Some matrices with nilpotent entries, and their determinants.

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The present note is really a Section in a forthcoming treatise [4] on differential forms in the context of Synthetic Differential Geometry (elaborating on [2], [3], [1]); but since the methods of this Section fall entirely within elementary linear algebra over a commutative ring R, we believe that it might be of more general interest, and worthwile a separate publication.

The base ring R over which we work is implicitly supposed to have a rich supply of nilpotent elements, in particular elements $d \in R$ with $d^2 = 0$, since otherwise the theory collapses to the "theory of 0-matrices".

For the applications which motivated the present research, R is the number line in a model of Synthetic Differential Geometry (SDG), but no assumptions in this direction are needed for what we develop here. The only extra assumption on R that we do make, is that "2 is cancellable in R", meaning that for all $x \in R$, $x + x = 0$ implies $x = 0$. This will be a standing assumption.

1 Matrices

We consider a commutative ring R. We use the word "vector space" as synonymous with "R-module", and "linear" means "R-linear". A vector space is called *finite dimensional* if it is linearly isomorphic to some R^n.

We begin by describing some equationally defined subsets of R, of R^n (=the vector space of n-dimensional coordinate vectors), and of $R^{m \cdot n}$ (=the vector space of $m \times n$-matrices over R).

The fundamental one is $D \subseteq R$,

$$D := \{x \in R \mid x^2 = 0\}.$$

More generally, for n a positive integer, we let $D(n) \subseteq R^n$ be the following set of n-dimensional coordinate vectors $\underline{x} = (x_1, \ldots, x_n)$:

$$D(n) := \{(x_1, \ldots, x_n) \in R^n \mid x_j x_{j'} = 0 \text{ for all } j, j' = 1, \ldots, n\},$$

in particular $(j = j')$, $x_j^2 = 0$, so that $D(n) \subseteq D^n \subseteq R^n$. The inclusion $D(n) \subseteq D^n$ will usually be a proper inclusion, except for $n = 1$. Note also that $D = D(1)$. Note that if $\underline{x}$ is in $D(m)$, then so is $\lambda \cdot \underline{x}$ for any $\lambda \in R$, in particular, $-\underline{x}$ is in $D(m)$ if $\underline{x}$ is. In general, $D(n)$ is not stable under addition.

The notation for D and $D(n)$ is the standard one of SDG. The following set $\widetilde{D}(m,n)$ was first described in [2] §I.16 and §I.18, with the aim of constructing a combinatorial notion of differential m-form.

The subset $\widetilde{D}(m,n) \subseteq R^{m \cdot n}$ is the following set of $m \times n$ matrices $[x_{ij}]$ $(m, n \geq 2)$:

$$\widetilde{D}(m,n) := \{[x_{ij}] \in R^{m \cdot n} \mid x_{ij} x_{i'j'} + x_{i'j} x_{ij'} = 0 \\ \text{for all } i, i' = 1, \ldots m \text{ and } j, j' = 1, \ldots, n\}.$$

– We note that the equations defining $\widetilde{D}(m,n)$ are row-column symmetric; equivalently, the transpose of a matrix in $\widetilde{D}(m,n)$ belongs to $\widetilde{D}(n,m)$. Also clearly any $p \times q$ submatrix of a matrix in $\widetilde{D}(m,n)$ belongs to $\widetilde{D}(p,q)$. For if the defining equations

$$x_{ij} x_{i'j'} + x_{i'j} x_{ij'} = 0 \tag{1}$$

hold for all indices i, i', j, j', they hold for any subset of them. And since each of the equations in (1) only involve (at most) four indices i, i', j, j', we see that for and $m \times n$ matrix to belong to $\widetilde{D}(m,n)$ it suffices that all of its 2×2 submatrices belong to $\widetilde{D}(2,2)$.

If $[x_{ij}] \in \widetilde{D}(m,n)$, we get in particular, by putting $i = i'$ in the defining equation (1), that for any $j, j' = 1, \ldots, n$

$$x_{ij} x_{ij'} + x_{ij} x_{ij'} = 0.$$

Since 2 is assumed cancellable in R, we deduce from this equation that $x_{ij} x_{ij'} = 0$, which is to say that the ith row of $[x_{ij}]$ belongs to $D(n)$. – Similarly, the jth column belongs to $D(m)$.

The equations (1) defining $\widetilde{D}(m,n)$ can be reformulated in terms of a certain bilinear map $\beta : R^n \times R^n \to R^{n^2}$, where $\beta(\underline{x},\underline{y})$ is the n^2-tuple whose jj' entry is $x_j y_{j'} + x_{j'} y_j$. Then an $m \times n$ matrix X ($m,n \geq 2$) is in $\widetilde{D}(m,n)$ if and only if $\beta(\underline{r}_i, \underline{r}_{i'}) = 0$ for all $i, i' = 1, \ldots, m$ ($\underline{r}_i$ denoting the ith row of X).

Note that this description is not row-column symmetric. But it has the advantage of making the following observation almost trivial:

Proposition 1 *If an $m \times n$ matrix X is in $\widetilde{D}(m,n)$, then the matrix X' formed by adjoining to X a row which is a linear combination of the rows of X, is in $\widetilde{D}(m+1,n)$.*

(There is of course a similar Proposition for columns.) Combining this Proposition with the observation that the rows of a matrix in $\widetilde{D}(p,n)$ are in $D(n)$, we therefore have

Proposition 2 *If X is a matrix in $\widetilde{D}(m,n)$, then any row in X is in $D(n)$, and also any linear combination of rows of X is in $D(n)$. – Similarly for columns.*

We have a "geometric" characterization of matrices in $\widetilde{D}(m,n)$, which depends on the following definition. We say that two vectors $\underline{x} = (x_1, \ldots, v_n)$ and $\underline{y} = (y_1, \ldots, y_n)$ in R^n are *neighbors* (more precisely, *first order neighbours*) if $\underline{x} - \underline{y} \in D(n)$. It is clearly a reflexive and symmetric relation. To say that $\underline{x} \in D(n)$ is thus equivalent to saying that $\underline{x}$ is a neighbour of the zero vector $0 \in R^n$. (This "neigbour"-relation is closely related to "the first neighbourhood of the diagonal" known for schemes in algebraic geometry, see e..g. [1]; this is a fundamental relation in SDG.)

The geometric characterization of $\widetilde{D}(m,n)$ is now the equivalence of 1) and 2) (or of 1) and 3)) in the following

Proposition 3 *Given an $m \times n$ matrix $X = [x_{ij}]$ ($m,n \geq 2$). Then the following three conditions are equivalent: 1) the matrix belongs to $\widetilde{D}(m,n)$; 2) each of its rows is a neigbour of $0 \in R^n$, and any two rows are mutual neighbours; 3) each of its columns is a neigbour of $0 \in R^m$, and any two columns are mutual neighbours. 2') any linear combination of the rows of X is in $D(n)$; 3') any linear combination of the columns of X is in $D(m)$.*

Proof. We have already observed (Proposition 2) that 1) implies 2'), which in turn trivially implies 2).

Conversely, assume the condition 2). Let $\underline{r}_i$ denote the ith row of the matrix. Then the condition 2) in particular says that the $\underline{r}_i$ and $\underline{r}_{i'}$ are neighbours; this means that for any pair of column indices j, j',

$$(\underline{r}_i - \underline{r}_{i'})_j \cdot (\underline{r}_i - \underline{r}_{i'})_{j'} = 0$$

where for a vector $\underline{x} \in R^n$, $\underline{x}_j$ denotes its jth coordinate. So $(x_{ij} - x_{i'j}) \cdot (x_{ij'} - x_{i'j'}) = 0$. Multiplying out, we get

$$x_{ij}x_{ij'} - x_{ij}x_{i'j'} - x_{i'j}x_{ij'} + x_{i'j}x_{i'j'} = 0. \qquad (2)$$

The first term vanishes because $\underline{r}_i \in D(n)$, and the last term vanishes because $\underline{r}_{i'} \in D(n)$. The two middle terms therefore vanish together, proving that the defining equations (1) for $\widetilde{D}(m,n)$ hold for the matrix. This proves equivalence of 1), 2), and 2'). The equivalence of 1), 3), and 3') now follows because of the row-column symmetry of the equations defining $\widetilde{D}(m,n)$.

Remark. The condition 2) in this Proposition was the motivation for the consideration of $\widetilde{D}(m,n)$, since the condition says that the m rows of the matrix, together with the zero row, form an *infinitesimal m-simplex*, i.e. an $m+1$-tuple of mutual neighbour points, in R^n; see [2] I.18 and [3]. (In the context of SDG, the theory of differential m-forms, in its combinatorial formulation, has for its basic input-quantities such infinitesimal m-simplices. The notion of infinitesimal m-simplex, and of affine combinations of the vertices of such, make invariant sense in any manifold N, due to some of the algebraic stability properties (in the spirit of Proposition 13 below) which $\widetilde{D}(m,n)$ enjoys.)

2 Stability properties

We begin with a "coordinate free" characterization of $D(n) \subseteq R^n$. Recall that we assume that 2 is cancellable in R. (Another characterization is given in Proposition 7 below.)

Proposition 4 *Let $\underline{x} \in R^n$. Then $\underline{x} \in D(n)$ if and only if for any linear $\alpha : R^n \to R$, $\alpha(x) \in D$.*

Proof. Assume $\underline{x} \in D(n)$. Let α have matrix $(a_1, \ldots, a_n)$, so that $\alpha(\underline{x}) = \sum_j a_j x_j$. Then

$$(\alpha(\underline{x}))^2 = (\sum_j a_j x_j)(\sum_{j'} a_{j'} x_{j'}),$$

which is a sum of n^2 terms $a_j x_j a_{j'} x_{j'} = a_j a_{j'} x_j x_{j'}$, each of which vanish because $x_j x_{j'} = 0$.

Conversely, assume $\alpha(\underline{x}) \in D$ for all linear $\alpha : R^n \to R$. Taking α to be proj_j (=projection onto the jth coordinate), the assumption gives that $x_j^2 = 0$. Then taking α to be $\text{proj}_j + \text{proj}_{j'}$, the assumption gives that $(x_j + x_{j'})^2 = 0$. In view of $x_j^2 = 0$ and $x_{j'}^2 = 0$, this says $2x_j x_{j'} = 0$, and since 2 is cancellable, $x_j x_{j'} = 0$.

The following is an immediate Corollary:

Proposition 5 *Let $f : R^n \to R^m$ be a linear map. Then f maps $D(n)$ into $D(m)$.*

Proof. Let $\underline{x} \in D(n)$. To see that $f(\underline{x}) \in D(m)$, it suffices, by Proposition 4, to see that for any linear functional $\alpha : R^m \to R$, we have $\alpha(f(\underline{x})) \in D$. But $\alpha \circ f$ is a linear functional on R^n, and thus takes $\underline{x}$ into D, by the Proposition 4 again.

The set of matrices $\widetilde{D}(m, n)$ was defined for $m, n \geq 2$ only, but it will make statements easier if we extend the definition by putting $\widetilde{D}(1, n) = D(n), \widetilde{D}(m, 1) = D(m)$ (here, of course, we identify R^p with the set of $1 \times p$ matrices, or $p \times 1$ matrices, as appropriate). By Proposition 2, the assertion that $p \times q$ submatrices of matrices in $\widetilde{D}(m, n)$ are in $\widetilde{D}(p, q)$ retains its validity, also for p or $q = 1$.

Proposition 6 *Let $X \in \widetilde{D}(m, n)$. Then for any $p \times m$ matrix P, $P \cdot X \in \widetilde{D}(p, n)$; and for any $n \times q$-matrix Q, $X \cdot Q \in \widetilde{D}(m, q)$.*

Proof. Because of the row-column symmetry of the property of being in $\widetilde{D}(k, l)$, it suffices to prove one of the two statements of the Proposition, say, the first. So consider the $p \times n$ matrix $P \cdot X$. Each of its rows is a linear combination of rows from X, hence is in $D(n)$, by Proposition 2. But also any linear combinatinon of rows in $P \cdot X$ is in $D(n)$, since a linear combination of linear combinations of some vectors is again a linear combination of these vectors. So the result follows from Proposition 3.

Here is an alternative characterization of $D(n) \subseteq R^n$:

Proposition 7 *Let* $\underline{x} \in R^n$*. Then the following conditions are equivalent:*

1) $\underline{x} \in D(n)$*;*

2) for any bilinear $\phi : R^n \times R^n \to R$*,* $\phi(\underline{x}, \underline{x}) = 0$*;*

3) for any symmetric bilinear $\psi : R^n \times R^n \to R$*,* $\psi(\underline{x}, \underline{x}) = 0$*.*

Proof. Any bilinear $\phi : R^n \times R^n \to R$ may be written $\psi + \phi_a$ with ψ bilinear symmetric and ϕ_a bilinear alternating, in particular, $\phi_a(\underline{y}, \underline{y}) = 0$ for any $\underline{y}$. Therefore, 2) and 3) are equivalent. Assume 2). For any pair of indices $i, i' = 1, \ldots, n$, we have the bilinear map

$$(\underline{x}, \underline{y}) \mapsto x_i \cdot y_{i'}. \tag{3}$$

The assumption 2) applied to this bilinear map and to the given $\underline{x}$ gives that $x_i \cdot x_{i'} = 0$ for all such pairs i, i', and this is the defining set of equations for $D(n)$, so $\underline{x} \in D(n)$, proving 1). Finally, 1) implies 2), since any bilinear $R^n \times R^n \to R$ is a linear combination of the special bilinear maps listed in (3).

3 Coordinate free aspects

Consider an arbitrary vector space (= R-module) V. We let $D_s(V) \subseteq V$ be the set defined by

$$\{v \in V \mid \exists \text{ linear } f : R^n \to V \text{ (for some } n) \text{ and } \exists \underline{x} \in D(n) \text{ with } f(\underline{x}) = v\}.$$

Also, we let $D_w(V) \subseteq V$ be the set defined by

$$\{v \in V \mid \forall \text{ linear } \phi : V \to R, \quad \phi(v) \in D\}. \tag{4}$$

From Proposition 4 follows immediately that $D_s(V) \subseteq D_w(V)$ (whence the subscripts s and w, for "strong" and "weak"). However,

Proposition 8 *If* V *is finite dimensional (i.e. if* $V \cong R^m$ *for some* m*),* $D_s(V) = D_w(V)$ *(denoted* $D(V)$*); for* $V = R^m$*,* $D(V) = D(m)$*.*

(An alternative characterization of $D(V)$, in terms of quadratic maps, may be obtained from a coordinate free version of Proposition 7 above.)

Proof. Since both constructions $D_s(-)$ and $D_w(-)$ are preserved under linear isomorphisms, it suffices to prove the result for $V = R^m$, i.e. to prove

$D(m) = D_s(R^m) = D_w(R^m)$. Clearly $D(m) \subseteq D_s(R^m)$; for, the witnessing f may be taken to be the identity map. Also $D_s(R^m) \subseteq D_w(R^m)$, as observed for a general V. And finally $D_w(R^m) \subseteq D(m)$ by Proposition 4.

Since $m \times n$ matrices may be identified with linear maps $R^n \to R^m$, we would like a characterization of the matrices in $\widetilde{D}(m,n)$ in terms of the vector space $Lin(R^n, R^m)$.

Let V and W be finite dimensional vector spaces ($V \cong R^n$, $W \cong R^m$, say).

Proposition 9 *For a linear map $F : V \to W$, the following conditions are equivalent:*

1) for all $v \in V$, $F(v) \in D(W)$.

2) for all $v \in V$ and all linear functionals $y : W \to R$, $y(F(v)) \in D$.

3) (if $V = R^n$, $W = R^m$): $F \in \widetilde{D}(m,n)$.

Proof. The equivalence of 1) and 2) follows from Proposition 8, applied to $F(v)$; 3) implies 1), by Proposition 6. Finally (assming $V = R^n$, $W = R^m$), to say that 1) holds is now equivalent to saying that the matrix product $F \cdot v$ is in $D(m)$ for any n-dimensional column vector v, or, equivalently, that any linear combination of the columns of F is in $D(m)$. This implies by Proposition 3 that $F \in \widetilde{D}(m,n)$.

For arbitrary finite dimensional vector spaces V and W, we may now define a subset $\widetilde{D}(V,W) \subseteq Lin(V,W)$ by saying that $F \in \widetilde{D}(V,W)$ if the equivalent conditions 1) and 2) in the Proposition hold. Then $\widetilde{D}(R^n, R^m) = \widetilde{D}(m,n)$ (note the unfortunate interchange of the order of the arguments.) Also, under the identification of V with $Lin(R,V)$, $D(V)$ gets identified with $\widetilde{D}(R,V)$.

Note that if V and W are finite dimensional, $Lin(V,W)$ is finite dimensional, and so $D(Lin(V,W)) \subseteq Lin(V,W)$ makes sense; it will in general be strictly smaller than $\widetilde{D}(V,W)$; in matrix terms, let $V = R^n, W = R^m$, and let $A = [a_{ij}] \in Lin(V,W)$. Then to say that $A \in D(Lin(V,W)$ is to say that $a_{ij}a_{i'j'} = 0$ for all i, i', j, j', which is a strictly stronger assertion than (1) (the fact that it is strictly stronger follows from the description of the "generic" matrix in $\widetilde{D}(m,n)$ given at the end of the next Section.)

Let us finally record the "ideal-" properties of Proposition 6 when expressed in coordinate free terms; V, W, as well as U, U', denote finite dimensional vector spaces.

Proposition 10 *Let $F \in \widetilde{D}(V, W)$. Then for any linear maps $P : W \to U$ and $Q : U' \to V$, $P \circ F \circ Q \in \widetilde{D}(U', U)$.*

4 Determinants

We now consider square matrices, say $n \times n$. They form the R-algebra $gl(n)$; the subset $\widetilde{D}(n, n) \subseteq gl(n)$ satisfies the ideal property, Proposition 6, (but it is not an ideal, since it is not stable under addition). Recall that $X \in \widetilde{D}(n, n)$ means that the equations (1) hold. Some of the determinant theory depends only on a smaller set of equations, namely on the equations

$$x_{ij}x_{i'j'} + x_{i'j}x_{ij'} = 0 \tag{5}$$

for $i \neq i'$ and $j \neq j'$. For brevity, we call a matrix satisfying this restricted set of equations a *special* matrix. Thus, a 2×2 matrix $[x_{ij}]$ is special if

$$x_{11}x_{22} + x_{12}x_{21} = 0;$$

a matrix is special iff all its 2×2 submatrices are special. Unlike matrices in $\widetilde{D}(n, n)$ (which always are nilpotent), special matrices may be invertible, to wit for instance the 2×2 matrix over $\mathbb{Q}$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Recall that the *trace* of an $n \times n$ matrix X is the *sum* of its diagonal entries, $\mathrm{tr}(X) = \sum_i x_{ii}$. The *product* of the diagonal entries is usually not very interesting, but it will be significant here; for brevity, we call it the *multiplicative trace* of the matrix,

$$\mathrm{tr}_m(X) := \prod_i x_{ii}.$$

Proposition 11 *For special matrices (in particular for matrices in $\widetilde{D}(n, n)$), multiplicative trace is a multilinear alternating function of the columns (or of the rows) of the matrix.*

Proof. We do the column case. Multilinearity is clear. For the alternating property, it suffices to see that if we interchange two columns of a special matrix, then the multiplicative trace changes sign. For simplicity of notation, let us consider interchange of the two first columns of a special matrix X, with resulting matrix X'. Then

$$\mathrm{tr}_m(X) = x_{11}x_{22}u$$

where u is the product $x_{33} \cdot \ldots \cdot x_{nn}$, and

$$\mathrm{tr}_m(X') = x_{12}x_{21}u,$$

with the same u. These two expressions differ by sign, by (5), and this proves the Proposition.

Recall the standard formla for the determinant of an $n \times n$ matrix X,

$$\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)}. \tag{6}$$

The product in the σth term may be viewed as $\mathrm{tr}_m(X^\sigma)$, where X^σ comes about by permuting the n columns of X according to σ.

Thus, we can write the standard formula for the determinant of any $n \times n$ matrix X as follows:

$$\det(X) = \sum_{\sigma} \mathrm{sign}(\sigma)\, \mathrm{tr}_m(X^\sigma).$$

If X is special, it follows from the Proposition that

$$\mathrm{tr}_m(X^\sigma) = \mathrm{sign}(\sigma)\, \mathrm{tr}_m(X);$$

since $\mathrm{sign}(\sigma) \cdot \mathrm{sign}(\sigma) = 1$, we have that all the $n!$ terms in the sum (6) are equal, namely equal to $\mathrm{tr}_m(X)$.

So we get in particular

Corollary 12 *If X is a special $n \times n$ matrix, in particular, if $X \in \widetilde{D}(n, n)$, then we have*

$$\det(X) = n!\ \mathrm{tr}_m(X).$$

Remark. The contention of this section is that for a matrix $X \in \widetilde{D}(n,n)$ ($n \geq 2$), its determinant is of interest. Clearly, over suitable rings R, there do exist non-zero matrices in $\widetilde{D}(n,n)$, – take e.g. the $n \times n$ matrix all of whose entries are equal to $d \in R$, where $d \in R$ has $d^2 = 0$. This matrix, however, has determinant zero. Do there, for suitable R, exist $X \in \widetilde{D}(n,n)$ with non-zero determinant ? The answer is yes, namely one may take R to be the commutative k-algebra containing the *generic* $X \in \widetilde{D}(n,n)$ (here, k is a field of characteristic 0). By this, we mean the k-algebra

$$R := k[X_{11}, X_{12}, \ldots, X_{nn}]/J$$

obtained from the polynomial k-algebra in n^2 indeterminates X_{ij}, by dividing out the ideal J, where J is generated by the defining equations (1) for $\widetilde{D}(n,n)$. In this ring R, the matrix $[X_{ij}]$ formed by the indeterminates satisfies the defining equations for being in $\widetilde{D}(n,n)$, by construction (in fact, it is what one would call the *generic* such matrix, for k-algebras); and its determinant is non-zero, by Theorem I.16.4 in [2]. For instance, if $n = 2$, the theorem quoted implies that R, as a vector space over k, is 6-dimensional, having for its basis the (classes modulo J of) the six polynomials

$$1, X_{11}, X_{12}, X_{21}, X_{22}, \begin{vmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{vmatrix}.$$

More generally, the k-algebra R containing the generic matrix X in $\widetilde{D}(m,n)$ is finite dimensional, having for its basis the determinants of all $p \times p$-submatrices of X (the 0×0-matrix is taken to be he constant polynomial 1); see loc.cit.

5 Non-linear aspects

Assume that $g : R^m \to R^l$ is a map, not necessarily linear. Then if X is an $m \times n$ matrix, we get an $l \times n$ matrix $g \cdot X$ by applying g to each of the n columns of X. If g is linear, so given by an $l \times m$ matrix, $g \cdot X$ is just the standard matrix product of g and X.

If $\underline{a} \in R^n$ (viewed as a column matrix), $X \cdot \underline{a} \in R^m$ is a linear combination of the columns of X (with coefficients the entries of $\underline{a}$). Any linear map $g : R^m \to R^l$ preserves linear combinations, which in matrix theoretic formulation says

$$g \cdot (X \cdot \underline{a}) = (g \cdot X) \cdot \underline{a}, \tag{7}$$

which is just the associative law for matrix multiplication. A crucial property of matrices $X \in \widetilde{D}(m,n)$ is the following Proposition:

Proposition 13 *Let $X \in \widetilde{D}(m,n)$, and let $g : R^m \to R^l$ be a 0-preserving polynomial map. Then g preserves linear combinations of the columns of X, i.e. the law (7) holds.*

Proof. It is enough to consider the case where $l = 1$. To say that g is a 0-preserving polynomial map is to say that

$$g(u) = g_1(u) + g_2(u,u) + \ldots + g_p(u,\ldots,u)$$

with $g_k : R^m \times \ldots \times R^m \to R$ k-linear symmetric. We shall do the case of "degree-2" polynomials only, so

$$g(u) = g_1(u) + g_2(u,u)$$

with g_1 linear and g_2 bilinear symmetric. Since (7) holds for $g = g_1$, it suffices to see that it holds for the g given by $u \mapsto g_2(u,u)$; it does so, because both sides of (7) then give 0, as we shall argue. First $X \cdot \underline{a} \in D(m)$, by Proposition 2, and it is therefore killed by $u \mapsto g_2(u,u)$, by Proposition 7. On the other hand, the matrix $g_2 \cdot X$ has for its columns $g_2(c_j, c_j)$, and since $g_2(-,-)$ is symmetric bilinear, these columns are all 0, again by Propositions 2 and 7.

Remark. Consider for a moment the real numbers $\mathbb{R}$. If $g : \mathbb{R}^m \to \mathbb{R}^l$ is a smooth zero preserving map, then it may be written $g_l + h$ with g_l linear, and h a remainder of the form $u \mapsto g_2(u,u) \cdot k(u)$ with g_2 bilinear symmetric (and k smooth). This assumption on g (except the smoothness), makes sense also for a general commutative ring R instead of $\mathbb{R}$. Inspecting the proof of Proposition 13, we see that we might as well have proved the following Proposition; we did not present it as our "primary" formulation, because its seems like a more ad hoc result. It is, however, in this form that it is applied in SDG. (In fact, in SDG, the decomposition assumed in the Proposition obtains for *any* zero-preserving map $g : R^m \to R^l$.)

Proposition 14 *Let $g : R^m \to R^l$ be a zero preserving map, and assume g may be written $g_l + h$ with g_l linear, and h a remainder of the form $u \mapsto g_2(u,u) \cdot k(u)$ with g_2 bilinear symmetric. If $X \in \widetilde{D}(m,n)$, g preserves linear combinations of the columns of X, i.e. the law (7) holds.*

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