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Spaces with local equivalence relations, and their monodromy

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Abstract

We elaborate a suggestion of Grothendieck, and study the invariant sheaves for a local equivalence relation on a space (e.g., a foliation). One of our purposes is to compare this to the standard model for the leaf-(quotient-)space of a foliation, given by the holonomy groupoid. To this end, we prove that, under suitable connectedness assumptions, Grothendieck's invariant sheaves can be described in terms of a closely related, but different, "monodromy" groupoid.

Our second purpose is to prove that every étale groupoid arises this way.

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Introduction

The idea of an equivalence relation given locally on a topological space was introduced by Ehresmann [7], and by Grothendieck and Verdier in SGA4 [1], as a possible way to study foliations. In the framework of SGA4, these so-called local equivalence relations naturally come together with particular kinds of sheaves on the space, namely those sheaves which locally admit transport along the equivalence classes of the local equivalence relation. In particular, it was conjectured in SGA4 that under suitable (rather abstract) conditions, these sheaves collectively have particularly nice properties; more precisely, they form a topos, in fact of a special kind: a so-called étendue.

The purpose of this paper is to review and develop these suggestions made in SGA4. Our first main result (after recalling the notion of local equivalence relation r on a space

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M, and the notion of r-sheaf on M) is a simple geometric description of these r-sheaves: under certain connectedness assumptions on r (namely that locally the equivalence classes are simply connected, see the end of Section 1), they are precisely the sheaves on M which are covering projections for the *leaf* topology on M induced by r (see Theorem 2.8 below).

Next, we associate (in Section 3) a topological groupoid to a local equivalence relation r, satisfying the connectedness assumptions, the so-called *monodromy groupoid* of r, denoted $\Pi(M,r)$. This construction generalizes that of the monodromy groupoid of a foliation, considered in [22], and, e.g., in [21,4]. The simple geometric description of r-sheaves just mentioned enables us to show that the r-sheaves are precisely the sheaves admitting a continuous action by this monodromy groupoid, Theorem 3.6.

In Section 4, we construct a topological groupoid $\boldsymbol{G}$, smaller than the monodromy groupoid $\Pi(M,r)$, but determining the same category of equivariant sheaves. In particular, the category of all r-sheaves on M is also equivalent to the category of sheaves equipped with a continuous action by this smaller groupoid $\boldsymbol{G}$. This groupoid $\boldsymbol{G}$ has the special property that its source and target maps are local homeomorphisms (so $\boldsymbol{G}$ is an "étale groupoid").

From the equivalence already mentioned between the category of r-sheaves on M and the category of sheaves with a $\boldsymbol{G}$-action,

$$\mathrm{sh}(M,r) \simeq (\boldsymbol{G}\text{-sheaves}), \tag{1}$$

it is clear that $\mathrm{sh}(M,r)$ is an étendue topos, as conjectured by Grothendieck and Verdier. (These topos theoretic aspects will be deferred to the Appendix of this paper.)

We observe that the equivalence of categories (1) provides us with a topological description of the weak homotopy type of the topos $\mathrm{sh}(M,r)$. Indeed, by a result of [19], the topos of $\boldsymbol{G}$-sheaves has the same weak homotopy type as the classifying space $B\boldsymbol{G}$ of the groupoid $\boldsymbol{G}$. It follows, by the essential equivalence between the groupoids $\boldsymbol{G}$ and $\Pi(m,r)$, that $\mathrm{sh}(M,r)$ has the same weak homotopy type as the classifying space $B\Pi(M,r)$ of the monodromy groupoid.

After this analysis, we can achieve the two goals stated in the Abstract.

As to the comparison with the holonomy, the standard constructions [7,32,21] of the holonomy groupoid of a foliation apply also to a local equivalence relation r on a space M, so as to define a groupoid $\mathrm{Hol}(M,r)$, which serves as a model for the quotient space of M by r.

Using our equivalence between Grothendieck's category of r-sheaves and that of monodromy-equivariant sheaves, the comparison of r-sheaves with the standard "holonomy" quotient space can therefore be achieved, by utilizing a more or less well known surjection from monodromy to holonomy; in fact, we will give an indirect construction of the holonomy groupoid as a quotient of the monodromy groupoid, as constructed here. It will be clear that these two groupoids are in general quite different, as are their categories of sheaves (and also the weak homotopy types of their classifying spaces).

As to the second purpose stated in the Abstract, we prove that for any étale topological groupoid $\boldsymbol{H}$ (i.e., source and target maps are local homeomorphisms), one can construct a suitable space M equipped with a local equivalence relation r, so that $\boldsymbol{H}$ is essentially equivalent to the monodromy groupoid of r (and hence $\mathrm{sh}(M,r) \simeq (\boldsymbol{H}\text{-sheaves})$). This shows that every étale groupoid occurs as the monodromy groupoid of a local equivalence relation on a topological space. A topos theoretic formulation of this result is given in the Appendix.

Topos theoretic aspects of local equivalence relations are also discussed in [14]. Here an analogue of Theorem A.4 is proved for *locales*, by a construction quite different from the one in Section 5, and no study is made of the associated groupoids.

1. Local equivalence relations

The motivating example is that of a foliation on a manifold, cf., e.g., [15]. Recall that the leaves of the foliation may be given locally as the level sets of submersions, thus (still locally) as the equivalence classes of suitable equivalence relations. The set of leaves, topologized by the quotient topology, is generally too coarse an object for studying the transversal structure, and several finer types of mathematical structures have been proposed, cf., e.g., [3,8,11,20,23]. The theory proposed by Grothendieck and Verdier [1] for this study is based on their notion of local equivalence relation: Consider for a given topological space M and any open subset $U \subseteq M$ the set $\mathcal{E}(U)$ of all equivalence relations $R \subseteq U \times U$ on U. For $V \subseteq U$, there is an evident restriction map $\mathcal{E}(U) \to \mathcal{E}(V)$, sending a relation $R \subseteq U \times U$ to its restriction $R|V := R \cap (V \times V)$, and this gives $\mathcal{E}$ the structure of a presheaf on M; it is not a sheaf, in general.

Definition 1.1. A local equivalence relation on the space M is a global section r of the sheaf $\widetilde{\mathcal{E}}$ associated to the presheaf $\mathcal{E}$.

We recall how the global sections of a sheaf $\widetilde{P}$ associated to a presheaf P on a space M are constructed. In the literature, this is usually done by constructing the sheaf space (local homeomorphism to M), consisting of germs of "elements" of P, at the various points. For the present purpose, a description in terms of atlases is more appropriate: Given any presheaf P on a space M, an atlas in P, or more precisely, an atlas for a global section of $\widetilde{P}$, consists of a family

$$\mathcal{U} = \{(U_i, p_i) \mid i \in I\}, \tag{2}$$

where the U_i's form an open covering of M, while $p_i \in P(U_i)$ for each i, and which satisfy the local compatibility condition that, for all i, j, there exists a cover of $U_i \cap U_j$ by open sets W for which $p_i|W = p_j|W$.

If $\mathcal{U}$ is an atlas, as in (2), the individual (U_i, p_i)'s are called its *charts*. An atlas $\mathcal{V} = \{(V_j, q_j) \mid j \in J\}$ is said to *refine* $\mathcal{U}$ if for each index $j \in J$, there exists an index $i \in I$ such that $V_j \subseteq U_i$ and $p_i|V_j = q_j$, i.e., (V_j, q_j) is a subchart of (U_i, p_i). Every global section r of the sheaf $\widetilde{P}$ associated to P is given by some atlas in P; conversely, every atlas in P determines a global section, while two atlases define the same r iff they have a common refinement. Given a global section r, a pair (U, p), where $p \in P(U)$, may be called a chart for r if either of the two equivalent conditions hold: (1) there exists some atlas $\mathcal{U}$ for r with (U, p) as a member, (2) for every $x \in U$, the germ of p at x equals $r(x)$.

Returning to the special case of the presheaf $\mathcal{E}$ and its associated sheaf $\widetilde{\mathcal{E}}$, a local equivalence relation r on M may thus be given by an open cover $M = \bigcup U_i$, and for each i an equivalence relation R_i on U_i, such that each point $x \in U_i \cap U_j$ has a neighbourhood W on which R_i and R_j agree. Clearly any foliation on a manifold M determines a local equivalence relation r on M; other examples will be discussed in the course of the paper.

One of Ehresmann's approaches to the foundations of foliation theory [7] goes via the consideration of a topological space equipped with a further, "fine", topology. Such fine topologies appear also in the context of local equivalence relations and have been considered in [1, p. 487] and in [28]. We shall need the following elaboration of this idea. Let R be an equivalence on a topological space M. Then we can introduce a new finer topology $M_{R\text{-fine}}$ on M by taking as basic open sets any set of the form $U \cap Q$, where Q is an equivalence class for R and U is open for the original topology. Thus this topology is the coarsest for which the original open sets as well as the equivalence classes for R are open; and $M_{R\text{-fine}}$ is topologically the disjoint union of the equivalence classes for R, each of them with its subspace topology from M. If r is a local equivalence relation on M, we define a new finer topology $M_{r\text{-fine}}$ on M by letting the basic open subsets be sets of the form $V \cap Q$ where V is open in M, and Q an equivalence class for R, where (V, R) is some chart for r. Such equivalence classes are called *plaques*. Since $(U \cap V, R|(U \cap V))$ is also a chart for r if (U, R) is, these basic open sets may also be described just as the plaques.

Proposition 1.2. *Let r be a local equivalence relation on a space M, and let (U, R) be some chart for r. Then the inclusion $U_{R\text{-fine}} \hookrightarrow M_{r\text{-fine}}$ is continuous, and makes $U_{R\text{-fine}}$ an open subspace of $M_{r\text{-fine}}$. In particular, if r is given by a global equivalence relation R on M, then $M_{R\text{-fine}} = M_{r\text{-fine}}$.*

Proof. Consider a basic open set in $M_{r\text{-fine}}$, given by an equivalence class Q for S, where (V, S) is a chart for r. To prove continuity of the inclusion, we must prove $Q \cap U$ open in $U_{R\text{-fine}}$. Let $x \in Q \cap U$. Since R and S have the same germ at x (namely $r(x)$), we may find an open $W \subseteq M$ with $x \in W \subseteq U \cap V$, and such that $R|W = S|W$. Let xR denote the equivalence class of x for R, and similarly xS. Then $xR \cap W$ is open in $U_{R\text{-fine}}$ and $xR \cap W = xS \cap W \subseteq xS \cap U$, so $xR \cap W$ witnesses that x is an interior point, for the R-fine topology, of $xS \cap U = Q \cap U$, and this proves continuity. Since the inclusion $U_{R\text{-fine}} \hookrightarrow M_{r\text{-fine}}$ by construction of $M_{r\text{-fine}}$ takes basic open sets to open sets, it is clear that this inclusion is open. $\square$

Recall that an equivalence relation R on a space M is called *open* if the saturation of any open set under the equivalence relation is again open; equivalently, if the quotienting map $M \to M/R$ is an open map; or again equivalently, if the projection map $R \to M$

given by $(x, y) \mapsto x$ is open (where $R \subseteq M \times M$ is given the subspace topology). We shall only consider open equivalence relations. An equivalence relation R on a space M is called *connected* if its equivalence classes are connected subsets of M, and *locally connected* if M has a basis of open sets U such that the restriction of R to U is connected; similarly for simply connected and locally simply connected, and for path connected and locally path connected.

Now some properties of local equivalence relations on a space M are defined in terms of properties of atlases $\mathcal{U}$ and their individual charts (U, R). A chart (U, R) is open if R is an open equivalence relation. Any subchart of an open chart is clearly open. An atlas is open if all its charts are. All refinements of open atlases are automatically open. A local equivalence relation is open if it has an open atlas. An atlas is locally (simply) connected if all its charts are, or equivalently, if every refinement of it may be refined by a (simply) connected atlas, i.e., one where all charts are (simply) connected. Similarly for locally path connected. A local equivalence relation is locally (simply) connected if it has some locally (simply) connected atlas, or, equivalently, if every atlas for it can be refined by a (simply) connected one. Similarly for locally path connected.

The local equivalence relation associated with a foliation on a manifold is locally connected, locally simply connected (and open). On the other hand, the singular foliation on the plane given by concentric circles around the origin, comes from a local (even global) equivalence relation which is locally connected, but not locally simply connected.

2. Sheaves with transport

Fundamental to the Grothendieck–Verdier approach is the notion of a sheaf equipped with transport along, or action by, a local equivalence relation; and this in turn derives from the notion of transport along a "global" equivalence relation, so we recall this concept (it is actually just a special case of the notion of action by a topological groupoid): Let R be an equivalence relation on a space M, and let $p: E \to M$ be a space over M. A *transport along R in E* is given by a continuous map

$$\nabla : R \times_M E \to E$$

satisfying

$$p\big(\nabla\big((y,x)(e)\big)\big) = y$$

(for $(y, x) \in R$ and $p(e) = x$), as well as the "unit" and "cocycle" conditions (writing $\nabla_{y,x}(e)$ for $\nabla((y, x)(e))$)

$$\nabla_{x,x}(e) = e,$$

$$\nabla_{z,y}\big(\nabla_{y,x}(e)\big) = \nabla_{z,x}(e).$$

Remark 1. Observe that if a sheaf E carries such transport by R, then the restriction of E to each equivalence class Ry is a constant sheaf.

We proceed to discuss action or transport on sheaves by local equivalence relations, which is our main concern. Suppose we are given a topological space M and a sheaf E on it. We shall consider sheaves on M in terms of the sheaf *spaces* $E \to M$, a space $E \to M$ being a sheaf iff the map $E \to M$ is a *local homeomorphism*, also called an *étale* map. Suppose we are given such a sheaf $p: E \to M$. For any open subset $U \subseteq M$, we consider the set $T_E(U)$ consisting of pairs (R, ∇), where R is an equivalence relation on U and ∇ is a transport along $R|U$ on $E|U = p^{-1}(U)$. For $V \subseteq U$, there is an evident restriction map $T_E(U) \to T_E(V)$, and this gives T_E the structure of a presheaf on M. Furthermore, there is a "forgetful" map $p: T_E \to \mathcal{E}$, given by $(R, q) \mapsto R \in \mathcal{E}(U)$ for $(R, q) \in T_E(U)$. Let $\widetilde{T_E}$ and $\widetilde{\mathcal{E}}$ be the associated sheaves, so that p induces a map of sheaves $p: \widetilde{T_E} \to \widetilde{\mathcal{E}}$. Consider a fixed local equivalence relation r on M, i.e., a global section r of $\widetilde{\mathcal{E}}$.

Definition 2.1. An r-transport on the sheaf E is a global section t of $\widetilde{T_E}$ such that $p(t) = r$. An r-sheaf on M is a sheaf on M equipped with r-transport.

Remark 2. Recall from Section 1 the fine topology $M_{r\text{-fine}}$ on a space equipped with a local equivalence relation r; since r is fixed in what follows,we omit it from notation: we shall in fact write M_{leaf} instead of $M_{r\text{-fine}}$ to stress the relationship to foliation theory, where this topology is usually called the leaf topology. Let $p: E \to M$ be a sheaf on M. Write E_{leaf} for the space E but equipped with that unique fine topology making the given map $p: E \to M$ an étale map into M_{leaf}. If $p: E \to M$ has the structure of an r-sheaf, then p is a covering projection when viewed as a map $E_{\text{leaf}} \to M_{\text{leaf}}$ (i.e., E_{leaf} is a locally constant sheaf on M_{leaf}). This follows from Remark 1.

We shall study only the case of a locally connected r, where the situation simplifies considerably, due to

Theorem 2.2. *Let r be a locally connected local equivalence relation on a space M. Then any sheaf E on M has at most one r-transport.*

Proof. Suppose s and s' are two transports on the sheaf $p: E \to M$ for the local equivalence relation r on M. Consider any point $x \in M$; we will show that s and s' have the same germ at x. To this end, choose charts (U, R, ∇) and (U, R, ∇') for s and s', with the same underlying chart (U, R) for r, where U is a neighbourhood of x. Since r is assumed to be locally connected, we can choose U so small that (U, R) is a connected chart. Let y be any point, and consider the two maps (where E_y denotes the stalk of E over y, and Ry the equivalence class of y)

$$\nabla_y, \nabla'_y : Ry \times E_y \to E|Ry$$

given by $\nabla_y(x, e) = \nabla_{x,y}(e)$ and similarly for ∇'. By the unit and cocycle conditions for transport, ∇_y has a continuous inverse ∇_y^{-1} defined by $\nabla_y^{-1}(e') = (p(e'), \nabla_{y,p(e')}(e'))$.

Consider for each $e \in E_y$, the embedding $Ry \to Ry \times E_y$ given by $z \mapsto (z, e)$. Then the composite

$$Ry \longrightarrow Ry \times E_y \xrightarrow{\nabla'_y} E|Ry \xrightarrow{\nabla_y^{-1}} Ry \times E_y \xrightarrow{\text{proj}} E_y$$

is a continuous map from the connected space Ry into the discrete space E_y, hence is a constant map, necessarily with value e since it takes the value e at y. Thus $\nabla_y^{-1}\nabla'_y(z, e) = (z, e)$, for any $z \in yR$. It follows that $\nabla_y = \nabla'_y$. □

Remark 3. If r is locally connected, then for a sheaf E to be an r-sheaf, it suffices to give transports on E by each of the equivalence relations in an atlas for r; local compatibility is then automatic, by the uniqueness.

From this, it also follows that the property of being an r-sheaf is a local property, i.e., if the base space M is covered by open sets U such that the restriction of the sheaf $E \to M$ to each U is an $r|U$-sheaf, then E is an r-sheaf.

It is possible to define the notion of transport preserving map between two r-sheaves on M, so as to obtain a category, equipped with a faithful functor to the category $\text{sh}(M)$ of all sheaves on M. For the case of a locally connected local equivalence relation, this functor can be proved full and faithful. Since we shall be interested only in such local equivalence relations, we take the easy way out and put:

Definition 2.3. Let r be a locally connected local equivalence relation on a space M. The category $\text{sh}(M, r)$ is the full subcategory of $\text{sh}(M)$ determined by those sheaves that admit an r-transport (necessarily unique, by Theorem 2.2).

Example 2.4. Let M be a space equipped with an open local equivalence relation r, and let T be an arbitrary space. Let $U \subseteq M$ be open. A function $f : U \to T$ is *r-invariant* in $x \in U$ if there exists an open V, $x \in V \subseteq U$ and a chart (V, R) for r for which f is constant on the equivalence classes of R; f is called r-invariant on U if it is r-invariant in all $x \in U$. The germs of r-invariant T-valued functions form a subsheaf of the sheaf of germs of all T-valued functions on M. Let us denote it $F(r, T)$.

Since the equivalence classes in all charts for r (the plaques) form a basis for M_{leaf}, it follows that if $f : U \to T$ is r-invariant, then it is locally constant on U_{leaf} ($= U$, equipped with the topology induced from M_{leaf}). The converse holds if r is assumed locally connected; for if $x \in U$ is an arbitrary point, we may pick a chart (V, R) for r with $x \in V \subseteq U$, and with connected equivalence classes. These are then open connected sets in U_{leaf}, so f is constant on them, so f is r-invariant in x.

If r is open (not necessarily locally connected), we shall equip $\widetilde{F} = F(r, T)$ with the structure of an r-sheaf. For any open chart (U, R), there is a canonical map $\theta = \theta_{U,R}$,

$$\theta : R \times_U (\widetilde{F}|U) \to \widetilde{F}|U,$$

and these $\theta_{U,R}$ will form an atlas for an r-structure on $\widetilde{F}$. To define $\theta_{U,R}$, let $(y, x) \in R$, and let $f_x \in \widetilde{F}_x$, $(x \in U)$. Then there is an open V with $x \in V \subseteq U$ and a continuous function $f: V \to T$ which is invariant for $R|V$ and has the given f_x as germ at x. Then f extends uniquely to an R-invariant $f^+: V^+ \to T$, where V^+ is the saturation of V under R. It is an open set, since R is an open equivalence relation, and f^+ is continuous since $V/(R|V)$ is homeomorphic to $V^+/(R|V^+)$ (by openness of R and hence of the quotienting map). We define $\theta((y,x)(f_x))$ as the germ of f^+ at y. Thus in the neighbourhood of $((y,x), f_x)$ given by $V^+ \times V$, and f^+ (the latter being a section of $\widetilde{F}$, thus defining a neighbourhood of $f_x \in \widetilde{F}$), θ is given by

$$(y', x', f_{x'}) \mapsto f^+ y',$$

and thus is clearly continuous. The unit- and associative laws are clear. If (U', R') is a subchart of (U, R), then clearly $\theta_{U,R}$ and $\theta_{U',R'}$ agree on U', and from this the compatibility conditions for the $\theta_{U,R}$'s follow.

One may also consider a vector field X on a manifold M; the local integral curves for it define a local equivalence relation r (actually a one-dimensional foliation, possibly with singularities). The sheaf of germs of smooth functions f on M which satisfy the partial differential equation $X(f) = 0$ is an r-sheaf.

For the rest of this section, we shall be interested in locally connected and open equivalence relations, local as well as global.

Theorem 2.5. *Let R be an open, connected, locally connected equivalence relation on a space M, and let $p: E \to M$ be a sheaf whose restriction to each equivalence class is constant. Then E admits an R-transport* (*and conversely, by Remark* 1).

Remark. A third equivalent condition is that E "descends" to M/R, meaning that it is of the form q^*F for some sheaf F om M/R, where $q: M \to M/R$ is the quotienting map. This condition will not play any role in the main body of the paper.

Proof. The assertion that $E|Rx$ is a constant sheaf means that there is given an action $\theta: R \times_M E \to E$ whose restriction to each equivalence class Rx is continuous. The assertion to be proved is that θ is actually continuous on the whole of its domain. Let $(y,x) \in R$, and let $e \in E$ with $p(e) = x$. We prove continuity of the action θ in the point $((y,x), e)$. Let X denote the equivalence class of x (and y). For each $t \in X$, there exists a neighbourhood V of t in M, and a section s over V with

$$s(t) = \theta(t,x)(e),$$

and such that $R|V$ has connected equivalence classes (this uses local connectedness of R). Therefore by connectedness of X, there exists a chain of such neighbourhoods $U_n, \ldots, U_0$ with $y \in U_n$, $x \in U_0$, with $U_i \cap U_{i-1} \cap X$ nonempty, and with sections $s_i: U_i \to E$ with $s_i(t_i) = \theta(t_i, x)(e)$ for suitable $t_i \in U_i \cap X$. Since $s_i(-)$ and $\theta(-,x)(e)$ are continuous sections of E over the connected set $U_i \cap X$, and since they agree in the point t_i, they agree on all of $U_i \cap X$,

$$s_i(t) = \theta\big((t,y), e\big) \quad \text{for all } t \in U_i \cap X.$$

The right hand side here does not depend on i, so in particular s_i and s_{i-1} agree on $U_i \cap U_{i-1} \cap X$, and since this set is nonempty, say $z_i \in U_i \cap U_{i-1} \cap X$, it follows from étaleness of p that we may find open subsets V_i with $z_i \in V_i \subseteq U_i \cap U_{i-1}$, on which s_i and s_{i-1} agree. For any open set W, write WR for its saturation under R; it is an open set, because R is an open equivalence relation. Again, because R is open, we may pick a sequence of open sets $W_{n+1}, \ldots, W_0$ around $y, z_n, \ldots, z_1, x$, respectively, with

$$\begin{aligned}
&y \in W_{n+1} \subseteq U_n,\\
&z_n \in W_n \subseteq V_n \cap W_{n+1}R,\\
&\vdots\\
&z_i \in W_i \subseteq V_i \cap W_{i+1}R,\\
&\vdots\\
&z_1 \in W_1 \subseteq V_1 \cap W_2R,\\
&x \in W_0 \subseteq U_0 \cap W_1R.
\end{aligned}$$

We claim that

$$s_n(y') = \theta(y', x')(e') \tag{3}$$

for all $y' \in W_{n+1}$, $x' \in W_0$, and $e' \in s_0(W_0)$ with $p(e') = x'$ and $y'Rx'$. From this, the continuity of θ in the neighbourhood of $((y, x), e)$ given by W_{n+1}, $W_0, s_0(W_0)$, immediately follows. To prove (3), pick $w_1 \in W_1$ with w_1Rx' and pick inductively $w_i \in W_i$ with w_iRw_{i-1}. All the points $y', w_n, \ldots, w_1, x'$ belong to the same equivalence class, X', say. Now note that E restricts to a constant sheaf over each equivalence class, and s_0 and $\theta(-, x')(e')$ are continuous sections of E over the connected set $U_0 \cap X'$, and they agree in x', so they agree on all of $U_0 \cap X'$, in particular in w_1: $s_0(w_1) = \theta(w_1, x')(e') =: e_1$; also s_1 and $\theta(-, w_1)(e_1)$ are continuous sections of E over the connected set $U_1 \cap X'$, and they agree in w_1, since

$$s_1(w_1) = s_0(w_1) = e_1 = \theta(w_1, w_1)(e_1)$$

(the first equality since $w_1 \in V_1$ where s_0 and s_1 agree), so they agree on all of $U_1 \cap X'$, in particular in w_2:

$$s_1(w_2) = \theta(w_2, w_1)(e_1) = \theta(w_2, w_1)\theta(w_1, x')(e') = \theta(w_2, x')(e').$$

Proceeding in this way, we get (3) after n steps, and the theorem is proved. □

For local equivalence relations, we then get as a corollary:

Theorem 2.6. *Let r be an open, locally connected local equivalence relation on a space M, and let E be a sheaf on M. Then the following conditions are equivalent:*

(1) *E is an r-sheaf;*

(2) *there exists an atlas $\mathcal{U}$ for r such that E is a $\mathcal{U}$-sheaf, i.e., $\mathcal{U}$ consists of charts (U, R) with the property that $E|xR$ is a constant sheaf for each $x \in U$ (where xR denotes the equivalence class of x under R).*

Proof. Assume (1), and take an atlas for the r-structure on E. For each chart (U, R) of it, R acts on $E|U$, so $E|xR$ is a constant sheaf. Conversely, assume (2). So there is an atlas $\mathcal{U}$ such that E is a $\mathcal{U}$-sheaf. Then clearly for any refinement $\mathcal{U}'$ of $\mathcal{U}$, E is a $\mathcal{U}'$-sheaf. By the assumptions on r, we may find a refinement $\mathcal{U}'$ of $\mathcal{U}$ whose charts (U, R) are open, connected, and locally connected. Since E is a $\mathcal{U}'$-sheaf, it is constant on each of the equivalence classes Rx for each chart (U, R), and by the previous theorem, therefore, $E|U$ carries R-transport. By local connectedness of r, this means that E is an r-sheaf (cf. Remark 3). □

Corollary 2.7. *Let r, E, and M be as in Theorem 2.6. If $E_{\text{leaf}} \to M_{\text{leaf}}$ is a constant sheaf, $E \to M$ is an r-sheaf.*

Proof. For any atlas for r, each plaque xR is not only a subspace of M, but also of M_{leaf}, by Proposition 1.2. Therefore $E|xR = E_{\text{leaf}}|xR$, which is constant. Thus the theorem implies that E is an r-sheaf. □

In general, the assumption of constancy in this corollary cannot be replaced by local constancy. But it obviously can if r is furthermore assumed locally *simply* connected:

Theorem 2.8. *Let r be a local equivalence relation on a space M, and assume that r is locally simply connected (as well as locally connected and open). Let $E \to M$ be a sheaf. Then E is an r-sheaf if and only if $E_{\text{leaf}} \to M_{\text{leaf}}$ is a covering projection (= locally constant sheaf).*

Proof. The implication $\Rightarrow$ was already argued in Remark 2, and does not depend on the special assumptions on r. Conversely, assume that $E_{\text{leaf}} \to M_{\text{leaf}}$ is a covering projection. Choose any simply connected atlas $\mathcal{U}$ for r. Then for each chart (U, R) of it, any equivalence class xR is not only a subspace of M, but of M_{leaf}, by Proposition 1.2. Therefore E restricts to a covering projection on each such equivalence class xR. But these equivalence classes are simply connected, so $E|xR$ is a constant sheaf. By the previous theorem, therefore, E is an r-sheaf. □

Remark 4. From the proof, we in fact see a little more: if $E \to M$ is an r-sheaf, and $\mathcal{U}$ is any simply connected atlas for r, then E is a $\mathcal{U}$-sheaf, i.e., the r-action can be defined on the atlas $\mathcal{U}$. In other words, with self-explanatory notation, the inclusion $\mathrm{sh}(M,\mathcal{U}) \subseteq \mathrm{sh}(M, r)$ is an equality.

Let us remark that Theorem 2.8 is not in general true without the simply-connectedness assumption on r:

Example 2.9. Consider the (open, locally connected) equivalence relation R on the complex plane $\mathbb{C}$ whose equivalence classes are $\{0\}$ and the circles with center in the origin, and consider the local equivalence relation r to which it gives rise. Consider the punctured Riemann surface for $\sqrt{z}$; it is a double cover of $\mathbb{C} - \{0\}$, and thus is a sheaf on $\mathbb{C}$

which is a covering projection over each of the leaves of r. However, it is not an r-sheaf; for an atlas for an r-action on it would have some chart (U, R', q) with $0 \in U$, and U would contain a circular disk D around 0 on which R' agrees with R. It would then follow that E restricted to D is constant along the leaves (circles) inside D, whereas E is only locally constant along these leaves (being a nontrivial double covering space of them).

Example 2.10. We finish this section by a discussion of local equivalence relations corresponding to foliations arising by suspension.

First recall Grothendieck's notion of G-sheaf [9]: if G is a discrete group acting on a space X (not necessarily faithfully), then there is a category (a topos, in fact) $\mathrm{sh}(X, G)$, consisting of sheaves $E \to X$ over X equipped with a G-action θ compatible with the given G-action on X. There is an obvious forgetful functor $\mathrm{sh}(X, G) \to \mathrm{sh}(X)$. If G acts "freely", in the strong sense that $(\mathrm{act}, \mathrm{proj}) : X \times G \to X \times X$ is a subspace inclusion, it follows from descent theory (cf., e.g., [13, Theorem D], [25], or [31] (or [9, p. 199], for the case where the action is free and proper)) that

$$\mathrm{sh}(X, G) \simeq \mathrm{sh}(X/G),$$

where X/G is the quotient space.

If $h : X \to X$ is a homeomorphism, it may be identified with an action by $\mathbb{Z}$ on X, and we may write $\mathrm{sh}(X, h)$ instead of $\mathrm{sh}(X, \mathbb{Z})$.

For such h, we get a *free* (in the strong sense), and proper action $\overline{h}$ by $\mathbb{Z}$ on $X \times \mathbb{R}$, given by $\overline{h}(x, r) = (h(x), r + 1)$. Let us denote the orbit space for this action by $\widehat{X}$. Since the action is proper, the quotienting map $q : X \times \mathbb{R} \to \widehat{X}$ is a covering projection, in particular étale. The local equivalence relation r' on $X \times \mathbb{R}$ whose leaves are the sets $\{x\} \times \mathbb{R}$ is compatible with the action $\overline{h}$, and hence induces a well-defined local equivalence relation r on $\widehat{X}$. (This is the local equivalence relation given by the so-called suspension of the homeomorphism h on X.) We shall prove that

$$\mathrm{sh}(\widehat{X}, r) \simeq \mathrm{sh}(X, h).$$

There is an obvious full and faithful functor $\mathrm{sh}(X) \to \mathrm{sh}(X \times \mathbb{R})$ taking a sheaf E on X to the sheaf $E \times \mathbb{R}$ on $X \times \mathbb{R}$. It lifts to a full and faithful functor $\mathrm{sh}(X, h) \to \mathrm{sh}(X \times \mathbb{R}, \overline{h})$ given by $(E \to X, \theta) \mapsto (E \times \mathbb{R} \to X \times \mathbb{R}, \overline{\theta})$, where $\overline{\theta}(x, r) = (\theta(x), r + 1)$. It is straightforward to see that if a sheaf on $X \times \mathbb{R}$ is in the image of $\mathrm{sh}(X) \to \mathrm{sh}(X \times \mathbb{R})$ and carries a $\mathbb{Z}$-action compatible with $\overline{h}$, then it comes from a sheaf in $\mathrm{sh}(X, h)$; in other words, we have a pull-back diagram of categories

$$\begin{array}{ccccc}
\mathrm{sh}(X) & \hookrightarrow & \mathrm{sh}(X \times \mathbb{R}) & & \\
\uparrow & & \uparrow & \nwarrow^{q^*} & \\
\mathrm{sh}(X, h) & \hookrightarrow & \mathrm{sh}(X \times \mathbb{R}, \overline{h}) & \xrightarrow{\simeq} & \mathrm{sh}(\widehat{X})
\end{array}$$

On the other hand, we shall prove that $\mathrm{sh}(\widehat{X}, r)$ can be obtained by essentially the same pull-back. Recall that we considered for a local equivalence relation on M the fine topology M_{leaf} on M, and for any local homeomorphism $f : N \to M$ the induced fine topology N_{leaf} on N. It comes from a unique local equivalence relation r' on N (which we may denote $f^*(r)$), and r' is locally (simply) connected if r is. From the fact that being an r-sheaf is a local property (cf. Remark 3), it is immediate that for a sheaf E on M, if it is an r-sheaf, then $f^*(E)$ is an r'-sheaf, and vice versa provided the local homeomorphism f is surjective. This can be expressed by stating that a certain commutative diagram of categories is a pull-back, which for the case at hand (with f being the surjective local homeomorphism $X \times \mathbb{R} \to \widehat{X}$) is the diagram

$$\begin{array}{ccccc} \mathrm{sh}(X) & \xrightarrow{\simeq} & \mathrm{sh}(X \times \mathbb{R}, r') & \hookrightarrow & \mathrm{sh}(X \times \mathbb{R}) \\ & & \uparrow & & \uparrow{\scriptstyle q^*} \\ & & \mathrm{sh}(\widehat{X}, r) & \hookrightarrow & \mathrm{sh}(\widehat{X}) \end{array}$$

Comparing these two pull-backs then gives the desired equivalence $\mathrm{sh}(X, h) \simeq \mathrm{sh}(\widehat{X}, r)$.

3. The monodromy groupoid

Let r be a locally simply connected (and locally path connected, open) local equivalence relation on a space M, fixed throughout this section. We shall construct two topological groupoids with M as space of objects, generalizing the monodromy and holonomy groupoid of a foliation. The monodromy groupoid will for the present purpose be the more important one, since its "equivariant sheaves" will be proved to be precisely the r-sheaves. Recall from, e.g., [17,19] that a topological groupoid $\boldsymbol{G}$ is given by a space G_0 ("of objects") and a space G_1 ("of arrows"), together with continuous maps d_0 and $d_1 : G_1 \to G_0$ associating to each arrow its domain and codomain, a continuous map associating to each composable pair of arrows their composite, and a continuous map associating to each object an "identity arrow" for this object, and such that the usual unit and associative laws hold; also, one requires the existence of a continuous "inversion" map $G_1 \to G_1$ associating to each arrow an inverse for it. Recall that for any topological groupoid $\boldsymbol{G} = G_1 \rightrightarrows G_0$, we have a notion of a $\boldsymbol{G}$-equivariant sheaf (or sheaf with a (right) $\boldsymbol{G}$-action), namely a sheaf $E \to G_0$ with a continuous map $E \times_{G_0} G_1 \to E$, satisfying the evident unit and associative laws. Also there is an evident notion of when a map between two such sheaves is $\boldsymbol{G}$-equivariant; so we have a category of such sheaves. (This category $\boldsymbol{BG}$ is in fact a topos, called the classifying topos of $\boldsymbol{G}$, cf. the Appendix.) We shall prove that the r-sheaves are exactly the $\boldsymbol{G}$-equivariant sheaves for the monodromy groupoid which we are going to construct, cf. Theorem 3.6 below.

The monodromy groupoid is constructed as a topological groupoid consisting of homotopy classes of leafwise paths (much as in [21], who, however, only treats the manifold case). Recall that when we have a local equivalence relation r on a topological space M, the latter aquires a further topology M_{leaf}, finer than the original one. The following consideration applies to any such topological space M equipped with a further, finer, topology M'. Since we have a continuous identity map $M' \to M$, we have a continuous (and injective) map $M'^I \to M^I$ between the path spaces. We endow its image $P \subseteq M^I$ with the subspace topology (inherited from the compact-open topology on M^I). For the special case where $M' = M_{\text{leaf}}$ for a local equivalence relation r on M, we write $P(M,r)$ for P. We shall consider an explicit description of a basis for the topology on $P(M,r)$ below.

Assigning to each path α its domain $\alpha(0)$ and its codomain $\alpha(1)$ makes M^I into a topological oriented graph $M^I \rightrightarrows M$, which further carries a natural multiplication structure (concatenation of paths), and a natural section $M \to M^I$ (formation of constant paths). It also admits a reparametrization action by the monoid of continuous maps $I \to I$, in particular an inversion, namely reparametrization by $t \mapsto 1-t$. This whole structure comes close to being a topological groupoid, except that the associativity etc. laws hold only up to standard explicit homotopies. Now the subspace $P(M,r)$ of M^I is evidently stable under this structure, and thus itself is a topological oriented graph $P \rightrightarrows M$. For any space $E \to M$ over M it thus makes sense to talk about continuous actions on it by M^I or by $P(M,r)$.

Let $E \to M$ be a topological space over M, and let $E' \to M'$ be its pull-back along the bijective continuous $M' \to M$; thus, E' is just E, but with a finer topology. From the continuous map $M'^I \to P$, and the fact that E' is defined by a pull-back, it is clear that any continuous action a by P on E gives rise to a continuous action a' of M'^I on E' (set theoretically, a' is the same mapping as a).

From standard covering space theory, we quote:

Proposition 3.1. *Let $E \to X$ be a sheaf (= local homeomorphism) on a locally simply connected space X. Then it admits a continuous action by X^I iff it is a covering space over X. If P is a subspace of X^I (containing with every path also every continuous reparametrization of it), then a sheaf $E \to X$ admits at most one continuous action by P.*

We then have the following extension of Theorem 2.8; r is a locally simply connected local equivalence relation on a space M, so that M_{leaf} is a locally simply connected space; we also assume that r is locally path connected.

Theorem 3.2. *Let $E \to M$ be a sheaf on M. Then the following conditions are equivalent:*

(1) *E is an r-sheaf.*
(2) *$E \to M$ pulls back along $M_{\text{leaf}} \to M$ to a covering space $E_{\text{leaf}} \to M_{\text{leaf}}$.*
(3) *$E \to M$ admits a continuous action by $P(M,r)$.*

Proof. The equivalence of the two first conditions is expressed in Theorem 2.8. If (3) holds, $E_{\text{leaf}} \to M_{\text{leaf}}$ admits a continuous action by M^I_{leaf}, by the general remarks above, hence is a covering space, by Proposition 3.1. Finally, assume (2). The evaluation map $\text{ev}: P(M,r) \times I \to M$ (sending (ξ, t) to $\xi(t)$) is continuous, since $P(M,r)$ has the subspace topology from M^I. Now pull $E \to M$ back along ev to obtain a sheaf E' on $P(M,r) \times I$,

$$\begin{array}{ccc} E' & \xrightarrow{\varepsilon} & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ P(M,r) \times I & \xrightarrow[\text{ev}]{} & M \end{array}$$

The assumption that $E \to M$ pulls back to a covering space on M_{leaf} implies that, for each $\xi : I \to M$ which is continuous for M_{leaf}, the pull-back of $E \to M$ along ξ yields a locally constant sheaf E_ξ on I; since I is simply connected, E_ξ is in fact constant. This in turn implies that the sheaf $E' \to P \times I$ is constant along each subspace $\{\xi\} \times I$. By Theorem 2.5, it follows that the equivalence relation, whose equivalence classes are the subsets $\{\xi\} \times I$, acts in a continuous way; denote this action by ∇, as in Section 2. But now it is clear that path lifting, i.e., the action a of $P(M,r)$ on E, can be described explicitly in terms of ∇:

$$a(e,\xi) = \varepsilon\big(\nabla_{((\xi,1),(\xi,0))}(e,(\xi,0))\big),$$

for $p(e) = \xi(0) = \text{ev}(\xi,0)$. From this the continuity of a is clear, thus proving (3). This proves the theorem. □

We shall now investigate some topological properties of the topological graph $P(M,r)$, which will allow us to construct the monodromy groupoid (as a topological groupoid) as a quotient of it.

For any topological space X, one of the standard descriptions of the compact open topology on the space X^I of continuous paths is as the topology where the basic open sets are given in terms of finite lists $(U_1,\ldots,U_n)$ of open subsets of X (which may even be taken from any prescribed basis for X), this list defining the basic open set $N_X(U_1,\ldots,U_n)$ of paths $\alpha : I \to X$ satisfying $\alpha([(i-1)/n, i/n]) \subseteq U_i$ for $i = 1,\ldots,n$. Thus in particular as basic open subsets of the space $P(M,r) \subseteq M^I$ we may take the sets $P(M,r) \cap N_M(U_1,\ldots,U_n)$, such that each U_i is underlying set of a chart (U_i, R_i) for r, i.e., as the set of continuous $\alpha : I \to M$ which satisfy (4):

$$\alpha \text{ is continuous for } M_{\text{leaf}}, \text{ and } \alpha\left(\left[\frac{i-1}{n}\right], \frac{i}{n}\right) \subseteq U_i. \tag{4}$$

(This set we denote by $N_M((U_1,R_1),\ldots,(U_n,R_n))$.) This set can equally well be described as the set of continuous $\alpha : I \to M$ satisfying (5):

$$\alpha\left(\left[\frac{i-1}{n}, \frac{i}{n}\right]\right) \subseteq \alpha\left(\frac{i}{n}\right)R_i \quad \left(= \alpha\left(\frac{i-1}{n}\right)R_i\right). \tag{5}$$

Using this explicit description of the topology on $P(M,r)$, we can now easily prove

Proposition 3.3. *The domain and codomain maps $P(M,r) \rightrightarrows M$ are open. The multiplication (concatenation of paths) is an open map*

$$P(M,r) \times_M P(M,r) \to P(M,r).$$

Proof. Let $\alpha_0 \in P(M,r)$ be a path from x to y. Consider a basic neighbourhood V of α_0, given as the set of paths α satisfying (4) or equivalently (5), but where we now further assume that the R_i-equivalence classes are path connected (using the assumption of local path-connectedness of r). Since each R_i is an open equivalence relation, we may, as in the proof of Theorem 2.5, find an M-open neighbourhood N_x around x such that for each $x_1 \in N_x$, there is a chain $z_1, z_2, \ldots, z_n$ with $z_1 = x_1$, with $z_j \in U_{j-1} \cap U_j$ for $j = 2, \ldots, n$, with $x_1 R_1 z_2, z_2 R_2 z_3, \ldots, z_{n-1} R_{n-1} z_n$, and with $z_n \in U_n$. Since each R_i has path connected equivalence classes, we may choose paths $\alpha_1, \ldots, \alpha_{n-1}$ with α_i a path from z_i to z_{i+1} inside the equivalence class $z_i R_i$. Also let α_n be the path with constant value z_n. Concatenating these n paths yields a path α with $\alpha(0) = x_1$, and $\alpha([(i-1)/n, i/n]) \subseteq z_i R_i$ for $i = 1, \ldots, n$. This path therefore satisfies (5). So every $x_1 \in N_x$ is the domain of a path $\alpha \in V$, and thus $d_0 : P(M,r) \to M$ is open. The codomain formation d_1 is treated similarly. Finally, let us consider the multiplication (path concatenation). It is clear that if $\alpha, \beta \in M^I$ can be concatenated and have their concatenation $\alpha * \beta \in P(M,r)$, then both α and β are in $P(M,r)$. From this follows that the diagram

$$\begin{array}{ccc} P(M,r) \times_M P(M,r) & \hookrightarrow & M^I \times_M M^I \\ \downarrow{*} & & \downarrow{*} \\ P(M,r) & \hookrightarrow & M^I \end{array}$$

is in fact a pull-back of spaces; since the multiplication structure $*$ on M^I (right hand vertical map in the diagram) is known (and easily seen) to be open, then so is the multiplication structure on $P(M,r)$ (left hand vertical map in the diagram). This proves the proposition. □

We now discuss leafwise homotopy. (We will always mean homotopy relative to end-points.) Thus a homotopy between two paths α and β in a space X is a continuous map $h : I \times I \to X$, restricting to α and β on the two horizontal edges of $I \times I$, and to constant maps on the two vertical edges. In particular $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. We say that α and $\beta \in P(M,r)$ are *leafwise homotopic* if they are homotopic by a homotopy in M_{leaf}. Let $h : I \times I \to M_{\text{leaf}}$ be such a homotopy. Since the covering of M_{leaf} by plaques Q is an open covering, $I \times I$ is covered by open subsets of the form $h^{-1}(Q)$; finding a Lebesgue number for this covering, we conclude that there is a natural number n such that the restriction of h to each of the small squares $[(i-1)/n, i/n] \times [(j-1)/n, j/n] \subseteq I \times I$ factors through some plaque $Q_{i,j}$, given by a chart $(U_{i,j}, R_{i,j})$.

We will show that these leafwise homotopy classes of paths admit a continuous composition operation. The proof is based on the following proposition.

Proposition 3.4. *Assume the local equivalence relation r on M is locally simply connected. Then the equivalence relation of leafwise homotopy on $P(M,r)$ is an open equivalence relation.*

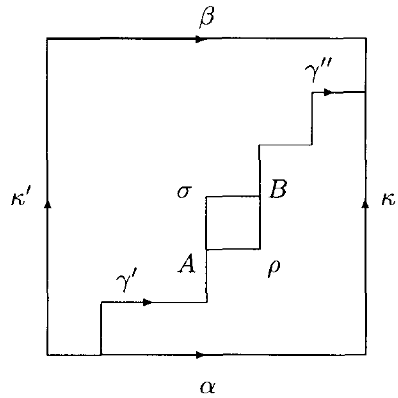
Proof. Let us write $\sim$ for the relation "leafwise homotopic to". Let us write $\alpha \rightsquigarrow \beta$ for the relation: $\alpha \sim \beta$, and for every open U around α, there is an open V around β such that for every $\delta \in V$ there is a $\gamma \in U$ with $\gamma \sim \delta$. Thus to say that $\sim$ is an open equivalence relation is to say that $\alpha \sim \beta$ implies $\alpha \rightsquigarrow \beta$. The relation of $\rightsquigarrow$ is evidently reflexive and transitive. It is probably not symmetric, in general.

Lemma 3.5. *Let κ denote the constant path at $\alpha(1)$, where $\alpha \in P(M,r)$. Then $\alpha \rightsquigarrow \alpha * \kappa$ and $\alpha * \kappa \rightsquigarrow \alpha$. Also, for κ the constant path at $\alpha(0)$, $\alpha \rightsquigarrow \kappa * \alpha$ and $\kappa * \alpha \rightsquigarrow \alpha$. Finally, if $\alpha \rightsquigarrow \beta$, then $\gamma' * \alpha * \gamma'' \rightsquigarrow \gamma' * \beta * \gamma''$, for any γ' and γ'' where it makes sense.*

Proof. Given a neighbourhood $U = N_M(U_1, \ldots, U_n) \cap P(M,r)$ of α, the neighbourhood of $\alpha * \kappa$ given by $V = N_M(U_1, \ldots U_n, U_n, \ldots, U_n) \cap P(M,r)$ (with n extra copies of U_n) will serve, since any path $\delta \in V$ is leafwise homotopic (namely by a pure reparametrization) to a path in U. On the other hand, given a neighbourhood U around $\alpha * \kappa$, it contains one of the form $N_M(U_1, \ldots, U_{2n})$, and since κ is constant $= \alpha(1)$, it follows that the $n+1$ last of these U_i's have the point $\alpha(1)$ in common, so there is an open set W with $\alpha(1) \in W \subseteq U_n \cap U_n \cap \cdots \cap U_{2n}$. The set $N_M(U_1, \ldots, U_n, W) \cap P(M,r)$ will now serve as V, for clearly if $\delta \in V$ for this V, then for the constant path κ at the endpoint of δ, $\delta * \kappa \in N_M(U_1, \ldots, U_{n-1}, W, W, \ldots, W) \subseteq N_M(U_1, \ldots, U_{n-1}, U_n, \ldots, U_{2n})$; and $\delta \sim \delta * \kappa$ by a reparametrization homotopy. The second assertion follows similarly, and the third is an elementary consequence of the fact (Proposition 3.3) that the multiplication $*$ is continuous and open. This proves the lemma. □

Now let $\alpha \sim \beta$ by virtue of a homotopy $h: I \times I \to M$, as described above, but where we now assume that the plaques $Q_{i,j}$ into which the n^2 small squares are mapped by h are simply connected (using local simply connectedness of r). By the above, $\alpha \rightsquigarrow \alpha * \kappa$, and $\kappa' * \beta \rightsquigarrow \beta$ (where κ and κ' are the relevant constant paths), so it suffices to see that $\alpha * \kappa \rightsquigarrow \kappa' * \beta$. Between these two paths we can (in many ways) interpolate a sequence of $2n$ paths, each of which is a restriction of h to a rectangular zig-zag through the grid lines of the division of $I \times I$ into the n^2 small squares; and each of the two consecutive terms in the sequence agree, except that they take opposite routes around one of the elementary squares, as for example displayed in the below figure.

Call the common part prior to A of these two paths γ', and the common part after B γ'', and call the two paths from A to B ρ and σ, respectively. By Lemma 3.5, it suffices to prove that $\rho \rightsquigarrow \sigma$. For simplicity, let (W, R) denote a chart with the plaque $Q_{i,j}$ as one of its equivalence classes (where i, j is the index pair corresponding to the little square from A to B). Consider an open neighbourhood of ρ. It contains one of the form $U = N_M((U_1, R_1), \ldots (U_n, R_n))$, with the R_i's having path connected equivalence classes, and with $R_i = R|U_i$ for any i. Proceeding as in the proof of Proposition 3.3, we may find a neighbourhood N_x around $x = \rho(0)$ such that every $x' \in N_x$ is the beginning point of a path $\gamma \in U$; we may even find such γ from x' to any prescribed end point in U_n R-equivalent to x' (using path connectedness of the equivalence classes



of R_n). Clearly, such γ stays inside one equivalence class of R. Now let V be the open neighbourhood of σ defined as

$$V = N_M(W, R) \cap d_0^{-1}(N_x) \cap d_1^{-1}(U_n).$$

Any path δ in this neighbourhood has its endpoints R-equivalent, and in N_x and U_n, respectively, and there is therefore, by the above, a path $\gamma \in U$ with the same two endpoints. But since the equivalence classes of R are simply connected, γ and δ are homotopic inside one equivalence class of R, and hence in particular $\gamma \sim \delta$. This proves $\rho \rightsquigarrow \sigma$, and thus openness of the relation $\sim$ of "leafwise homotopy". Proposition 3.4 is proved. □

Let us denote by $\Pi_1(M, r)$ the quotient space of $P(M, r)$ under the equivalence relation "leafwise homotopy". By Proposition 3.4 the quotienting map $P(M, r) \to \Pi_1(M, r)$ is an open surjection, and hence the pull-back of it along any map is again an open surjection. This finally enables us to see that composition of paths, as well as formation of d_0 and d_1 of paths, being compatible with the equivalence relation, induce continuous maps $*: \Pi_1(M, r) \times_M \Pi_1(M, r) \to \Pi_1(M, r)$ and $d_0, d_1 : \Pi_1(M, r) \to M$. Note also that it follows from Proposition 3.3 that they are open maps. Finally, inversion of paths in $P(M, r)$ (reparametrization by $t \mapsto 1 - t$) induces a continuous inversion in $\Pi(M, r)$. All this structure, together with formation of constant paths, $M \to P(M, r) \to \Pi_1(M, r)$, makes $\Pi_1(M, r)$ into a topological groupoid; this is the *monodromy groupoid* of the local equivalence relation.

Clearly any continuous action of $P(M, r)$ on a sheaf $E \to M$ is invariant under homotopy, so induces an action of $\Pi_1(M, r)$. This action is again continuous, because the quotient map $P(M, r) \to \Pi_1(M, r)$ (and hence any pull-back of it) is an open surjection. Thus a sheaf $E \to M$ carries a continuous action by $\Pi_1(M, r)$ iff it carries one by the topological graph $P(M, r)$ (in both cases, the action is unique if it exists), and combining this with Theorem 3.2, we see that the subcategory $\text{sh}(M, r) \subseteq \text{sh}(M)$ of r-sheaves, described in various equivalent ways in this theorem, can furthermore be described as the full subcategory of $\text{sh}(M)$ consisting of sheaves which admit a (necessarily unique) continuous action by the monodromy groupoid $\Pi_1(M, r)$. We therefore have

Theorem 3.6. *Let r be a locally simply connected local equivalence relation on a space M. Then the monodromy groupoid $\Pi_1(M,r)$, as described above, is a topological groupoid; and a sheaf on M is an r-sheaf if and only if it is $\Pi_1(M,r)$-equivariant.*

Remark 5. The theorem expresses that the two subcategories of $\mathrm{sh}(M)$, $\mathrm{sh}(M,r)$ and $\boldsymbol{B}(\Pi_1(M,r))$, are equivalent. In particular, since the latter is a Grothendieck topos, we conclude that $\mathrm{sh}(M,r)$ is. Further topos theoretic implications of the theorem may be found in Appendix A.

4. Étale monodromy and holonomy

To get further information about the equivalence $\mathrm{sh}(M,r) \simeq \boldsymbol{B}(\Pi_1(M,r))$ of Theorem 3.6, we shall prove (Theorem 4.1 below) that the topological groupoid $\Pi_1(M,r)$ is *essentially* equivalent (in a sense to be made precise below) to an étale topological groupoid $\boldsymbol{G}$ (a notion that we shall also recall). In the foliation case, one would construct $\boldsymbol{G}$ as a full subcategory of $\Pi_1(M,r)$ with space of objects some *complete transversal* $T \subseteq M$ of the foliation. Such a procedure is not available in the present generality, where instead we must let $\boldsymbol{G}$ have for its space of objects a certain "local quotient space" of M (which, just as for the case of a transversal, is noncanonical, but depends on a choice of an atlas for r).

Recall that a morphism of topological groupoids $p: \boldsymbol{H} \to \boldsymbol{G}$ is called *full and faithful* if the space of arrows H_1 of $\boldsymbol{H}$ is the pull-back of G_1 along $p_0 \times p_0$ (the relevant diagram appears in (11) below), and is called an *essential equivalence* [18] if further p_0 is "essentially surjective on objects"—the exact meaning of this need not concern us here, since in the cases to be considered, p_0 will actually be an open surjection, which is a stronger condition.

Full and faithful functors appear in particular when constructing "full images" (or "inverse images", in the terminology of [17, p. 11]), a construction which we shall now recall in slightly more general form. Let $d_0, d_1 : P \to M$ be continuous maps making $P \rightrightarrows M$ into a topological oriented graph. If $\phi: \widetilde{M} \to M$ is any continuous map, we may form the topological graph $\widetilde{P} \rightrightarrows \widetilde{M}$ obtained by pulling $(d_0, d_1): P \to M \times M$ back along $\phi \times \phi : \widetilde{M} \times \widetilde{M} \to M \times M$. We may denote it $\phi^*(P \rightrightarrows M)$ or just $\widetilde{P} \rightrightarrows \widetilde{M}$. Since open surjections are stable under pull-back and composition, an easy diagrammatic argument shows that if $d_0 : P \to M$ and ϕ are open surjections, then so is the structural map $\widetilde{d}_0$ of $\phi^*(P)$. And if $P \rightrightarrows M$ is part of the structure of a topological multiplicative graph, respectively of a topological groupoid, then a similar structure is induced on $\widetilde{P}$.

Theorem 4.1. *For any locally simply connected (locally path connected, open) local equivalence relation r on a space M, the monodromy groupoid $\Pi_1(M,r)$ of Theorem 3.6 is essentially equivalent to an étale groupoid.*

Proof. Let $\{(U_i, R_i) \mid i \in I\}$ be a simply connected atlas for the local equivalence relation r on M. Let $\widetilde{M} = \coprod U_i$. Thus we have an open (even étale) surjection $\phi : \widetilde{M} \to M$, and one may form the topological graph $\widetilde{P} \rightrightarrows \widetilde{M} = \phi^*(P(M,r) \rightrightarrows M)$. Its arrows are thus *labelled* leafwise paths (α, i, j) where $(i,j) \in I \times I$ (the "label"), and where α is a leafwise path with $\alpha(0) \in U_i$ and $\alpha(1) \in U_j$. Note that the space of arrows $\widetilde{P}$ of this topological graph is a disjoint union of the open subsets $d_0^{-1}(U_i) \cap d_1^{-1}(U_j)$ of $P(M,r)$.

Now the map $(d_0, d_1) : P \to M \times M$ for $P = P(M,r)$ factors across the map $h : P(M,r) \to \Pi_1(M,r) = \Pi$, as displayed in the right hand column in the following diagram

$$\begin{array}{ccccc}
 & {\scriptstyle q_1}\swarrow & \widetilde{P} & \longrightarrow & P \\
 & & \downarrow{\scriptstyle \tilde{h}} & & \downarrow{\scriptstyle h} \\
G_1 & \longleftarrow & \widetilde{\Pi} & \longrightarrow & \Pi \\
\downarrow{\scriptstyle (\overline{d_0},\overline{d_1})} & & \downarrow{\scriptstyle (\tilde{d}_0,\tilde{d}_1)} & & \downarrow{\scriptstyle (d_0,d_1)} \\
G_0 \times G_0 & \xleftarrow[q_0 \times q_0]{} & \widetilde{M} \times \widetilde{M} & \xrightarrow[\phi \times \phi]{} & M \times M
\end{array} \tag{6}$$

and we may therefore perform the pulling back that defines $\widetilde{P}$ in two stages, i.e., by constructing it from the two pull-backs that appear on the right in the diagram. Since h is an open surjection, then so is $\tilde{h}$.

We introduce now an equivalence relation $\sim$ on $\widetilde{P}$ by putting $(\alpha, i, j) \sim (\alpha', i', j')$ if $i = i'$, $j = j'$, and if α is leafwise homotopic to $\overrightarrow{xx'} * \alpha' * \overrightarrow{y'y}$, for *some* (hence *any*, by simply-connectedness) path $\overrightarrow{xx'}$ from $x = \alpha(0)$ to $x' = \alpha'(0)$ inside xR_i, and for some (hence any) path $\overrightarrow{y'y}$ from $y' = \alpha'(1)$ to $y = \alpha(1)$, inside yR_j.

Note that the equivalence relation $\sim$ is a disjoint union of equivalence relations on the individual $d_0^{-1}(U_i) \cap d_1^{-1}(U_j)$'s; and each of these individual equivalence relations is open, due to openness of the relation of leafwise homotopy, and openness of the concatenation map $*$. Thus $\sim$ is an open equivalence relation on $\widetilde{P}$. Let us denote the quotient space G_1. If we also let G_0 denote $\coprod (U_i/R_i)$ (which is a quotient $q_0 : \widetilde{M} \to G_0$ of $\widetilde{M}$), we have well-defined continuous maps $\overline{d_0}, \overline{d_1} : G_1 \to G_0$ obtained from the structural maps $\tilde{d}_0, \tilde{d}_1$ for the topological graph $\widetilde{P} \rightrightarrows \widetilde{M}$, and since these structural maps are open, then so are $\overline{d_0}, \overline{d_1}$. Let us denote the topological graph $G_1 \rightrightarrows G_0$ by $\boldsymbol{G}$.

Now because the equivalence relation $\sim$ includes leafwise homotopy, we evidently may factor the quotient map $\widetilde{P} \to G_1$ across $\widetilde{\Pi}$ as displayed in the diagram, with both factors open surjections. The factor q_1 is compatible with the structural maps, so that we get a morphism q of topological graphs

$$q : \phi^*\big(\Pi_1(M,r)\big) \to \boldsymbol{G}.$$

There is (set theoretically) a unique multiplicative structure on the graph $\boldsymbol{G}$ making q into a homomorphism of *multiplicative* graphs; and using that all the maps displayed in the relevant square (lower left hand square of (6)) are open surjections, one concludes that the multiplication on G is in fact continuous. And since $\phi^*(\Pi_1(M,r))$ is a topological groupoid, so is G.

We shall prove that the morphism q of topological groupoids is an essential equivalence. Clearly, q_0 is an open surjection. So it remains to be seen that q is full and faithful, i.e., that the lower left hand square in the diagram is a pull-back. Set theoretically, this is clear. To see that it is also topologically so, observe that the whole picture in question is a disjoint union over the set of labels (i,j); fix one such label (i,j) and suppress it from notation. So consider a point $[\alpha] = \widetilde{h}(\alpha) \in \widetilde{\Pi}$, i.e., a leafwise homotopy class of a path α with $\alpha(0) \in U_i$ and $\alpha(j) \in U_j$, and consider an open neighbourhood $h(O_\alpha)$ around it, where O_α is an open neighbourhood around α in $P(M,r)$. We may assume that

$$O_\alpha = N_M\big((V_1,S_1),\dots,(V_n,S_n)\big)$$

with $V_1 \subseteq U_i$ and $V_n \subseteq U_j$, and with $S_1 = R_i|V_1$ and $S_n = R_j|V_n$. To see that the topology on $\widetilde{\Pi}$ is not finer than the topology induced from G_1 and $\widetilde{M} \times \widetilde{M}$, it suffices to see that

$$\widetilde{d}_0^{-1}(V_1) \cap \widetilde{d}_1^{-1}(V_n) \cap q_1^{-1}q_1 h(O'_\alpha) \subseteq h(O_\alpha) \tag{7}$$

for some open set O'_α around α in $P(M,r)$. Let $\widehat{V}_1$ be the saturation of V_1 under the equivalence relation R_i, and $\widehat{S}_1 = R_i|\widehat{V}_1$; similarly, $\widehat{V}_n$ is the R_j-saturation of V_n and $\widehat{S}_n = R_j|\widehat{V}_n$. Let O'_α be the basic open set

$$O'_\alpha = N_M\big((\widehat{V}_1,\widehat{S}_1),(V_1,S_1),\dots,(V_n,S_n),(\widehat{V}_n,\widehat{S}_n)\big).$$

We easily see that $h(O'_\alpha)$ is saturated for $\sim$, i.e., equals $q_1^{-1}q_1hO'_\alpha$. For, if $\beta \in O'_\alpha$, then $\overrightarrow{x'x} * \beta * \overrightarrow{y'y}$ is easily reparametrized (and hence leafwise homotopic) to a path γ likewise belonging to O'_α, for any paths $\overrightarrow{x'x}$ and $\overrightarrow{yy'}$ inside one equivalence class of R_i (respectively R_j), since such paths also belong to one equivalence class of $\widehat{S}_1$ (respectively $\widehat{S}_n$).

Now if $\beta \in O'_\alpha$ and furthermore $\beta(0) \in V_1$, $\beta(1) \in V_n$, we may, by simply-connectedness of the equivalence classes of $\widehat{S}_1$, deform the restriction of β to $[0, 1/(n+2)]$ by a leafwise homotopy into a path entirely in $V_1 \subseteq \widehat{V}_1$, and similarly for the restriction of β to $[(n+1)/(n+2), 1]$, and then by a reparametrization deform it further into a path in O_α. These three deformations together yield a leafwise homotopy of β into a path in O_α, proving the inclusion (7).

We finally prove that $\overline{d}_0 : G_1 \to G_0$ (and hence $\overline{d}_1$) is locally injective. Let an element in G_1 be represented by $(\alpha,i,j) \in \widetilde{P}$. It suffices to find an open neighbourhood N around (α,i,j) such that if $(\beta,i,j) \in N$ and $\alpha(0)R_i\beta(0)$, then $(\alpha,i,j) \sim (\beta,i,j)$. We first take a neighbourhood of $\alpha \in P(M,r)$ of form

$$N\big((V_1,S_1),\dots,(V_n,S_n)\big) \tag{8}$$

with each (V_k,S_k) a simply connected chart for r, and with $V_1 \subseteq U_i$ and $V_n \subseteq U_j$ as charts, i.e., with S_1 the restriction of R_i and S_n the restriction of R_j. Then $\alpha(k/n) \in V_k \cap V_{k+1}$ for each $k = 1,\dots,n-1$; we may therefore find an open W_k with $\alpha(k/n) \in$

$W_k \subseteq V_k \cap V_{k+1}$ and such that S_k and S_{k+1} agree on W_k and have connected equivalence classes there. Let N be the open set consisting of (β, i, j) which satisfy

$$\beta \in N\big((V_1, S_1), \ldots, (V_n, S_n)\big) \quad \text{and} \quad \beta\left(\frac{k}{n}\right) \in W_k \ (k = 1, \ldots, n-1).$$

Assume now that the elements in G_1 represented by (α, i, j) and (β, i, j) have common $\overline{d}_0$-value; we want to prove $(\alpha, i, j) \sim (\beta, i, j)$. The assumption of common $\overline{d}_0$-value means that $\alpha(0) \sim \beta(0)$. Since $\alpha(0)$ and $\beta(0)$ are in V_1, $\alpha(0)S_1\beta(0)$. It follows that $\alpha(1/n)S_1\beta(1/n)$, and since both these points by construction of N are in W_1 where S_1 and S_2 agree, we have $\alpha(1/n)S_2\beta(1/n)$. Continuing in this way, we conclude $\alpha(1)S_n\beta(1)$. Let $\overrightarrow{x'x}$ and $\overrightarrow{yy'}$ be paths inside the equivalence classes of S_1 and S_n, respectively, connecting $x' = \beta(0)$ with $x = \alpha(0)$ (respectively $y = \alpha(1)$ with $y' = \beta(1)$). We claim that

$$\overrightarrow{x'x} * \alpha * \overrightarrow{yy'} \quad \text{is leafwise homotopic to } \beta. \tag{9}$$

For, we may connect $\alpha(k/n)$ to $\beta(k/n)$ by a path inside their common equivalence class of $S_k|W_k = S_{k+1}|W_k$ $(k = 1, \ldots, n-1)$, and using simply-connectedness of the equivalence classes of the S_k's, we can then, in n steps, deform $\overrightarrow{x'x} * \alpha * \overrightarrow{yy'}$ to β by a leafwise homotopy. Thus $(\alpha, i, j) \sim (\beta, i, j)$, so they represent the same arrow in G_1. This shows that $\overline{d}_0$ is injective when restricted to the open neighbourhood of $[(\alpha, i, j)]$ represented by N.

Together with the fact, already proved, that $\overline{d}_0$ is open, it now follows that $\overline{d}_0$ is étale. We remind the reader that a topological groupoid is called an *étale* groupoid if its d_0-map (and hence its d_1-map) is étale. Thus our groupoid $\boldsymbol{G}$ is étale. This proves the theorem. □

Since the monodromy groupoid is essentially equivalent to an étale topological groupoid $\boldsymbol{G}$, and the categories of invariant sheaves for two essentially equivalent groupoids are equivalent (cf. [18]), we thus conclude from the Theorems 4.1 and 3.6:

Corollary 4.2. *Let r be a locally simply connected (and locally connected, open) local equivalence relation on a space M. Then the category $\mathrm{sh}(M, r)$ is equivalent to the category of $\boldsymbol{G}$-sheaves, for a suitable étale topological groupoid.*

This implies that $\mathrm{sh}(M, r)$ is in fact an étendue topos, in the sense of Grothendieck, cf. Appendix A. Let us also note that the existence of an equivalence $\Pi_1(M, r) \simeq \boldsymbol{G}$ by [12] implies that $\Pi_1(M, r)$ is an *étale-complete* groupoid, i.e., can be reconstructed as the topological groupoid of points of its classifying topos $\boldsymbol{B}\Pi_1(M, r) \simeq \mathrm{sh}(M, r)$; cf. [18] for this notion.

We shall, for use below, explicitly record an information contained in the proof of the above theorem, namely the information how the local section s which is a right inverse for $\overline{d}_0|N$ looks in a sufficiently small neighbourhood O of $[x, i] = q_0(x, i)$; if (x', i) has x' so near x that there exists a path $\beta \in N$ with $\beta(0) = x'$, then the formula (9) shows that the $\sim$-equivalence class of β only depends on the equivalence class of (x', i), so that

$$[x', i] \mapsto [(\beta, i, j)]$$

is a well-defined map

$$O/S_1 \to N/\sim .$$

(The set of (x', i) for which there exists a path $\beta \in N$ with $\beta(0) = x'$ is open, by openness of $d_0 : \widetilde{P} \to \widetilde{M}$. Thus $O/S_1 = O/R_i$ is open in G_0.)

Holonomy

The rest of this section is not used anywhere else in this paper, and is included only to contrast the holonomy with the monodromy.

It is well known that the holonomy groupoid of a foliation is the standard model for the quotient "space", i.e., the "space" of leaves. There are constructions of this holonomy groupoid ([7,32,21,2], following a suggestion of Pradines) directly in terms of the foliations, and these constructions can also be adapted for local equivalence relations. Here, however, in order to compare the holonomy with the monodromy discussed above, we will give a roundabout construction of the holonomy groupoids, namely in terms of the monodromy groupoids.

For any étale topological groupoid $\boldsymbol{G} = (G_1 \rightrightarrows G_0)$, one can construct a morphism of groupoids $\boldsymbol{G} \to \Gamma G_0$, where ΓG_0 is the (étale) topological groupoid of germs of local homeomorphisms of G_0, namely by associating to an arrow $g : x \to y$ in $\boldsymbol{G}$ the germ at x of the map $d_1 \circ s$ where s is a local section of $d_0 : G_1 \to G_0$ with $s(x) = g$. The morphism $\boldsymbol{G} \to \Gamma G_0$ is the identity map on the space of objects and is étale on the space of arrows. The image $J\boldsymbol{G}$ of $\boldsymbol{G} \to \Gamma G_0$ is thus an open subgroupoid of ΓG_0.

Returning to the specific étale groupoid $\boldsymbol{G}$ constructed above, and recalling the description of local sections of $\overline{d}_0$ given above, it follows that the germ $\in \Gamma G_0$ at $[x, i]$ associated to the arrow $[(\alpha, i, j)] \in G_1$ (where $\alpha(0) = x, \alpha(1) = y$) may be described by

$$[x', i] \mapsto [y', j]$$

($[\ldots]$ denoting R_i-equivalence class, respectively R_j-equivalence class), where $[x', i] \in O$ and $y' = \beta(1)$ for some path $\beta \in N$ with $\beta(0) = x'$. We call this germ $\mathrm{germ}_x(\alpha, i, j)$. Since the description of N did not depend on the specific label (i, j) (when V_1 and V_n are chosen small enough, the same N will work for any other label (i', j') with $x \in U_{i'}$, $y \in U_{j'}$), it follows that if two leafwise paths α_1 and α_2 from x to y have the property that (α_1, i, j) defines the same germ as (α_2, i, j), then (α_1, i', j') defines the same germ as (α_2, i', j'). In other words, there is a well defined equivalence relation hol on $P(M, r)$ (coarser than leafwise homotopy, thus also descending to an equivalence relation on $\Pi(M, r)$), such that, for two paths α_1 and α_2 from x to y

$$\alpha_1 \text{ hol } \alpha_2 \quad \text{iff} \quad \mathrm{germ}_x(\alpha_1, i, j) = \mathrm{germ}_x(\alpha_2, i, j)$$

for some (hence any) label (i, j) with $x \in U_i$, $y \in U_j$.

Let us denote the quotient $\Pi(M,r)/\text{hol}$ by $\text{Hol}(M,r)$. From the construction given of Hol it is clear that $\text{Hol}(M,r)$ sits in a diagram

$$\begin{array}{ccccc}
G_1 & \longleftarrow & \widetilde{\Pi} & \longrightarrow & \Pi \\
\downarrow & & \downarrow{\scriptstyle \widetilde{q}} & & \downarrow \\
JG & \longleftarrow & q_0^* JG & \longrightarrow & \text{Hol} \\
\downarrow & & \downarrow & & \downarrow \\
G_0 \times G_0 & \xleftarrow[q_0 \times q_0]{} & \widetilde{M} \times \widetilde{M} & \longrightarrow & M \times M
\end{array}$$

in which all four squares are pull-backs; now $\widetilde{q}$ is étale, so $\Pi(M,r) \to \text{Hol}(M,r)$ is étale. Composition in JG thus induces a continuous composition on $\text{Hol}(M,r)$, and $\Pi(M,r) \to \text{Hol}(M,r)$ is thus an *étale* homomorphism of groupoids.

It is clear that the construction of $\text{Hol}(M,r)$, unlike that of (G and) JG, does not depend on the choice of atlas (U_i, R_i).

We shall prove, however, that, modulo essential equivalence, the étale groupoid JG is determined by (M,r) alone. Any other choice of atlas would give another étale groupoid G', essentially equivalent to Π, so G and G' are essentially equivalent. By the construction of [18, Section 7], this implies that there is a topological groupoid H and essential equivalence functors $H \to G$ and $H \to G'$, with H étale since G and G' are. So to prove the essential equivalence of G and G', it suffices to prove

Proposition 4.3. *If $f : H \to G$ is an essential equivalence between étale topological groupoids, then JH is essentially equivalent to JG.*

Proof. As shown by Pronk [24, Lemma 1.3.2], the assumptions of the proposition imply that $f_0 : H_0 \to G_0$ is étale. Therefore, it is clear that the square

$$\begin{array}{ccc}
\Gamma H_0 & \longrightarrow & \Gamma G_0 \\
\downarrow & & \downarrow \\
H_0 \times H_0 & \longrightarrow & G_0 \times G_0
\end{array} \tag{10}$$

is a pull-back. Also, the assumption that f is an essential equivalence functor implies that

$$\begin{array}{ccc}
H_1 & \longrightarrow & G_1 \\
\downarrow & & \downarrow \\
H_0 \times H_0 & \longrightarrow & G_0 \times G_0
\end{array} \tag{11}$$

is a pull-back. It follows that we have a pull-back

$$\begin{array}{ccc}
H_1 & \longrightarrow & G_1 \\
\downarrow & & \downarrow \\
\Gamma H_0 & \longrightarrow & \Gamma G_0
\end{array} \tag{12}$$

The left hand vertical map here associates to an arrow $h \in H_1$ the homeomorphism germ on H_0 to which it gives rise. Similarly for $\boldsymbol{G}$. But the space of arrows of $J\boldsymbol{H}$, respectively $J\boldsymbol{G}$, are just the images of these vertical maps (with subspace topology from ΓH_0, respectively ΓG_0). The image factorization of a pull-back diagram is for trivial reasons a pull-back, so

$$\begin{array}{ccc} (J\boldsymbol{H})_1 & \longrightarrow & (J\boldsymbol{G})_1 \\ \downarrow & & \downarrow \\ \Gamma H_0 & \longrightarrow & \Gamma G_0 \end{array} \tag{13}$$

is a pull-back. Concatenating this with the pull-back diagram (10) yields a pull-back diagram witnessing that $J\boldsymbol{H} \to J\boldsymbol{G}$ is full and faithful. Since $\boldsymbol{H} \to \boldsymbol{G}$ is essentially surjective on objects, it follows that so is $J\boldsymbol{H} \to J\boldsymbol{G}$ (in fact, the space of objects of $J\boldsymbol{H}$ agrees with that of $\boldsymbol{H}$, and similarly for $\boldsymbol{G}$). □

5. Constructing local equivalence relations from groupoids

In this section, we shall prove a converse of our construction of an étale groupoid out of a local equivalence relation, namely we shall prove the following result. (In the next section, we will reinterpret this result as a characterization theorem in topos theory.)

Theorem 5.1. *For any étale topological groupoid $\boldsymbol{G}$, there exists a topological space M, equipped with a locally connected, locally simply connected local equivalence relation r, such that there is an equivalence of categories*

$$\boldsymbol{BG} \simeq \mathrm{sh}(M, r)$$

between $\boldsymbol{G}$-equivariant sheaves and r-sheaves on M.

The space M in the statement of the theorem will in fact be constructed as (a variant of) the classifying space $B\boldsymbol{G}$ of the topological groupoid $\boldsymbol{G}$, which generalizes Milnor's construction of the classifying space for a topological group (cf. [30]). As in the case of groups, one has on $B\boldsymbol{G}$ a ("universal") principal $\boldsymbol{G}$-bundle $E\boldsymbol{G}$,

$$\begin{array}{ccc} E\boldsymbol{G} & \xrightarrow{\pi} & G_0 \\ \downarrow{\scriptstyle \phi} & & \\ B\boldsymbol{G} & & \end{array} \tag{14}$$

and since $G_1 \to G_0$ is etale, then so is ϕ, see below.

The strategy is to consider the equivalence relation R_π on $E\boldsymbol{G}$ given by $(x, y) \in R_\pi$ iff $\pi(x) = \pi(y)$. It provides an atlas for a local equivalence relation r_π on $E\boldsymbol{G}$ which we shall prove "descends" to a local equivalence relation r on $B\boldsymbol{G}$ (meaning that $\phi^*(r) = r_\pi$), and this r will be proved to have the property claimed in the theorem.

We begin by some general considerations concerning principal $\boldsymbol{G}$-bundles.

Recall that a left $\boldsymbol{G}$-bundle over a space M is a space P equipped with a map $\phi : P \to M$ and a continuous fibrewise left action by $\boldsymbol{G}$, given by maps

$$P \xrightarrow{\pi} G_0$$

$$G_1 \times_{G_0} P \xrightarrow{\cdot} P .$$

To say that the action is fibrewise means that

$$\phi(g \cdot p) = \phi(p)$$

for any $p \in P$ and $g \in G_1$ for which $g \cdot p$ is defined, i.e., for which $d_0(g) = \pi(p)$. Also, the action should be unitary and associative, in the evident sense. The $\boldsymbol{G}$-bundle is called *principal* if $\phi : P \to M$ is an open surjection, and the map

$$G_1 \times_{G_0} P \to P \times_M P$$

given by $(g, p) \mapsto (g \cdot p, p)$ is a homeomorphism. Thus, in such a principal bundle, for any two points p, q in the same fibre of $\phi : P \to M$, there exists a unique arrow $g : \pi(p) \to \pi(q)$ in $\boldsymbol{G}$ with $g \cdot p = q$. It is not difficult to prove that for such a principal $\boldsymbol{G}$-bundle, the étaleness of the structural maps for $\boldsymbol{G}$ implies the étaleness of ϕ (cf., e.g., [19]).

Recall also that when P is a $\boldsymbol{G}$-bundle, as above, one may construct a new topological groupoid $P_{\boldsymbol{G}}$ with P as space of objects, and with arrows $p \to q$ those arrows $g \in G_1$ for which $g \cdot p$ is defined and equals q; so the space of objects of $P_{\boldsymbol{G}}$ is $G_1 \times_{G_0} P$. There is an evident functor $\pi : P_{\boldsymbol{G}} \to \boldsymbol{G}$ given on objects by $\pi : P \to G_0$. Pull-back along this functor defines a functor taking $\boldsymbol{G}$-equivariant sheaves to $P_{\boldsymbol{G}}$-equivariant ones; we denote it

$$\boldsymbol{B}\pi^* : \boldsymbol{BG} \to \boldsymbol{B}P_{\boldsymbol{G}}.$$

We next consider, for a principal bundle as above, the local equivalence relation r_π on P obtained from the global equivalence relation R_π on P given by $(x, y) \in R_\pi$ iff $\pi(x) = \pi(y)$. We will prove that this local equivalence relation descends to an local equivalence relation on M:

Lemma 5.2. *There exists a (unique) local equivalence relation r on M such that $r_\pi = \phi^*(r)$.*

Proof. As we pointed out above, the map $\phi : P \to M$ is necessarily étale, and hence has enough local sections. For any open set $U \subseteq M$ and section $a : U \to P$ of ϕ, consider the equivalence relation R_a on U having as equivalence classes the sets

$$\phi\big(\pi^{-1}(t) \cap U\big), \quad t \in G_0.$$

It suffices to show that these equivalence relations R_a, for all open $U \subseteq M$ and all sections a, are locally compatible. For in this case they together define a local equivalence relation r on M which evidently has the property that $\phi^*(r) = r_\pi$ on P. So, for compatibility, consider any point $x \in M$ and two sections $a : U \to P$ and $b : V \to P$ of

ϕ defined on neighbourhoods U and V of x. We need to find a smaller neighbourhood $W \subseteq U \cap V$ of x on which R_a and R_b agree. To this end, let $g : \pi(a(x)) \to \pi(b(x))$ be the unique arrow in G_1 with

$$g \cdot a(x) = b(x).$$

Since $d_0 : G_1 \to G_0$ is an étale map, there is a small open neighbourhood K of $\pi(a(x))$ in G_0 so that there is a section $\widetilde{g} : K \to G_1$ of d_0 with $\widetilde{g}(\pi(a(x))) = g$. Choose $W \subseteq U \cap V$ so small that for any point $p \in W$,

$$\widetilde{g}\big(\pi\big(a(p)\big)\big) \cdot a(p) = b(p).$$

For any two points $p, q \in W$ with $(p, q) \in R_a$, i.e., with $\pi(a(p)) = \pi(a(q))$, one then has

$$\begin{aligned} \pi\big(b(p)\big) &= \pi\big(\widetilde{g}\big(\pi(a(p))\big) \cdot a(p)\big) = d_1\widetilde{g}\big(\pi(a(p))\big) \\ &= d_1\widetilde{g}\big(\pi(a(q))\big) = \pi\big(\widetilde{g}\big(\pi(a(q))\big) \cdot a(q)\big) = \pi\big(b(q)\big), \end{aligned}$$

hence $(p, q) \in R_b$. This shows that $R_a \cap (W \times W) \subseteq R_b$. A symmetric argument will produce a neighbourhood W' so that $R_b \cap (W' \times W') \subseteq R_a$, so that R_a and R_b agree on $W \cap W'$, as required.

This proves the lemma. □

If the map π has sufficiently good properties, the local equivalence relation r_π on P will be open and locally path connected, and hence so will the local equivalence relation r on M, so that our theory applies; in particular, we will have full subcategories $\mathrm{sh}(M, r) \subseteq \mathrm{sh}(M)$ and $\mathrm{sh}(P, r_\pi) \subseteq \mathrm{sh}(P)$. Finally, we let $\boldsymbol{B}(P_G, r_\pi)$ be the category of those P_G-equivariant sheaves which, as sheaves on P, are r_π-invariant.

The various categories and functors considered fit into the diagram below. In this, the top composite is just π^*. Note that a sheaf of form $\pi^*(F)$ on P is evidently r_π-invariant, so that π^* factors as indicated; similarly for $\boldsymbol{B}\pi^* : \boldsymbol{BG} \to \boldsymbol{B}P_G$.

$$\begin{array}{ccccc}
\mathrm{sh}(P) & \longleftarrow & \mathrm{sh}(P, r_\pi) & \xleftarrow{(1)} & \mathrm{sh}(G_0) \\
\uparrow & & \uparrow & & \uparrow \\
\boldsymbol{B}(P_G) & \longleftarrow & \boldsymbol{B}(P_G, r_\pi) & \xleftarrow[(2)]{} & \boldsymbol{BG} \\
{\scriptstyle (3)}\uparrow & & \uparrow{\scriptstyle (4)} & & \\
\mathrm{sh}(M) & \longleftarrow & \mathrm{sh}(M, r) & &
\end{array}$$

Consider now the case where π is an open map with connected fibres. Then the functor $\pi^* : \mathrm{sh}(G_0) \to \mathrm{sh}(P)$ is in fact full and faithful. Also the functor $\boldsymbol{B}\pi^* : \boldsymbol{BG} \to \boldsymbol{B}P_G$ is full and faithful; this is an easy diagram chase argument utilizing that π^* is full and faithful, and that the space of arrows of P_G is $\pi^*(G_1)$ (so all the horizontal arrows in the diagram represent full inclusions). A similar argument proves that if for a sheaf F on G_0 the sheaf $\pi^*(F)$ on P is provided with a P_G-action, then this action comes from a unique $\boldsymbol{G}$-action on F, via π^*. In other words, the composite of the two top horizontal squares in the diagram is a pull-back of categories.

Also the composite of the two left hand squares is a pull-back; this follows from the fact that r and r_π correspond under the local homeomorphism ϕ, and that the property of being an r-sheaf (respectively r_π-sheaf) is a local property. Finally, the top left hand square is a pull-back, by definition of the category $\boldsymbol{B}(P_G, r_\pi)$. From elementary properties of pull-back diagrams, it follows that the two remaining squares are also pull-backs.

We now have

Lemma 5.3. *Let $P \to M$ be a principal bundle for the étale topological groupoid $\boldsymbol{G}$, with structural map $\pi : P \to G_0$. Assume that the map π is open and such that r_π is locally connected, and assume that the fibres of π are connected and simply connected. Then the funcors* (2) *and* (4) *in the diagram are equivalences of categories, and in particular* $\mathrm{sh}(M, r) \simeq \boldsymbol{BG}$.

Proof. The proof hinges on consideration of the above diagram, in conjunction with the following result. This result is a corollary of the Theorem 2.5, and may be of use in other contexts, so that we state it in form of a separate proposition:

Proposition 5.4. *Let $\pi : P \to X$ be any open continuous map such that the fibres of it are connected and simply connected; also assume that the local equivalence relation r_π on P induced by π is locally connected. Then $\pi^* : \mathrm{sh}(X) \to \mathrm{sh}(P)$ establishes an equivalence*

$$\mathrm{sh}(X) \simeq \mathrm{sh}(P, r_\pi).$$

Proof. It is clear that any sheaf of the form $\pi^*(F)$ carries the structure of the r_π-sheaf. Conversely, assume $E \to P$ is an r_π-sheaf. So there exists a covering of P by open sets U_i such that the equivalence classes of $R_\pi | U_i$ are connected, and E is constant along these equivalence classes (where R_π, as above, is the global equivalence relation (kernel pair) of π). Thus E is locally constant along the equivalence classes of R_π. But these are simply connected, by assumption, so E is constant along them. Theorem 2.5 now implies that R_π act continuously on E, and thus by descent theory (cf. the discussion in the example at the end of Section 2), E is of form $\pi^*(F)$ for some sheaf F on X. This proves the proposition. □

From the proposition it now follows that the functor (1) in the diagram is an equivalence of categories, and since the square under it is a pull-back, it follows that the functor (2) is an equivalence. Also, the functor (3) is an equivalence; for, the assumption that the $\boldsymbol{G}$-action on P is principal is equivalent to the statement that the action groupoid P_G is isomorphic to the equivalence relation $P \times_M P$, but then $\boldsymbol{B}(P_G) \simeq \boldsymbol{B}(P \times_M P)$, the category of actions of this equivalence relation on sheaves on P. But by descent theory, (as in the proof of the Proposition), $\boldsymbol{B}(P \times_M P) \simeq \mathrm{sh}(M)$, via the functor (3), so (3) is an equivalence. Since the square next to it is a pull-back, it follows that the functor (4) is an equivalence of categories. This proves Lemma 5.3. □

Remark. The local equivalence relations r and r_π are both locally simply connected if there exists a basis of open sets U in P with the property that they intersect each fiber in a connected and simply connected (or empty) set.

To prove Theorem 5.1, it now suffices to construct a suitable principal $\boldsymbol{G}$-bundle. We will use a variant of the standard universal bundle $E\boldsymbol{G} \to B\boldsymbol{G}$ of (Milnor–)Buffet–Lor [5], considered in [19]. Its base space

$$M = B\boldsymbol{G}$$

is the usual [30] classifying space, constructed as the geometric realization of the nerve $N(\boldsymbol{G})$ of $\boldsymbol{G}$. (This nerve is a simplicial *space*, and the appropriate realization is the "thick" one of [30].) For the construction of the total space $P = E\boldsymbol{G}$ of the bundle, consider the topological groupoid $\mathrm{Dec}(\boldsymbol{G})$, the objects of which are the arrows $g : x \to y$ in $\boldsymbol{G}$, while the arrows from g to g' are the commutative triangles in $\boldsymbol{G}$ of form $g \circ h = g'$; thus there is a unique arrow $h = g^{-1} \circ g'$ from g' to g in $\mathrm{Dec}(\boldsymbol{G})$ iff g and g' have the same codomain. $\mathrm{Dec}(\boldsymbol{G})$ inherits an evident toplogy from $\boldsymbol{G}$. We define

$$P = B\,\mathrm{Dec}(\boldsymbol{G})$$

to be the classifying space of this groupoid. The domain map of $\boldsymbol{G}$ defines a homomorphism $\mathrm{Dec}(\boldsymbol{G}) \to \boldsymbol{G}$, hence a map of classifying spaces $\phi : P \to M$.

The groupoid $\boldsymbol{G}$ acts on P by composition. In fact, it acts on the simplicial space $N(\mathrm{Dec}(B\boldsymbol{G}))$, whose n-simplices are the strings

$$y \xleftarrow{g} x_0 \xleftarrow{h_1} \cdots \xleftarrow{h_n} x_n \,.$$

The structure map $\pi_n : N_n(\mathrm{Dec}(B\boldsymbol{G})) \to G_0$ sends such a string to the point y, while an arrow acts on this string by left composition,

$$u \cdot \langle g, h_1, \ldots, h_n \rangle = \langle ug, h_1, \ldots, h_n \rangle.$$

In this way, each space $N_n(\mathrm{Dec}(\boldsymbol{G}))$ comes equipped with a $\boldsymbol{G}$-action. This action, with the projection maps $\phi_n : N_n(\mathrm{Dec}(\boldsymbol{G})) \to N_n(\boldsymbol{G})$ sending $\langle g, h_1, \ldots, h_n \rangle$ to $\langle h_1, \ldots, h_n \rangle$, makes $N_n(\mathrm{Dec}(\boldsymbol{G}))$ into a principal $\boldsymbol{G}$-bundle over $N_n(\boldsymbol{G})$. By geometric realization, one thus obtains a principal $\boldsymbol{G}$-bundle structure on $\phi : P \to M$.

Observe that since the groupoid $\boldsymbol{G}$ is étale, so are the maps $\pi_n : N_n(\mathrm{Dec}(\boldsymbol{G})) \to G_0$. Thus $N_\bullet(\mathrm{Dec}(\boldsymbol{G}))$ is in fact a simplicial *sheaf* over G_0. Its fiber over a point $y \in G_0$ is the nerve of the (discrete) groupoid $\boldsymbol{G}/y$, and the fiber $\pi^{-1}(y)$ of $\pi : P \to G_0$ is the classifying space $B(\boldsymbol{G}/y)$ of this groupoid. Since $\boldsymbol{G}/y$ has a terminal object, this space is evidently contractible.

The following lemma shows that the map $\pi : P \to G_0$ is open and has the property that the local equivalence relation r_π defined by it (as above) is locally path connected (cf. the preceding remark), so that all conditions in Lemma 5.3 are in fact satisfied by the bundle constructed.

Lemma 5.5. *Let $E_\bullet$ be a simplicial sheaf on a space B, and let $|E_\bullet|$ be its geometric realization, with canonical map $p : |E_\bullet| \to B$. Then p is an open map, and $|E_\bullet|$ has a basis of open sets which intersect each fiber of p in a contractible set.*

Proof. The proof is a straightforward adaption of the standard construction of contractible neighbourhoods in the geometric realization of a simplicial *set*, see, e.g., [10, p. 45–47].

In detail, let us write $p_n : E_n \to B$ for the étale map corresponding to the sheaf E_n on B, and let us say that an open set $W \subseteq E_n$ is *p-small* if $p_n|W$ is a homeomorphism. With the standard skeletal filtration $|E|^{(n)}$ of $|E|$, there is for each n a canonical quotient map $\sigma^n : E_n \times \Delta^n \to |E|^{(n)}$, whose restriction to $E_n \times \partial\Delta^n$ maps into $|E|^{n-1}$, to give a pushout

$$\begin{array}{ccc} E_n \times \partial\Delta^n & \xrightarrow{\sigma_n} & |E|^{(n-1)} \\ \downarrow & & \downarrow \\ E_n \times \Delta^n & \xrightarrow[\sigma_n]{} & |E|^{(n)} \end{array}$$

By construction, $|E|$ has the inductive topolgy with respect to this skeletal filtration. Since each projection $E_n \times \Delta^n \to E_n \to B$ is clearly an open map, so is the map $|E|^{(n)} \to B$ for each n; so $p: E \to B$ is open. Next, following [10], we describe a procedure for extending an open set $U_{n-1} \subseteq |E|^{(n-1)}$ to an open set $U_n \subseteq |E|^{(n)}$: for $x \in E_n$ and each $t \in \Delta^n$ with $\sigma^n(x,t) \in U_{n-1}$, choose a small (in the above sense) neighbourhood W_x of x in E_n and a convex open neighbourhood V_t of t in Δ^n, such that $\sigma^n(W_x \times (V_t \cap \partial\Delta^n)) \subseteq U_{n-1}$. Let U_n be the union of the images under σ^n of all these open sets $W_x \times V_t \subseteq E_n \times \Delta^n$:

$$U_n = \bigcup \{\sigma^n(W_x \times V_t) \mid (x,t) \in E_n \times \Delta^n,\ \sigma^n(x,t) \in U_{n-1}\}.$$

Now suppose ξ is any point of $|E|$. Then there exists a smallest k such that $\xi = \sigma^k(x_0, t_0)$ for some x_0 in E_k and some interior point t_0 of Δ^k. Let $U_k = \sigma^k(W \times V)$, where W is a p-small neighbourhood of x_0 in E_k and V is a convex open set in Δ^k containing t. Starting from this set U_k, define $U_k \subseteq U_{k+1} \subseteq \cdots$ by the procedure just described, and let $U = \bigcup_{n \geqslant k} U_n$. This set U is an open neighbourhood of ξ in $|E|$, and the collection of all open neighbourhoods constructed in this way is a basis for the topology on $|E|$. Moreover, since all neighbourhoods in the various E_n used in the construction are p-small, the intersection U_b of such an open set $U \subseteq |E|$ with a fiber $p^{-1}(b)$ is an open set of standard form in the realization $|E|_b$ of the simplicial set E_b, hence is contractible, see [10, p. 47].

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Appendix A

In the previous sections, we have proved that every locally simply connected local equivalence relation on a space M has a monodromy groupoid essentially equivalent to an étale groupoid (Theorem 3.6), and conversely, that every étale topological groupoid arises this way (Theorem 5.1). In this appendix, we will rephrase these results in the language of topos theory.

Recall (cf., e.g., [16, p. 127]) that a *topos* is a category (equivalent to the category) of sheaves on a small site. One can equivalently define a topos to be a category which satisfies the exactness conditions of "Giraud's theorem" ([1, p. 303] and [16, p. 575]). In the following, we will use no other topoi than those listed in the following examples.

Example A.1. (a) For any topological space X, the category $\mathrm{sh}(X)$ of sheaves on X is a topos.

(b) For any topological groupoid $\boldsymbol{G}$, the category $\boldsymbol{BG}$ of all $\boldsymbol{G}$-equivariant sheaves is a topos (the *classifying topos* of $\boldsymbol{G}$).

(c) If $\mathcal{E}$ is a topos, and U is an object of $\mathcal{E}$, then the "comma category" $\mathcal{E}/U$ is a topos (see, e.g., [16, p. 190] and [1, p. 365]).

Recall also (from, e.g., [16, p. 348]) the definition of a (geometric) *morphism* $f:\mathcal{E} \to \mathcal{F}$ of topoi, as a pair of functors $f_*:\mathcal{E} \to \mathcal{F}$ and $f^*:\mathcal{F} \to \mathcal{E}$ (called direct and inverse image, respectively), such that f^* is left exact, and left adjoint to f_*. This morphism f is said to be an *equivalence*, denoted

$$f:\mathcal{E} \simeq \mathcal{F},$$

if f^* and f_* together define an equivalence of categories. Again, we will only use some simple examples:

Example A.2. (a) A continuous map $f:X \to Y$ of topological space induces a morphism of topoi $f:\mathrm{sh}(X) \to \mathrm{sh}(Y)$, [16, p. 348].

(b) A continuous homomorphism $\phi:\boldsymbol{G} \to \boldsymbol{H}$ between topological groupoids induces a morphism between their classifying topoi $\boldsymbol{B}\phi:\boldsymbol{BG} \to \boldsymbol{BH}$. If $\phi:\boldsymbol{G} \to \boldsymbol{H}$ is an essential equivalence, then $\boldsymbol{B}\phi:\boldsymbol{BG} \simeq \boldsymbol{BH}$ (cf. Section 4 above and [18]).

(c) if $\mathcal{E}$ is a topos and U is an object of $\mathcal{E}$, the functor $Y \mapsto (p_1:U \times Y \to U)$ is the inverse image of a geometric morphism $\mathcal{E}/U \to \mathcal{E}$.

Definition A.3 [1, p. 482]. A topos $\mathcal{E}$ is called an *étendue* if there exists an object $U \in \mathcal{E}$ and a topological space such that

(i) $U \to 1_{\mathcal{E}}$ is epi ($1_{\mathcal{E}}$ is the final object of $\mathcal{E}$);

(ii) there exists an equivalence of topoi $\mathcal{E}/U \simeq \mathrm{sh}(X)$.

These étendue topoi were introduced by Grothendieck and Verdier in the context of foliations and other local equivalence relations. In particular, it was conjectured in [1, p. 489] that for a suitable local equivalence relation r on a space M, the category $\mathrm{sh}(M,r)$ of r-invariant sheaves is an étendue if r satisfies certain (rather abstract) conditions.

The following theorem provides two concrete characterizations of étendue topoi. The first description, expressed by the equivalence (i) $\Leftrightarrow$ (ii), is proved in [1]. The second characterization, expressed by (ii) $\Leftrightarrow$ (iii), summarizes our earlier results. In particular, the implication (iii) $\Rightarrow$ (i) proves a form of the conjecture of Grothendieck and Verdier.

Theorem A.4. *For a topos $\mathcal{E}$, the following are equivalent:*

(i) *$\mathcal{E}$ is an étendue topos.*

(ii) *There exists an étale topological groupoid $\boldsymbol{G}$ such that $\mathcal{E}$ is equivalent to the category $\boldsymbol{BG}$ of $\boldsymbol{G}$-equivariant sheaves.*

(iii) *There exists a space M and an (open, locally path connected) locally simply connected local equivalence relation r on M such that $\mathcal{E}$ is equivalent to the category* $\mathrm{sh}(M,r)$ *of r-invariant sheaves on M.*

(A further, equivalent, condition, follows by combining [26] and [12].)

Proof. As said, (i) $\Leftrightarrow$ (ii) is in SGA4 ([1, p. 481]); the implication (iii) $\Rightarrow$ (ii) is Theorem 3.6 above, while Theorem 5.1 provides the implication (ii) $\Rightarrow$ (iii). □

As pointed out in the introduction, we can now apply the Comparison Theorem of [19] to characterize the weak homotopy type of $\mathrm{sh}(M,r)$.

Corollary A.5. *For any (locally path connected, open) locally simply connected local equivalence relation r on a space M, the topos* $\mathrm{sh}(M,r)$ *has the same weak homotopy type as the classifying space $B\Pi_1(M,r)$ of its monodromy groupoid.*

Proof. By Theorem 3.6, $\mathrm{sh}(M,r) \simeq \boldsymbol{BG}$ for an étale groupoid essentially equivalent to $\Pi_1(M,r)$. Furthermore, by loc. cit. the canonical map $B\boldsymbol{G} \to \boldsymbol{BG}$ is a weak homotopy equivalence. Finally, the essential equivalence $\boldsymbol{G} \to \Pi_1(M,r)$ induces a weak homotopy equivalence of classifying spaces $B\boldsymbol{G} \to B\Pi_1(M,r)$ (cf. [11]).

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