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EXPOSÉ 9

Remarks on the Maurer-Cartan forms

by

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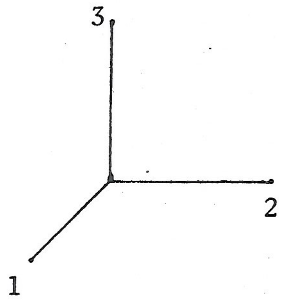
1. Maurer-Cartan for the set of frames

A frame in physical space consists of three rigid rods of length one meter, fitted rigidly together at a point so as to form right angles with each other. Furthermore, the three rods are numbers 1, 2, and 3, and in this order, they are positively oriented. So a frame looks like this:



The point where the three rods meet is called the base-point 0 of the frame.

The left Maurer-Cartan form associates to a given pair of neighbouring frames (0,1,2,3) and (0',1',2',3') the coordinates of the latter in terms of the former (clearly, a frame defines a coordinate system in space).

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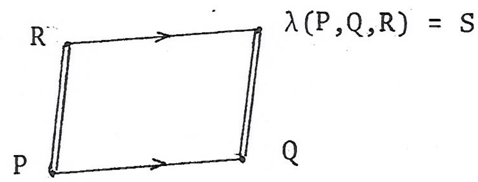
* The reader only interested in Maurer-Cartan for groups may proceed directly to §2.

The *right* Maurer-Cartan form associates to a pair of neighbouring frames (as above) the unique infinitesional *motion* that moves the first frame to the second. (It defines a motion of the whole space).

The algebraic structure which the set G of frames in space has, and which makes these two Maurer-Cartan forms possible, is a *pregroup* structure, [3], which means that we have a trinary operation $\lambda(P,Q,R)$ on the set of frames: given frames P,Q,R, let us connect P and R *rigidly* to each other by means of some long enough system of rigid rods, and let us then *move* P so as to coincide with Q. Then $\lambda(P,Q,R)$ is defined as the frame R has been moved to.

We may represent this state of affairs by a diagram in which points represent frames, double lines represent rigid connections, and arrows represent motions:

(1.1)



We now put ourselves in the context of synthetic differential geometry. Recall that a *vector field* on an object M is a map $X : M \times D \longrightarrow M$ with $X(m,0) = m \quad \forall m \in M$. A vector field X on G (= set of frames) is called *left* invariant if for any $P,Q \in G$ and $d \in D$ we have

$$\lambda(P,Q,X(P,d)) = X(Q,d) ,$$

which expresses that the coordinate expression of the infinitesional X-transforms of any frame P (expressed in terms of P) is the same everywhere (i.e. does not depend on P). Similarly, a vector field Y on G is right invariant if $\forall P,R \in G \quad \forall d \in D$:

$$\lambda(P, Y(P,d), R) = Y(R,d) ,$$

which means that the infinitesional transformations of Y are (infinitesional) motions of space.

Clearly, the notion of (left- and right-) invariant vector field on G depends only on the algebraic structure λ, which is what we call a pregroup structure; this means that λ satisfies six equations [3] which essentially say that "G is a group, but we have forgetten which element is the unit", and which imply that each tangent vector $t : D \longrightarrow G$ is member of precisely one left invariant vector field X (so $X(t(0),d) = t(d)$), and of precisely one right invariant vector field Y. (This is anyway clear from the interpretation 'coordinate expression', 'motion', respectively.)

The 'set' (internal object) of left (respectively right) invariant vector fields on an object G with a pregroup structure λ carries (if G is sufficiently nice, say infinitesimally linear ([2],[4],[6]) a vector space (= R-module) structure, and is closed under the Lie bracket formation of vector fields assuming G satisfies the Axiom 2 of [6]. We denote by $LG_{\#}$ and $LG^{\#}$, respectively, the Lie algebra (-object) of left (resp. right) invariant vector fields on G.

The left Maurer-Cartan form Ω on G is the differential 1-form on G with values in the vector space $LG_{\#}$, given as follows

$\Omega(t)$:= the unique left invariant vector field on G of which t is a member.

For the case where G = (frames in physical space), our previous remarks imply that $\Omega(t)(d)$ can be interpreted as the coordinate expression of $t(d)$ in terms of the frame (= coordinate system) $t(0)$. (This coordinate expression involves six numbers, which traditionally are denoted $\omega_1, \omega_2, \omega_3, \omega_{12}, \omega_{13}, \omega_{23}$; $LG_{\#}$ is in fact a 6-dimensional Lie algebra, with a canonical basis.)

The right Maurer-Cartan form is introduced dually, but we shall not consider it. The Theorem which we shall now state is valid verbatim for the right form also.

To any two tangent vectors $\xi, \eta : D \longrightarrow G$ at the same point $\xi(0) = \eta(0)$ (= P, say), we define a "2-tangent", [5] :

$$D \times D \xrightarrow{\ \xi \cdot \eta\ } G$$

by putting

$$(\xi \cdot \eta)\ (d_1,d_2) = \lambda(P,\ \xi(d_1),\ \eta(d_2)) .$$

It follows from the equational laws for λ that $(0,0) \mapsto P$, see [3] Axiom 1). We shall assume that the vector space $LG_{\#}$ is Euclidean ([1],[6]) so that exterior differentiation of differential forms with values in $LG_{\#}$ can be defined, according to [5] . Under these assumptions, we have "Maurer-Cartan's formula":

Theorem $(d\Omega)\ (\xi \cdot \eta) = -[\Omega(\xi), \Omega(\eta)]$.

To prove the theorem, we convert it into a theorem about groups. Now that P is fixed, we construct a binary operation on G by putting

$$Q \cdot R := \lambda(P,Q,R).$$

This is actually a group structure, due to the six equations of [3] , with P as its neutral element e (the structure λ can be reconstructed from the group structure as

$$\lambda(T,U,V) = U \cdot T^{-1} \cdot V).$$

It likewise follows easily from [3] that a vector field X on G is left invariant in the sense of the pregroup structure if and only if it is left invariant with respect to group structure, meaning

$$X(h \cdot g,d) = h \cdot X(g,d) \qquad \forall g,h \in G \quad \forall d \in D .$$

Finally, to the two tangent vectors ξ and η considered, which are now tangent vectors at e, the map $\xi \cdot \eta$ described above can now be described simply by

$$(\xi \cdot \eta)\ (d_1,d_2) = \xi(d_1) \cdot \eta(d_2) ,$$

using the group structure. It is now clear that the theorem above follows from the corresponding theorem for groups, which we state in the next § .

2. *Maurer-Cartan for groups*

Let G be a group with neutral element e. We work in the context of synthetic differential geometry, and we assume that G is infinitesimally linear [4] , [6] , and satisfies Axiom 2 of [6] . We also assume that the tangent vector space T_eG at e is Euclidean ([2],[6]) . There is a natural 1-1 correspondance between tangent vectors ξ at e

$$\xi : D \to G \qquad\qquad \xi(0) = e$$

and left invariant vector fields X on G

$$X : G \times D \to G \quad \text{s.t.} \quad X(g,0) = g \quad \forall g \in G$$
$$\text{and} \quad X(h \cdot g,d) = h \cdot X(g,d) \quad \forall g,h \in G,$$

given by associating to ξ the following X :

$$X(g,d) = g \cdot \xi(d) \quad .$$

A tangent vector t at an arbitrary $g \in G$ is a member of a unique left invariant vector field X given by $X(h,d) = h \cdot g^{-1} \cdot t(d)$, which in turn corresponds to the tangent vector ξ at e given by

$$\xi(d) = g^{-1} \cdot t(d) \quad (= t(0)^{-1} \cdot t(d)) \quad \forall d \in D \ .$$

The left Maurer-Cartan form Ω on G is the 1-form with values in T_eG , given by "$t \mapsto \xi$" , i.e.

$$\Omega(t) = [d \longmapsto t(0)^{-1} \cdot t(d)] \ . \tag{2.1}$$

(Alternatively, under the bijection above, $\Omega(t)$ is the unique left invariant vector field on G of which t is a member).

Note that for $\xi \in T_e G$, $\Omega(\xi) = \xi$. Also, for any $t : D \to G$ and any h, we have

(2.2) $$\Omega(h \cdot t) = \Omega(t) ,$$

(2.3) $$\Omega(t \cdot h) = h^{-1} \cdot \Omega(t) \cdot h ;$$

to see the latter, say, just apply the definition (2.1)

$$\begin{aligned} \Omega(t \cdot h) &= [d \longrightarrow (t(0) \cdot h)^{-1} \cdot (t(d) \cdot h)] \\ &= [d \longrightarrow h^{-1} \cdot t(0) \cdot t(d) \cdot h] \\ &= [d \longrightarrow h^{-1} \cdot \Omega(t) \cdot h] . \end{aligned}$$

The "set" of left invariant vector fields is stable under Lie bracket formation of vector fields, [2], and the identification of these with tangent vectors at $e \in G$ therefore gives rise to a Lie bracket on the "set" $T_e G$ of such tangent vectors. The latter Lie bracket can be described explicitely by

(2.0) $$[\xi, \eta]\,(d_1 \cdot d_2) = \xi(d_1) \cdot \eta(d_2) \cdot \xi(-d_1) \cdot \eta(-d_2)$$

which is also (modulo sign?) the formula of [6] .

For ξ, η tangent vectors at e, we define a map $\xi \cdot \eta : D \times D \longrightarrow G$ by

$$(\xi \cdot \eta)\,(d_1, d_2) = \xi(d_1) \cdot \eta(d_2)\ .$$

Theorem (Maurer-Cartan for groups). For $\xi, \eta \in T_eG$, we have

$$(2.4) \qquad (d\Omega)\,(\xi \cdot \eta) = -\,[\xi \cdot \eta]$$

Proof. It suffices to prove that for $\forall (d_1, d_2) \in D \times D$, the equation (2.4) holds when multiplied by $d_1 \cdot d_2$. (This is the principle of "cancelling universally quantified d's."). Now

$$d_1 \cdot d_2 \cdot d\Omega(\xi \cdot \eta) =$$

$$d_1\, \Omega(\xi) + d_2 \cdot \Omega(\xi(d_1) \cdot \eta) - d_1\, \Omega(\xi \cdot \eta(d_2)) - d_2 \cdot \Omega(\eta)$$

by the definition of coboundary for 1-forms [5] and the definition of $\xi \cdot \eta$ (recalling that $\xi(0) = \eta(0) = e$). Using (2.2), the second and fourth term cancel. Because of (2.3) (and $\Omega(\xi) = \xi$) we are left with

$$(2.5) \qquad d_1 \cdot (\xi - \eta(d_2)^{-1} \cdot \xi \cdot \eta(d_2))\ .$$

To this expression, we apply the following Lemma (putting $h = \eta(-d_2) = \eta(d_2)^{-1}$).

Lemma. Let $\xi \in T_eG$ and $h \in G$. Then, for any $d \in D$,

$$(2.6) \qquad (\xi - h \cdot \xi \cdot h^{-1})\,(d) = \xi(d) \cdot h \cdot \xi(-d) \cdot h^{-1}\ .$$

Proof. Note that $\xi - h \cdot \xi \cdot h^{-1}$ is a difference between two tangent vectors at $e \in G$, so the left hand side of the equation makes sense. To calculate the difference of tangent vectors, we must use infinitesimal linearity ([2], [4], [6]) of G and construct that unique.

$$\ell : D(2) \longrightarrow G$$

with

$$\ell(d,0) = \xi(d)$$

$$\ell(0,d) = (-h \cdot \xi \cdot h^{-1})(d) = (h \cdot \xi \cdot h^{-1})(-d)$$

then, by construction of fibrewise additive structure in tangent spaces, the left hand side of (2.6) will be given by $\ell(d,d)$. But we can manufacture such an ℓ explicitely, by putting.

$$\ell(d_1,d_2) = \xi(d_1) \cdot h \cdot \xi(-d_2) \cdot h^{-1} .$$

Putting $d_1 = d_2 = d$ gives the right hand side of (2.6).

Because of the lemma, (2.5) equals d_1 times the following tangent vector at e

$$\delta \longmapsto \xi(\delta) \cdot \eta(-d_2) \cdot \xi(-\ ) \cdot \eta(d_2) .$$

i.e. equals the tangent vector

$$\delta \longmapsto \xi(d_1 \cdot \delta) \cdot \eta(-d_2) \cdot \xi(-d_1 \cdot \delta) \cdot \eta(d_2). \tag{2.7}$$

On the other hand, $-d_1 \cdot d_2 \cdot [\xi, \eta]$ is the following tangent vector

$$\delta \longrightarrow (-d_1 \cdot d_2 \cdot [\xi, \eta])(\delta)$$
$$= [\xi, \eta]((d_1 \cdot \delta) \cdot (-d_2))$$
$$= \xi(d_1 \cdot \delta) \cdot \eta(-d_2) \cdot \xi(-d_1 \cdot \delta) \cdot \eta(d_2) .$$

by (2.0). This is the same as (2.7), proving the theorem.

We state a reformulation of the theorem which is more invariant (it also holds for the pregroup case). By [5] , for any 2-form θ on G and 2-vector $t : D \times D \to G$ the value $\theta(t)$ only depends on the restrictions t_1, t_2 of t to the two "axes" of $D \times D$, and we may denote this value $\theta(t_1, t_2)$.

For 1-forms w_1 and w_2 with values in a vector space equipped with a bilinear multiplication (denoted $[-, -]$, say), there is a unique 2-form $w_1 \wedge w_2$ such that

$$(w_1 \wedge w_2)(t) = [w_1(t_1), w_2(t_2)] - [w_1(t_2), w_2(t_1)]$$

where t, t_1, and t_2 are related as above. Now for the case where $t : D \times D \longrightarrow G$ is $\xi \cdot \eta$, as in the theorem, t_1 and t_2 are just ξ and η , respectively. So

$$d\Omega(\xi, \eta) = d\Omega(\xi . \eta)$$
$$= -[\Omega(\xi), \Omega(\eta)] \quad \text{(by the Theorem)}$$
$$= -\tfrac{1}{2}([\Omega(\xi), \Omega(\eta)] - [\Omega(\eta), \Omega(\xi)])$$

(since Lie bracket is skew)

$$= -\tfrac{1}{2}(\Omega \wedge \Omega)(\xi, \eta) .$$

Since this holds for all pairs of tangent vectors ξ, η at e, we conclude the equality of the two 2-forms in question, i.e. we have

$$d\Omega = -\tfrac{1}{2}\Omega \wedge \Omega \quad .$$

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