hat{a}} & & \downarrow{\scriptstyle\hat{a}} & & \downarrow{\scriptstyle\hat{a}} & & \\
H^{s-1}(X\times X, X\vee X) & \xrightarrow[d^*]{} & H^{s-1}(X_d,*) & \xrightarrow[\delta^*]{} & H^s(X\times X, X_d \cup (X\vee X)) & \longrightarrow & H^s(X\times X, X\vee X)\,. & &
\end{array}$$

An obvious diagram chase gives an additive relation (dotted arrow)

$$\hat{a}_d\colon\ H^r(X \times X, X \vee X) \to H^{s-1}(X,*)$$

defined on $\operatorname{Ker} d^* \cap \operatorname{Ker} \hat{a}$, and with indeterminacy $\operatorname{Im} d^* \cup \operatorname{Im} \hat{a}$. Denote by $\times$ the exterior cup product

$$H^*(X,*)\otimes H^*(X,*) \to H^*(X \times X, X \vee X)\,.$$

The connection between $\hat{a}_d$ and the $\overset{A}{\smile}$ is then given by

PROPOSITION 4.8 *Let* $\hat{x}\overset{A}{\smile}\hat{y}$ *be defined with* $\dim(\hat{x}) > 0$, $\dim(\hat{y}) > 0$ *and with* A *as in* (4.12). *Then* $\hat{a}_d(\hat{x} \times \hat{y})$ *is defined, and as sets of cohomology classes*

$$\hat{x}\overset{A}{\smile}\hat{y} \subseteq \hat{a}_d(\hat{x} \times \hat{y})\,.$$

PROOF. Let $u,v \in C^\cdot(X)$; then $u \times v = p_1^\#(u)\,p_2^\#(v)$, where p_1 and p_2 are the two projections $X \times X \to X$. If u and v are cocycles, then $u \times v$ represents $\hat{u} \times \hat{v}$. Also, $uv = d^\#(u \times v)$. By the special assumptions on $\hat{x},\hat{y}$, $\hat{x} \times \hat{y}$ is in the kernel for

$$H^*(X\times X,*) \to H^*(X_d\cup(X\vee X)),$$

and by the exact cohomology sequence for the pair $X\times X$, $X_d\cup(X\vee X)$, we may therefore find a $w'\in C^\cdot(X\times X,*)$ such that

$$\delta w' + x\times y \in C^\cdot(X\times X, X_d\cup(X\vee X)).$$

Then $j^*\{\delta w' + x\times y\} = \hat{x}\times\hat{y}$. By (2.5) and (2.12)

$$a(\delta w' + x\times y) = a\delta w' + \sum b_k(p^\# x)\, c_k(p^\# y) + \delta T(p^\# x, p^\# y) + \delta d(a; \delta w', x\times y + \delta w').$$

By assumption, there exist cochains $r_i, r_j \in C^\cdot(X)$ with

$$\delta r_i = b_i x, \quad \delta r_j = c_j y \quad i\in I,\ j\in J.$$

Therefore,

$$a(\delta w' + x\times y) = \delta[a(w') + \sum p^\#(r_i)\, c_i(p^\# y) + \sum b_j(p^\# x)\, p^\#(r_j) + T(p^\# x, p^\# y) + d(a; \delta w', x\times y + \delta w')].$$

From this follows that $\delta^{*-1}\{a(x\times y + \delta w')\}$—and therefore $\hat{a}_d(\hat{x}\times\hat{y})$—is represented by

$$\text{(4.14)} \qquad a(d^\# w') + \sum r_i c_i(y) + \sum b_j(x) r_j + T(x,y) \in C^\cdot(X_d).$$

On the other hand, since $\delta w' + x\times y \in C^\cdot(X\times X, X_d\cup(X\vee X))$,

$$d^\#\delta w' = d^\#(x\times y) = xy;$$

thus, in Definition (4.5) we may use $d^\# w'$ as the w of $\hat{x}\underset{\smile}{A}\hat{y}$. Doing this, we get (4.14). Thus we have proved that $\hat{a}_d(x\times\hat{y})$ and $\hat{x}\underset{\smile}{A}\hat{y}$ have a common representative. Using the fact that $\hat{x}$ and $\hat{y}$ are not of dimension zero, it is easy to see that the indeterminacy found in Proposition 4.6 is contained in the indeterminacy $\operatorname{Im} d^*\cup\operatorname{Im}\hat{a}$ for $\hat{a}_d$. This completes the proof.

A universal example P for $\underset{\smile}{A}$ acting in dimension (p,q) is constructed by means of a map f between generalized Eilenberg–MacLane complexes

$$f\colon\ K^{(p)}\times K^{(q)} \to \prod_I K^{(i)}\times\prod_J K^{(j)}\times\prod_\nu K^{(\nu)}.$$

Here $K^{(p)} = K(Z_2,p)$, $K^{(q)} = K(Z_2,q)$, $K^{(i)} = K(Z_2, p+\deg b_i)$, $K^{(j)} = K(Z_2, q+\deg c_j)$ and $K^{(\nu)} = K(Z_2, p+q+\deg a_\nu)$. The map f is defined such that

$$f^\#(x^{(i)}) = b_i x^{(p)},$$
$$f^\#(x^{(j)}) = c_j x^{(q)},$$
$$f^\#(x^{(\nu)}) = a_\nu(x^{(p)} x^{(q)}),$$

where $x^{(i)}$ denotes the fundamental class arising from the factor $K^{(i)}$, etc. We now construct P as the pull-back of f and the standard fibering with acyclic total space

$$\prod L^{(i)} \times \prod L^{(j)} \times \prod L^{(\nu)} \to \prod K^{(i)} \times \prod K^{(j)} \times \prod K^{(\nu)} ,$$

where $L^{(i)} = L(Z_2, p + \deg b_i - 1)$, etc.

5. Deviation from commutativity.

Since the comultiplication in the Steenrod algebra is commutative, there is a symmetric formula

$$\psi\hat{a} = [\sum \hat{c}_i' \otimes \hat{c}_i'' + \hat{b} \otimes \hat{b}] + \sum \hat{c}_i'' \otimes \hat{c}_i' . \tag{5.1}$$

Obviously, the middle term can only appear when $\deg \hat{a}$ is even. In this section we shall consider a $\underset{\smile}{A}$-product with the factorization $\hat{a} = \sum \hat{\alpha}_\nu \hat{a}_\nu + \hat{e}$ of $\hat{a}$, and the splitting of the Cartan formula sum indicated by square brackets in (5.1). As usual, let $N+1$ denote the excess of $\hat{e}$. The product $\hat{x} \underset{\smile}{A} \hat{y}$ is then defined if $\deg(\hat{x}\hat{y}) \leqq N$ and if there exist cochains w_ν, r_i, r, s_i such that

$$\begin{aligned} a_\nu(xy) &= \delta w_\nu , \\ c_i'(x) &= \delta r_i , \\ b(x) &= \delta r , \\ c_i'(y) &= \delta s_i . \end{aligned} \tag{5.2}$$

Now, since $\delta\big(w_\nu + a_\nu(x \cup_1 y) + d(a_\nu; xy, yx)\big) = a_\nu(yx)$, we see that in order to define $\hat{y} \underset{\smile}{A} \hat{x}$, it is only necessary to add the following requirement to (5.2): there exists s such that

$$b(y) = \delta s . \tag{5.3}$$

PROPOSITION 5.1. *If* (5.2) *and* (5.3) *hold for cocycles* x,y $\big(\deg(xy) \leqq N\big)$, *then* $\hat{x} \underset{\smile}{A} \hat{y} + \hat{y} \underset{\smile}{A} \hat{x}$ *is represented by* $E(x,y) + d(e; xy, yx)$, *where* $E \in Q^{1,1}$ *is the cochain operation defined by*

$$\begin{aligned} E(u,v) = {} & T(u,v) + T(v,u) + d(a; uv, vu, \delta u \cup_1 v, u \cup_1 \delta v) + \\ & + a(u \cup_1 v) + \sum c_i'(u) \cup_1 c_i''(v) + b(u) \cup_1 b(v) + \sum c_i''(u) \cup_1 c_i'(v) , \end{aligned} \tag{5.4}$$

where T *satisfies* (2.12).

PROOF. For reasons of degree, $e(x \cup_1 y) = 0$. Using the fact that x and y are cocycles, (5.4) is an immediate consequence of Definition 4.1 and the definition of $d(a)$ expressed by $d(\alpha_\nu)$, $d(a_\nu)$, $d(e)$.

PROPOSITION 5.2. *Let* F *be the cochain operation given in Lemma* 3.5, *then* $\nabla(E+F) = 0$.

PROOF. ∇F is given in Lemma 3.5. A straightforward computation gives the result, using the particular symmetric form of $\psi\hat{a}$ in (5.1).

Since $E+F \in Z(Q^{1,1})$, ε is defined on $E+F$. To get some information about $\varepsilon(E+F)$, we prove

PROPOSITION 5.3. *The operation $E+F$ is symmetric in the sense that*

$$(E+F)(x,y) \sim (E+F)(y,x)$$

for arbitrary cocycles x,y.

PROOF. It is easily seen from (3.6) that

$$F(x,y) = g(x)g(y)\varkappa(a)(xy) \sim g(y)g(x)\varkappa(a)(yx) = F(y,x) .$$

Furthermore, by (5.4)

$$\begin{aligned} E(x,y)+E(y,x) = {} & d(a;xy,yx) + d(a;yx,xy) + a(x\cup_1 y) + a(y\cup_1 x) + \\ & + \sum c_i'(x)\cup_1 c_i''(y) + b(x)\cup_1 b(y) + \sum c_i''(x)\cup_1 c_i'(y) + \\ & + \sum c_i'(y)\cup_1 c_i''(x) + b(y)\cup_1 b(x) + \sum c_i''(y)\cup_1 c_i'(x) . \end{aligned}$$

Using Lemma 3.6, this is seen to be equal to

$$\begin{aligned} & \delta C(xy,yx,0) + a(\delta(x\cup_2 y)) + \Delta d(a;x\cup_1 y, y\cup_1 x + x\cup_1 y) + \\ & \quad + \delta\big(\sum c_i'(x)\cup_2 c_i''(y) + b(x)\cup_2 b(y) + \sum c_i''(x)\cup_2 c_i'(y)\big) \end{aligned}$$

which is cohomologous to 0.

By the propositions 5.3 and 3.16, we infer the existence of cochain operations $\lambda_\nu', \lambda_\nu'', \eta \in Z(\mathcal{O}^1)$ such that for cocycles x,y

$$(E+F)(x,y) \sim \sum \lambda_\nu'(x)\,\lambda_\nu''(y) + \eta(x)\eta(y) + \sum \lambda_\nu'(y)\,\lambda_\nu''(x) .$$

Now the T in the definition of $\overset{A}{\smile}$ may be changed by anything in $\operatorname{Ker}\nabla$ giving some other $\overset{A}{\smile}$-product. For instance, we may add the term $\sum \lambda_\nu'(u)\,\lambda_\nu''(v)$. Hence with this change of T and therefore of E, we get

$$\varepsilon(E+F) = \hat{\eta}\otimes\hat{\eta} . \tag{5.5}$$

THEOREM 5.4. *There is a $\overset{A}{\smile}$-product such that for* $\deg(\hat{x}\hat{y}) < N$

$$\hat{x}\overset{A}{\smile}\hat{y} + \hat{y}\overset{A}{\smile}\hat{x} = \hat{\eta}(\hat{x})\,\hat{\eta}(\hat{y}) + g(x)g(y)\widehat{\varkappa(e)}(\hat{x}\hat{y}) \tag{5.6}$$

whenever the left hand side is defined. Furthermore, there are operations $Sq^{J(s)} \in \mathscr{A}$ such that for $\deg(\hat{x}\hat{y}) = N$

$$\hat{x}\overset{A}{\smile}\hat{y} + \hat{y}\overset{A}{\smile}\hat{x} = \hat{\eta}(\hat{x})\,\hat{\eta}(\hat{y}) + g(x)g(y)\widehat{\varkappa(e)}(\hat{x}\hat{y}) + \sum\big(Sq^{J(s)}(\hat{x}\hat{y})\big)^2 , \tag{5.6'}$$

whenever the left hand side is defined. Both equations hold modulo indeterminacy. The operation $\hat{\eta} \in \mathscr{A}$ satisfies

(i) $\hat{\eta}=0$ *if* $\deg \hat{a}$ *is even*,

(ii) *if* $\deg \hat{a}$ *is odd and if* $\hat{z} \in H^p(X)$ *is any p-dimensional cohomology class, then*

$$(p+1)\widehat{\varkappa(a)}\, Sq^p \hat{z} + \hat{a}\, Sq^{p-1}\hat{z} + Sq^{\frac{1}{2}(\deg \hat{a}-1)+p}\, \hat{\eta}\hat{z} = 0 . \tag{5.7}$$

PROOF. Let $\hat{x}$ and $\hat{y}$ be cohomology classes such that the left hand side of (5.6) or (5.6′) is defined. By (5.5) and Theorem 2.1 there is an operation $B \in Q^{1,1}$ with

$$\nabla B(u,v) = E(u,v) + F(u,v) + \eta(u)\, \eta(v) . \tag{5.8}$$

We use this equation on the cocycles x,y. By Lemma 3.5,

$$F(x,y) = g(x)g(y)\varkappa(a)(xy) .$$

So we get

$$E(x,y) \sim g(x)g(y)\varkappa(a)(xy) + \eta(x)\, \eta(y) .$$

But by Proposition 5.1, $\hat{x}\underset{\smile}{\hat{A}}\hat{y}+\hat{y}\underset{\smile}{\hat{A}}\hat{x}$ is represented by $E(x,y)+d(e;\, xy,yx)$. Now, just as in the proof of Lemma 4.5, $d(e;\, xy,yx)=0$ for $\deg(xy)<N$, and for $\deg(xy)=N$,

$$d(e;\, xy,yx) = \sum sq^{J(s)}(xy)\, sq^{J(s)}(yx) .$$

Using that $\varkappa$ is a derivation in $\mathscr{A}$, one may see that $\widehat{\varkappa(a)}(\hat{x}\hat{y}) \equiv \widehat{\varkappa(e)}(\hat{x}\hat{y})$ modulo the total indeterminacy. This proves (5.6) and (5.6′). If $\deg a$ is odd, there is no middle term in (5.1) ,so that for any cocycle x

$$\sum c_i'(x)\cup_1 c_i''(x) + \sum c_i''(x)\cup_1 c_i'(x) = \delta \sum c_i'(x)\cup_2 c_i''(x) .$$

Therefore, by (5.4)

$$E(x,x) \sim d(a;\, x^2,x^2) + a(x\cup_1 x) .$$

Thus by (5.8) and (3.6)

$$a(x\cup_1 x) + \big(g(x)+1\big)\varkappa(a)(x^2) + \big(\eta(x)\big)^2 \sim 0 . \tag{5.9}$$

This is true for any cocycle x. In particular, take x to be p-dimensional. Using the Definition (2.16) of reduced squares, (5.7) is an immediate consequence. This completes the proof.

For any concrete $\hat{a}$ it is in principle easy to solve the equation (5.7) for $\hat{\eta}$. We shall give one such computation. Let ζ be the dual of the squaring homomorphism ζ^* in the dual Hopf algebra $\mathscr{A}^*$ of $\mathscr{A}$. It has the property

$$\hat{\alpha}(\hat{x}^2) = \big(\zeta(\hat{\alpha})(\hat{x})\big)^2 .$$

PROPOSITION 5.5. *For* $\hat{a}=\hat{\alpha}\, Sq^{2i+1} Sq^{2I}$, $I=(i_1,\ldots,i_r)$, *the corresponding* $\hat{\eta}$ *is given by*

$$\hat{\eta} = (\deg I + i + 1)\,\zeta(\hat{\alpha})\, Sq^i\, Sq^I .$$

PROOF. Let q be an integer and let us apply the operation

$$Sq^{2i}\,Sq^{2I}\,Sq^{2q} + Sq^{\deg I+2q+i}\,Sq^{i}\,Sq^{I} \tag{5.10}$$

to the bacis class $\hat{x}=\hat{x}_{2q}\in H^{2q}(K(Z_2,2q))$. This obviously gives zero. The operation (5.10) can therefore be written in the form $\hat{d}=\sum Sq^K$, where K is admissible and of excess $\geqq 2q+1$. Since $\deg(K)$ is even and since for all admissible $K=(k_1,\ldots,k_r)$

$$\text{excess}\,(K) = 2k_1-\deg K\,,$$

we get that excess(K) is even. Therefore, excess$(K)\geqq 2q+2$. Since $Sq^1\,Sq^n=(n+1)\,Sq^{n+1}$ for all n, we get

$$\begin{aligned}\hat{a}\,Sq^{2q} &= \hat{\alpha}\,Sq^{2i+1}\,Sq^{2I}\,Sq^{2q}\\ &= \big(\deg(I)+i+1\big)\,\hat{\alpha}\,Sq^{\deg I+2q+i+1}\,Sq^{i}\,Sq^{I} + \hat{\alpha}\,Sq^1\hat{d}\,.\end{aligned}$$

Using this equation and (5.7) for $p=2q+1$ and

$$\hat{x} = \hat{x}_{2q+1} \in H^{2q+1}\big(K(Z_2,2q+1)\big)\,,$$

we get

$$\begin{aligned}(\hat{\eta}\hat{x})^2 &= (\deg I+i+1)\,\hat{\alpha}\,Sq^{\deg I+2q+i+1}\,Sq^{i}\,Sq^{I}\,\hat{x}\\ &= \big((\deg I+i+1)\,\zeta(\hat{\alpha})\,Sq^{i}\,Sq^{I}\,\hat{x}\big)^2\,.\end{aligned}$$

Hence $(\hat{c}\hat{x})^2=0$ for

$$\hat{c} = \hat{\eta} + (\deg I+i+1)\,\zeta(\hat{\alpha})\,Sq^{i}\,Sq^{I}\,.$$

Since $H^*\big(K(Z_2,p)\big)$ is a polynomial algebra, $\hat{c}(\hat{x})=0$. Since p was an arbitrary odd number, we get $\hat{c}=0$, which concludes the proof.

6. Connection with secondary operations in one variable.

In this section we shall investigate $\hat{x}\overset{A}{\smile}\hat{x}$ for certain dimensions of $\hat{x}$. Let A be as in (5.1).

CONVENTION 6.1 If $\deg a$ is even, put $m=\frac{1}{2}\deg a$. Then the operation τ in (6.3) below is zero, and no meaning is attached to the symbol $\overline{m}$. If $\deg a$ is odd, put $\overline{m}=\frac{1}{2}(\deg a-1)$. Then the middle term in (5.1) does not appear, and no meaning is attached to the symbols b and m.

Let n be a positive integer, fixed throughout this section. By the Cartan formula (5.1) we have for any $n-1$-dimensional cohomology class $\hat{x}$

$$\hat{a}\,Sq^{n-1}\,\hat{x}+Sq^{m+n-1}\,\hat{b}\,\hat{x} = \hat{a}(\hat{x}^2)+(\hat{b}\,\hat{x})^2 = 0\,.$$

Hence $\hat{a}\,Sq^{n-1}+Sq^{m+n-1}\,\hat{b}$ is of excess $\geqq n$ and may be written

$$\hat{a}\,Sq^{n-1}+Sq^{m+n-1}\,\hat{b} = \sum \hat{c}'_{\nu}\,Sq^{n+\deg c_\nu}\,\hat{c}_{\nu} + \hat{e} \tag{6.1}$$

with excess $\hat{e} \geqq n+1$. Using (6.1), we see that

$$r = asq^{n-1} + sq^{m+n-1}\, b + \sum c'_\nu sq^{n+\deg c_\nu}\, c_\nu + e \in Z(\mathcal{O}^1) \tag{6.2}$$

belongs to Kerε. Since excess $(\hat{e}) \geqq n+1$, the representative e can be chosen to be of excess $\geqq n+1$ in the sense of Section 2.

Now, consider the cochain operation $K \in Q^{1,1}$ given by

$$K(u,v) = a(u \cup_1 v) + bu \cup_1 bv + \sum c'_\nu(c_\nu(u)\, c_\nu(v))\,. \tag{6.3}$$

Using the Definition (2.16) of sq^i as a cochain operation, we get for an n-dimensional cocycle x

$$K(x,x) = r(x) \tag{6.4}$$

since $e(x) = 0$.

Our intention is to prove $\hat{x} \underset{\smile}{\Delta} \hat{x} = Qu^{R'}(\hat{x})$ for $\hat{x}$ of dimension $n-1$ and a suitable relation R': $\sum \hat{k}'_\nu \hat{k}''_\nu + l$. By the definition of $Qu^{R'}$ in [3], a cocycle representing $Qu^{R'}(\hat{x})$ is given by $\theta(x) + \sum k'_\nu(z_\nu)$, where $\Delta\theta = \sum k'_\nu k''_\nu + l$ and $\delta z = k''_\nu(x)$. The desired equality is obtained by the construction of a particular θ using cochain operations involved in the definition of $\underset{\smile}{\Delta}$.

PROPOSITION 6.2. *The following operation* $\tilde{K}$ *is in* $Z(Q^{1,1})$

$$\begin{aligned} \tilde{K}(u,v) = {} & K(u,v) + d(a;\, uv, vu, \delta u \cup_1 v, u \cup_1 \delta v)\, + \\ & + \sum d(c'_\nu;\, c_\nu(\delta u)\, c_\nu(v), c_\nu(u)\, c_\nu(\delta v)) + g(u)\, \varkappa(c'_\nu)(c_\nu u, c_\nu \delta v)\, + \\ & + \sum c'_i u \cup_1 c''_i v + \sum c'_i v \cup_1 c''_i u + T(u,v) + T(v,u) + F(u,v)\,, \end{aligned}$$

where T *satisfies* (2.12) *relative to* (5.1) *and* F *is given in Lemma* 3.5.

PROOF. ∇F is given explicitly in (3.5). A straightforward computation gives the assertion.

From the proposition now follows that ε is defined on $\tilde{K}$. To get some information on $\varepsilon\tilde{K}$, we prove in analogy with Proposition 5.3

PROPOSITION 6.3. *The operation* $\tilde{K}$ *is symmetric in the sense that*

$$\tilde{K}(x,y) \sim \tilde{K}(y,x)$$

for arbitrary cocycles x,y.

PROOF. By (3.6), $F(x,y) \sim F(y,x)$. Also,

$$a(x \cup_1 y) + a(y \cup_1 x) = \delta[a(x \cup_2 y) + d(a;\, x \cup_1 y, \delta(x \cup_2 y)]\,.$$

Similarly for the other terms in $K(x,y)$. Also,

$$d(a;\, xy, yx) \sim d(a;\, yx, xy)$$

by Lemma 3.6. The remaining terms cancel.

By Proposition 3.16, the symmetry of $\tilde{K}$ gives the existence of cochain operations γ_i', γ_i'', and $\tau \in Z(\mathcal{O}^1)$ such that

$$\varepsilon(\tilde{K})(\hat{x},\hat{y}) = \{\sum \gamma_i'(x)\, \gamma_i''(y) + \tau(x)\, \tau(y) + \sum \gamma_i'(y)\, \gamma_i''(x)\} .$$

We may use the same trick as in Section 5: Choosing a suitable T in the definition of $\tilde{K}$, we get

$$(6.5) \qquad \varepsilon(\tilde{K})(\hat{x},\hat{y}) = \{\tau(x)\, \tau(y)\} .$$

By Theorem 2.1 we choose an operation $B \in Q^{1,1}$ with

$$(6.6) \qquad \nabla B(u,v) = \tilde{K}(u,v) + \tau(u)\, \tau(v) .$$

If in this equation we put $u=v=x$, where x is an n-dimensional cocycle, we get by (6.4)

$$0 \sim (n+1)\varkappa(a)(x^2) + \tau(x)^2 .$$

Hence there is a relation r' of the form

$$r' = (n+1)\varkappa(a) sq^n + sq^{n+\bar{m}}\tau + e'$$

with excess $(e') \geqq n+1$ (in the sense of Section 2).

We shall use B to construct an operation θ with $\Delta\theta = r + r'$. First, we construct a cochain operation θ' defined on cochains of dimension $\leqq n-1$ and on cocycles of dimension n. We put

$$(6.7) \qquad \begin{aligned}
&\text{for } \deg u \leqq n-4: && \theta'(u) = 0 ,\\
&\text{for } \deg u = n-3: && \theta'(u) = \sum c_i'(u)\, c_i''(\delta u) ,\\
&\text{for } \deg u = n-2: && \theta'(u) = (n+1)\varkappa(a)(u\,\delta u) + T(u,\delta u) + \\
& && \quad + \sum c_i'(u) \cup_1 c''(\delta u) + \sum c_i''(u)\, c_i'(u) ,\\
&\text{for } \deg u = n-1: && \theta'(u) = B(u,\delta u) + T(u,u) + d(a;\, u^2 + u \cup_1 \delta u, u^2) + \\
& && \quad + \sum c_i'(u) \cup_1 c_i''(u) + V(u\,\delta u,0,0,u\,\delta u,0) + \\
& && \quad + C(\delta u\, u, u\,\delta u, \delta u \cup_1 \delta u) + C(u\,\delta u, u\,\delta u, 0) + \\
& && \quad + n[\varkappa^2(a)(u\,\delta u) + \varkappa(a) sq^{n-1} u] ,\\
&\left.\begin{array}{l}\text{for } \deg u = n\\ \text{and } \delta u = 0\end{array}\right\} : && \theta'(u) = B(u,u) + \sum \zeta' u \cup_1 \zeta'' u ,
\end{aligned}$$

V as in Lemma 3.2, C as in Lemma 3.6. By a lengthy but straightforward computation one may verify

PROPOSITION 6.4. *With the θ' defined in* (6.7),

$$\Delta\theta' = r + r'$$

whenever this makes sense (*i.e., for cochains of dimension* $<n$ *and for* n*-cocycles*).

By Lemma 3.4 in [3], there is an extension $\theta \in \mathcal{O}^1$ of θ' with

$$\Delta\theta = r + r' \tag{6.8}$$

such that

$$\theta(u) - \theta'(u) = \begin{cases} 0 & \text{for } \deg u < n-1\,, \\ \sum \alpha_1(u)\alpha_2(\delta u)\ldots\alpha_r(\delta u) & \text{for } \deg u = n-1\,, \end{cases}$$

where $\alpha_i \in Z(\mathcal{O}^1)$, and $r > 1$. In particular, $\theta(x) = \theta'(x)$ for $n-1$ cocycles x.

THEOREM 6.5. *Let A be a relation for $\hat{a} \in \mathcal{A}$*

$$A = \begin{cases} \sum \hat{\alpha}_\nu \hat{a}_\nu + \hat{e} \quad (\text{excess } \hat{e} \geqq 2n-1)\,, \\ [\hat{c}_i' \otimes \hat{c}_i'' + \hat{b} \otimes \hat{b}] + [\hat{c}_i'' \otimes \hat{c}_i']\,, \end{cases}$$

and let R be a relation

$$R = \sum \hat{\alpha}_\nu [\hat{a}_\nu Sq^{n-1}] + Sq^{m+n-1}\,\hat{b} + \hat{e}_0 \quad (\text{excess } \hat{e}_0 \geqq \hat{n})\,,$$

where n is a fixed integer and $m = \frac{1}{2} \deg \hat{a}$ (see Convention 6.1). (A relation of this form is given in (6.1)). Then there are operations $\overset{A}{\smile}$ and Qu^R with

$$\hat{x} \overset{A}{\smile} \hat{x} = Qu^R(\hat{x})$$

modulo the total indeterminacy, for all $n-1$-dimensional cohomology classes $\hat{x}$ for which $\hat{x} \overset{A}{\smile} \hat{x}$ is defined.

PROOF. Since $\hat{x} \overset{A}{\smile} \hat{x}$ is defined, there exist cochains w_ν, r_i, and s so that

$$\begin{aligned} \delta w_\nu &= a_\nu(xx) = a_\nu sq^{n-1}x\,, \\ \delta r_i &= c_i'(x)\,, \\ \delta r &= bx\,. \end{aligned}$$

Let

$$e_0 = \sum c_\nu' sq^{n+\deg c_\nu} c_\nu + e + r'\,.$$

Then e_0 is a representative of $\hat{e}_0$ of excess $\geqq n$. By (6.8)

$$\Delta\theta = \sum (\alpha_\nu a_\nu + e) sq^{n-1} + sq^{m+n-1}\, b + e_0\,,$$

and therefore θ defines a secondary operation Qu^R associated with R. A cocycle representing $Qu^R(\hat{x})$ is given by

$$\theta(x) + \sum \alpha_\nu(w_\nu) + sq^{m+n-1} r\,. \tag{6.9}$$

The operation T chosen in connection with (6.5) defines a $\overset{A}{\smile}$-product, and $\hat{x} \overset{A}{\smile} \hat{x}$ is then represented by

$$T(x,x) + \sum \alpha_\nu(w_\nu) + \sum r_i c_i''(x) + \sum c_i''(x) r_i + \sum rb(x)\,. \tag{6.10}$$

The difference between the cocycles (6.9) and (6.10) is by (6.7) seen to be

$$\delta\sum r_i \cup_1 c_i'' x + d(a; x^2, x^2) + n\varkappa(a) sq^{n-1} x \sim (n+1)\varkappa(a)(x^2) .$$

Altering Qu^R by the primary operation $(n+1)\varkappa(\hat{a}) Sq^{n-1}$, we get the theorem.

7. Coboundary and suspension.

In Definition 4.4 we introduced a functor S from the category of CSS-complexes to the category of sets; $S(X)$ consists of all tuples $[x,y \mid \{w_\nu\}, \{r_i\}, \{r_j\}]$ satisfying the conditions stated in the definition. These were called systems for x,y. Here we need to consider two kinds of generalized systems.

Let A and β be as in Section 4. Then, by $S_1(X)$, we denote the set of tuples

(7.1) $$s_1 = [x,y \mid \{w_\nu, q_\nu\}, \{r_i, t_i\}, \{r_j\}]$$

satisfying

(7.2) $$\begin{aligned} \delta y &= 0 , \\ \delta w_\nu &= a_\nu(xy) + q_\nu , \\ \delta r_i &= b_i(x) + t_i , \\ \delta r_j &= c_j(y) . \end{aligned}$$

By $S_2(X)$ we denote the set of the tuples

(7.3) $$s_2 = [x,y \mid \{w_\nu, q_\nu\}, \{r_i\}, \{r_j, t_j\}]$$

satisfying

$$\begin{aligned} \delta x &= 0 , \\ \delta w_\nu &= a_\nu(xy) + q_\nu , \\ \delta r_i &= b_i(x) , \\ \delta r_j &= c_j(y) + t_j . \end{aligned}$$

We define $\delta_i \colon S_i \to S_i$, $i=1,2$, by

$$\begin{aligned} \delta_1 s_1 &= [\delta x, y \mid \{q_\nu, 0\}, \{t_i, 0\}, \{r_j\}] , \\ \delta_2 s_2 &= [x, \delta y \mid \{q_\nu, 0\}, \{r_i\}, \{t_j, 0\}] . \end{aligned}$$

Note that the set $S(X)$ of systems from Definition 4.4 can be considered as a subset of both $S_1(X)$ and $S_2(X)$.

Definition 7.1. Let $s_i \in S_i(X)$, $i=1,2$, be as in (7.1) or (7.3). We put

$$\begin{aligned} c(s_i) = T(x,y) &+ \sum \alpha_\nu(w_\nu) + \sum r_i c_i(y) + \sum b_j(x) r_j + \\ &+ \sum d(\alpha_\nu; a_\nu(xy), q_\nu) + \sum \varkappa(\alpha_\nu)(q_\nu) . \end{aligned}$$

This is an extension of the natural transformation c defined in Definition 4.4. An easy computation yields

PROPOSITION 7.2. *Let* $s_i \in S_i(X)$ *be as in* (7.1) *or* (7.3). *Then*

$$\delta c(s_i) + c(\delta_i s_i) = e(xy) + g(x)\varkappa(a)(x\delta y) .$$

We remark that in case $s_i \in S$ (Definition 4.4), then the proposition gives $\delta c(s_i) = e(xy)$ as already seen in Section 4.

Let Y be a subcomplex of X and call the injection $i\colon Y \to X$. We shall consider

$$\delta^*\colon \; H^m(Y) \to H^{m+1}(X,Y) .$$

THEOREM 7.3. *Modulo the total indeterminacy*

$$\hat{x}\overset{A}{\smile}\delta^*\hat{y} = \delta^*(i^*\hat{x}\overset{A}{\smile}\hat{y}) + g(\hat{x})\varkappa(\hat{a})(\hat{x}\delta^*\hat{y}) ,$$

provided both sides are defined. Similarly for the equation

$$\delta^*\hat{x}\overset{A}{\smile}\hat{y} = \delta^*(\hat{x}\overset{A}{\smile}i^*\hat{y}) .$$

PROOF. If both sides are defined in the first equation, there is a generalized system s in $S_2(X)$,

$$s = [x,y \mid \{w_\nu,q_\nu\}, \{r_i\}, \{r_j,t_j\}] ,$$

$q_\nu, t_j \in C(X,Y)$ such that $\hat{x}\overset{A}{\smile}\delta^*\hat{y}$ is represented by $c(\delta_2 s)$ and $i^*\hat{x}\overset{A}{\smile}\hat{y}$ is represented by $c(i^\# s) = i^\# c(s)$. Since $e(xy) = 0$ for reasons of degree, Proposition 7.3 immediately yields the first equation. The proof of the second is similar, using a generalized system in $S_1(X)$. We remark that the lack of symmetry is to be found in Lemma 2.2.

We next prove an analogue of the fact that cup products in a suspension SX vanish.

THEOREM 7.4. *Let* $\hat{x}\overset{A}{\smile}\hat{y} \subseteq H^*(SX)$ *be defined. Then it is zero modulo indeterminacy.*

PROOF. In $H^*(SX)$, $\hat{x}\hat{y} = 0$. So, if $\hat{x}\overset{A'}{\smile}\hat{y}$ is defined, then so is $\hat{x}\overset{A}{\smile}\hat{y}$, where A' is the „degenerate" relation for $\hat{a}$ obtained from A by replacing $\hat{a} = \sum \hat{\alpha}_\nu \hat{a}_\nu + \hat{e}$ by $\hat{a} = \hat{a}\cdot 1 + 0$. Furthermore, $\overset{A'}{\smile}$ has less indeterminacy than $\overset{A}{\smile}$, so it suffices to prove the theorem for $\overset{A'}{\smile}$. Let $\delta w = xy$, $\delta r_i = b_i(x)$, $\delta r_j = c_j(y)$. Let CX denote the cone on X with the inclusion map $i\colon X \to CX$ and pinching map $p\colon CX \to SX$. The desuspension isomorphism

$$\sigma^{-1}\colon \; H^{n+1}(SX) \to H^n(X)$$

may be defined by means of the additive relation $i^\# \delta^{-1} p^\#$ on cochain stage. Since CX is acyclic, there is a cochain u with $\delta u = p^\# x$. Obviously, $b_i(u) + p^\# r_i$ is a cocycle so that we may find $\overline{r_i} \in C^\cdot(CX)$ with $d\overline{r_i} = b_i(u) + p^\# r_i$. Similarly, we can find $\overline{w} \in C^\cdot(CX)$ with

$$\delta\bar{w} = up^{\#}y + p^{\#}w \,. \tag{7.6}$$

Modifying w with a cocycle in $\sigma\{i^{\#}\bar{w}\}$, it is easily seen that a $\bar{w}$ satisfying (7.6) may be chosen with the further property $i^{\#}\bar{w}=0$. Now, consider the system $s \in S_1(CX)$

$$s = [u,p^{\#}y \mid \{\bar{w},p^{\#}w\},\{\bar{r}_i,p^{\#}r_i\},\{p^{\#}r_j\}] \,.$$

The operation $\hat{x}\overset{A}{\circlearrowleft}\hat{y}$ is represented by $c(s_0)$, where

$$s_0 = [x,y \mid \{w\},\{r_i\},\{r_j\}] \,.$$

Using Proposition 7.2, we get

$$p^{\#}c(s_0) = c(\delta_1 s) = \delta c(s)$$

so that $\sigma^{-1}\{c(s_0)\}$ is represented by $i^{\#}c(s)=c(i^{\#}s)$. But this is easily seen to be zero.

The last theorem in this section deals with the spectral sequence for a fibration p

$$F \xrightarrow{\;i\;} E \xrightarrow{\;p\;} B$$

with $H^*(E,*)=0$. The content of the theorem is indicated in the diagram below.

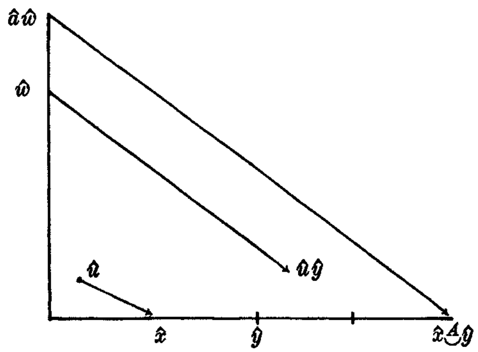
THEOREM 7.5. *Let $\hat{x},\hat{y} \in H^*(B)$ and let $\hat{x}\hat{y}=0$. Let $\hat{u} \in E_r(p)$ with $d_r(\hat{u})=\{\hat{x}\}$. Then there is a class $\hat{w} \in H^*(F)$ with*

$$d_t\{\hat{w}\} = \{\hat{u}\{\hat{y}\}\}, \qquad t = g(xy)-r \,.$$

Further, $\hat{a}(\hat{w}) \in H^*(F)$ is transgressive, and modulo total indeterminacy

$$\tau(\hat{a}(\hat{w})) = \hat{x}\overset{A}{\circlearrowleft}\hat{y} \,,$$

where A is the relation $\hat{a}=\hat{a}\cdot 1+0$, $\psi\hat{a}=\sum \hat{b}_i\otimes\hat{c}_i+\sum \hat{b}_j\otimes\hat{c}_j$.



PROOF. Let $xy = \delta b$ and $p^{\#}x = \delta u$, where u represents $\hat{u} \in E_r$. Since $C^{\cdot}(LX)$ is acyclic, there is a cochain w with $\delta w = up^{\#}y + p^{\#}b$. Let $\hat{w} \in H^*(F)$ be the class represented by $i^{\#}w$. Obviously $d_t\{\hat{w}\} = \{\hat{u}\cdot\{\hat{y}\}\}$, $t = g(xy) - r$. There are systems $s_0 \in S(B)$ and $s \in S_1(E)$ of the form

$$s_0 = [x,y \mid \{b\},\{r_i\},\{r_j\}] ,$$
$$s = [u,p^{\#}y \mid \{w,p^{\#}b\},\{\overline{r_i},p^{\#}r_i\},\{p^{\#}r_j\}] .$$

Then $\hat{a}(\hat{w})$ is represented by $c(i^{\#}s) = i^{\#}c(s)$. Since

$$\delta c(s) = c(\delta_1 s) = p^{\#}c(s_0) ,$$

we get the theorem.

8. Two relations between secondary products.

Let

$$(8.1) \qquad A: \begin{cases} \hat{a} = \sum \hat{\alpha}_\nu \hat{a}_\nu \\ \psi(\hat{a}) = \sum_{I'} \hat{a}_i' \otimes \hat{a}_i'' + \sum_{I''} \hat{a}_i' \otimes \hat{a}_i'', \quad I' \cup I'' = I , \end{cases}$$

$$(8.2) \qquad B: \begin{cases} \hat{b} = \sum \hat{\beta}_\mu \hat{b}_\mu \\ \psi(\hat{b}) = \sum_{J'} \hat{b}_j' \otimes \hat{b}_j'' + \sum_{J''} \hat{b}_j' \otimes \hat{b}_j'', \quad J' \cup J'' = J , \end{cases}$$

be relations for $\hat{a},\hat{b} \in \mathscr{A}$. The Cartan formula for $\hat{b}\hat{a}$ may be expressed

$$(8.3) \qquad \psi(\hat{b}\hat{a}) = \sum \hat{b}_j'\hat{a}_i' \otimes \hat{b}_j''\hat{a}_i'', \quad (j,i) \in J \times I .$$

In Lemma 3.7 we have given a construction of $d(ba)$. Using this and operations T_b,T_a satisfying 2.12, we shall here construct an operation $T_{ba} \in Q^{1,1}$.

LEMMA 8.1. *The following formula defines an operation T_{ba} satisfying* 2.12 *with respect to ba and* (8.3)

$$T_{ba}(x,y) = bT_a(x,y) + \sum_I T_b(a_i'x,a_i''y) + \sum R(x,y) ,$$

where $R(x,y)$, $R = XI,XII,\ldots,XV$, are the operations defined in Lemmas 3.11–3.15. *In XIII the operation A is replaced by T_a.*

PROOF. The proof is straightforward using Lemmas 3.11–3.15 and Lemma 3.7.

If x and y are cocycles, the operation T_{ba} simplifies to

$$(8.4) \qquad \begin{aligned} T_{ba}(x,y) = bT_a(x,y) &+ \sum_I T_b\big(a_i'(x),a_i''(y)\big) + \\ &+ d\big(b; a(xy),\ldots,a_i'xa_i''y,\ldots\big) + \\ &+ \sum_I \deg(a_i')g(y)\varkappa(b)(a_i'xa_i''y) . \end{aligned}$$

Consider the relation

$$(8.5)\qquad BA: \begin{cases} \hat{b}\hat{a} = \sum(\hat{b}\hat{\alpha}_\nu)\hat{a}_\nu\,, \\ \psi(\hat{b}\hat{a}) = \sum_{J\times I'}\hat{b}'_j\hat{a}'_i\otimes\hat{b}''_j\hat{a}''_i + \sum_{J\times I''}\hat{b}'_j\hat{a}'_i\otimes\hat{b}''_j\hat{a}''_i \end{cases}$$

for $\hat{b}\hat{a}$. For this relation we have

THEOREM 8.2. *Let $\hat{x},\hat{y}$ be cohomology classes. If $\hat{x}\overset{A}{\smile}\hat{y}$ is defined with A as in* (8.1), *then with BA as in* (8.5), *$\hat{x}\overset{BA}{\smile}\hat{y}$ is defined and*

$$b(\hat{x}\overset{A}{\smile}\hat{y}) = \hat{x}\overset{BA}{\smile}\hat{y}\,.$$

PROOF. Let

$$\theta(u) = d(b;\ldots,\alpha_\nu a_\nu(u),\ldots);$$

then

$$T(x,y) = T_{ba}(x,y) + \theta(xy) + d(\theta;\delta xy,x\delta y) + g(x)\varkappa(\theta)(x\delta y)\,,$$

where T_{ba} is as in Lemma 8.1 and $d(\theta)$ as in (2.8) satisfies

$$\nabla T(x,y) = \sum b\alpha_\nu a_\nu(xy) + \sum b'_j a'_i(x)\, b''_j a''_i(y) + \\ + d(\sum b\alpha_\nu a_\nu;\, \delta xy, x\delta y) + g(x)\varkappa(\sum b\alpha_\nu a_\nu)(x\delta y)\,.$$

Hence a cocycle representing $\hat{x}\overset{BA}{\smile}\hat{y}$ may be constructed using this T. The theorem now easily follows.

Now, let

$$(8.6)\qquad BA: \begin{cases} \sum\hat{\beta}_\mu(\hat{b}_\mu\hat{a}) \\ \psi(\hat{b}\hat{a}) = \sum_{J'\times I}\hat{b}'_j\hat{a}'_i\otimes\hat{b}''_j\hat{a}''_i + \sum_{J''\times I}\hat{b}'_j\hat{a}'_i\otimes\hat{b}''_j\hat{a}''_i\,. \end{cases}$$

Note the difference between (8.5) and (8.6).

THEOREM 8.3. *Let $\hat{x},\hat{y}$ be cohomology classes with both $\hat{x}\overset{BA}{\smile}\hat{y}$ and $\hat{a}'_i\hat{x}\overset{B}{\smile}\hat{a}''_i\hat{y}$ defined, B and BA as in* (8.2) *and* (8.6) *and $\hat{a}'_i,\hat{a}''_i$ as in* (8.1). *Then modulo the total indeterminacy*

$$\hat{x}\overset{BA}{\smile}\hat{y} = \sum_I \hat{a}'_i\hat{x}\overset{B}{\smile}\hat{a}''_i\hat{y} + \sum_I \deg(\hat{a}'_i)g(\hat{y})\varkappa(\hat{b})\left(\hat{a}'_i(\hat{x})\,\hat{a}''_i(\hat{y})\right).$$

PROOF. By assumption there are systems

$$s_i = [a'_i x, a''_i y \mid \{w_{\mu,i}\},\{r'_{j,i}\},\{r''_{j,i}\}]\,,$$
$$s = [x,y \mid \{\overline{w}_\mu\},\{r'_{j,i}\},\{r''_{j,i}\}]$$

with

$$\overline{w}_\mu = \sum_I w_{\mu,i} + b_\mu T_a(x,y) + d(b_\mu; a(xy),\ldots,a'_i(x)a''_i(y),\ldots)\,.$$

Then, using T_{ba} as in Lemma 8.1, a representative for

$$\hat{x}\overset{BA}{\smile}\hat{y} + \sum_I \hat{a}'_i(\hat{x})\overset{B}{\smile}\hat{a}''_i(\hat{y}) + \sum \deg(\hat{a}'_i)\, g(\hat{y})\varkappa(\hat{b})\left(\hat{a}'_i(\hat{x})\,\hat{a}''_i(\hat{y})\right)$$

is given by

(8.7)

$$
\begin{aligned}
c(s) &+ \sum c(s_i) + \sum \deg(a_i')g(y)\varkappa(b)\big(a_i'(x)\, a_i''(y)\big) \\
&= \sum\beta_\mu(\overline{w}_\mu) + \sum\beta_\mu(w_{\mu,i}) + \sum\beta_\mu b_\mu T_a(x,y) + \\
&\quad + \sum d(\beta_\mu b_\mu;\, a(xy),\ldots,a_i'(x)\, a_i''(y),\ldots) \\
&\sim \sum d(\beta_\mu b_\mu;\, a(xy),\ldots,a_i'x\, a_i''y,\ldots) + \\
&\quad + \sum d\big(\beta_\mu;\, b_\mu a(xy),\ldots,b_\mu(a_i'x\, a_i''y),\ldots,b_\mu\big(a(xy)+\sum a_i'x\, a_i''y\big)\big) + \\
&\quad + \sum \beta_\mu d(b_\mu;\, a(xy),\ldots,a_i'x\, a_i''y,\ldots) + \\
&\quad + \sum d\big(\beta_\mu;\, \delta d(b_\mu;\, a(xy),\ldots,a_i'x\, a_i''y,\ldots), \\
&\qquad\qquad d(b_\mu;\, \delta a(xy),\ldots,\delta(a_i'x\, a_i''y),\ldots)\big)\,.
\end{aligned}
$$

Replacing in this expression $a(xy)$ by u and $a_i'xa_i''y$ by u_i, we get an operation in $\mathcal{O}^t$, where t is one plus the cardinal of I. This operation is easily seen to be in $Z(\mathcal{O}^t)$. By ε it is mapped into

$$\sum_\mu \varkappa(\beta_\mu)\, b_\mu(u) + \sum_{\mu,\,i}\varkappa(\beta_\mu)\, b_\mu(u_i)\,.$$

Hence (8.7) is cohomologous to

$$\sum\varkappa(\beta_\mu)\, b_\mu a(xy) \;+\; \sum\varkappa(\beta_\mu)\, b_\mu(a_i'x\, a_i''y) \;\sim\; 0\,.$$

This proves the theorem.

9. Peterson-Stein formulas.

The two theorems in this section are analogous to the theorems 2.3 and 2.4 in [6]. We shall need the following generalization of the well-known functionalized cup product operation. Let $f\colon Y \to X$ be a map of CSS-complexes, and let $\hat{\gamma} \in \mathscr{A}$. Let $\hat{x}$ and $\hat{y}$ be cohomology classes on X with the properties

$$f^*(\hat{x}) = 0\,,$$
$$\hat{\gamma}(\hat{x}\hat{y}) = 0\,.$$

Then one may define a set $\hat{\gamma}_f(\hat{x},\hat{y})$ of cohomology classes in $H^*(Y)$, namely the set represented by cocycles of the form

$$\gamma(u\, f^{\#}y) + f^{\#}(w)\,,$$

where $x \in \hat{x}$, $y \in \hat{y}$, $\delta u = f^{\#}x$, $\delta w = \gamma(xy)$. One easily sees that $\hat{\gamma}_f(\hat{x}\hat{y})$ is a coset of the subgroup $\hat{\gamma}\big(H^*(Y)\,f^*(\hat{y})\big) + f^*\big(H^*(X)\big)$ in $H^*(Y)$. Similarly, if $f^*(\hat{y}) = 0$ and $\hat{\gamma}(\hat{x}\hat{y}) = 0$, one may define a set $\hat{\gamma}(\hat{x},\hat{y})_f$ of cohomology classes in $H^*(Y)$. If $\hat{\gamma} = 1 \in \mathscr{A}$, then $\hat{\gamma}_f(\hat{x},\hat{y})$ is the usual left-functionalized cup product, which we denote $\hat{x}_f\cup\hat{y}$. Similarly, if $\hat{\gamma} = 1$, $\hat{\gamma}(\hat{x},\hat{y})_f$ is the right-functionalized cup product $\hat{x}\cup_f\hat{y}$. Finally, if $\hat{y} = 1 \in H^0(X)$, then $\hat{\gamma}_f(\hat{x},\hat{y})$ is the usual primary functionalized operation $\hat{\gamma}_f(\hat{x})$.

Let A be the following relation for $\hat{a}$

$$(9.1)\qquad A:\begin{cases}\hat{a} = \sum\hat{\alpha}_\nu\hat{a}_\nu + \hat{e}, & \text{excess}\,\hat{e}=N+1\,,\\ \psi\hat{a} = \sum\hat{b}_i\otimes\hat{c}_i + \sum\hat{b}_j\otimes\hat{c}_j, & i\in I,\ j\in J\,.\end{cases}$$

THEOREM 9.1. *Let $f\colon Y\to X$ be a map of CSS-complexes and let $\hat{x},\hat{y}\in H^*(X)$. Let A be as in* (9.1). *Then the following equation holds modulo indeterminacy, provided every term is defined*

$$(9.2)\quad f^*(\hat{x}\overset{A}{\circlearrowleft}\hat{y}) = \sum\hat{\alpha}_\nu\big(\hat{a}_\nu(\hat{x},\hat{y})_f\big) + \sum_J f^*\big(\hat{b}_j(\hat{x})\big)\cdot(\hat{c}_j)_f(\hat{y}) + g(x)f^*\varkappa(\hat{a})\,(\hat{x}\,\hat{y})\,.$$

Similarly, for the equation

$$(9.3)\qquad f^*(\hat{x}\overset{A}{\circlearrowleft}\hat{y}) = \sum\hat{\alpha}_\nu\big(\hat{a}_{\nu f}(\hat{x},\hat{y})\big) + \sum_I(\hat{b}_i)_f(\hat{x})\,f^*\big(\hat{c}_i(\hat{y})\big)\,.$$

PROOF. We shall prove (9.2) only. The proof of (9.3) is similar. By assumption, there is a system $s\in S(X)$

$$s = [x,y\mid\{w_\nu\},\{r_i\},\{r_j\}]\,,$$

and a cochain $z\in C^\cdot(Y)$ with $\delta z=f^\# y$. Then $\hat{x}\overset{A}{\circlearrowleft}\hat{y}$ is represented by $c(s)$, $\hat{a}_\nu(\hat{x},\hat{y})_f$ is represented by $a_\nu\big(f^\#(x)\,z\big)+f^\# w_\nu$, and $(\hat{c}_j)_f(\hat{y})$ is represented by $c_j(z)+f^\# r_j$. Using these representatives, (2.12) easily gives (9.2).

THEOREM 9.2. *Let $f\colon Y\to X$ be a map of CSS-complexes and let $\hat{x},\hat{y}\in H^*(X)$. Let A be as in* (9.1). *Then the following equation holds modulo indeterminacy, provided every term is defined*

$$f^*\hat{x}\overset{A}{\circlearrowleft}f^*\hat{y} = \sum\hat{\alpha}_{\nu f}\big(\hat{a}_\nu(\hat{x}\hat{y})\big) + \sum_I\hat{b}_i\hat{x}_f\cup\hat{c}_i\hat{y} + \sum_J\hat{b}_j\hat{x}\cup_f\hat{c}_j\hat{y}\,.$$

PROOF. By assumption, there is a system $s\in S(Y)$

$$s = [f^\# x,f^\# y\mid\{w_\nu\},\{r_i\},\{r_j\}]\,.$$

Also, there are cochains $u_k\in C^\cdot(X)$ with

$$\delta u_k = b_k(x)\,c_k(y),\quad k\in I\cup J\,.$$

Finally, there are cochains $t_\nu\in C^\cdot(X)$ with

$$\delta t_\nu = \alpha_\nu a_\nu(xy)\,.$$

For t_1 we may use

$$t_1 = \sum_{\nu\neq 1}t_\nu + \sum_{I\cup J}u_k + T(x,y)\,.$$

Using these cochains to construct representatives, the theorem easily follows.

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