SOME PROBLEMS AND RESULTS IN SYNTHETIC FUNCTIONAL ANALYSIS*

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This somewhat tentative note aims at making a status about "functional analysis" in certain ringed toposes $\mathcal{E}, R$, in particular, duality theory for R-modules in $\mathcal{E}$. This is an area where knowledge still seems fragmentary, but where some of the known facts indicate the importance of such theory.

The first §, however, does not deal with specific toposes, but is of very general nature, making contact with the general theory of monads on closed categories, [3]-[6].

Most of the results and problems in this note were presented at the workshop. Closely related viewpoints were contained in Lawvere's workshop-contribution on Intensive and Extensive Quantities, which have also influenced the presentation given in the following pages.

1 Restricted double dualization monads

Let R be a ring in a topos $\mathcal{E}$ and let R-$\underline{Mod}$ denote the $\mathcal{E}$-category of R-modules (left R-modules, say). The $\mathcal{E}$-valued hom-functor for it is denoted $\underline{Hom}_R(-,-)$. If $X \in \mathcal{E}$ and $V \in R$-$\underline{Mod}$, there is a natural structure of R-module on $V^X = \Pi_X V$, and for any $U \in R$-$\underline{Mod}$

$$\underline{Hom}_R(U, V^X) \cong (\underline{Hom}_R(U, V))^X,$$

naturally in X, U, and V. This expresses that V^X is a cotensor of V with X (cf. e.g. [2]). Equivalently, it expresses that, for fixed V, we have a $\mathcal{E}$-strong left adjoint $V^{(-)} : \mathcal{E} \to (R\text{-}\underline{Mod})^{op}$ to the functor $\underline{Hom}_R(-, V)$. Thus we

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get by composition an $\mathcal{E}$-strong monad on $\mathcal{E}$, which is in an evident way a submonad of the monad "double dualization into V" (V just considered as an object of $\mathcal{E}$)

$$\underline{Hom}_R(V^X, V) \subseteq V^{V^X}.$$

(This latter monad arises similarly from the adjointness $V^{(-)} \dashv V^{(-)}$, cf. e.g. [4].) Hence the name "*restricted* double dualization".

We want to describe the monad-theoretic combinators [3], [5] and [6], for these monads. We shall be interested in the case $V = R$ mainly, and in fact mainly for the case where R is commutative. We introduce the notation

$$E(X) := \underline{Hom}_R(R^X, R).$$

It should be thought of as the R-module of distributions (with compact support) on X; this remark will be elaborated in §3. For the description of the combinators, we use notation as if $\mathcal{E}$ were the category of sets. Also, if $\phi \in E(X)$, $f \in R^X$, we shall sometimes write

$$\int f(s)\, d\phi(s) \quad \text{for} \quad \phi(f). \tag{1.1}$$

The *unit* $\eta = \eta_X : X \to E(X)$ of the monad E is easily seen to be given by, for $x \in X$

$$\eta(x)(f) = f(x).$$

Similarly, the *multiplication* $\mu = \mu_X : E(E(X)) \to E(X)$ is easily seen to be

$$(\mu(\Phi))(f) = \Phi(E(X) \xrightarrow{\mathrm{ev}_f} R)$$

where $\mathrm{ev}_f(\phi) = \phi(f)$.

The *"tensorial strength"* $t' = t_{X,Y}$ of the functor E (cf. [3] and [6])

$$E(X) \times Y \xrightarrow{t'} E(X \times Y)$$

is given by

$$t'(\phi, y)(f) = \phi(x \mapsto f(x, y))$$

where $f \in R^{X \times Y}$, and similarly for $t'' : X \times E(Y) \to E(X \times Y)$. Note that in the notation (1.1), we may write

$$t'(\phi, y)(f) = \int f(x, y)\, d\phi(x).$$

Combining the two tensorial-strength combinators t' and t'' with the multiplication μ, we obtain the *monoidal structure* ψ on E, given as the composite

$$E(X) \times E(Y) \xrightarrow{t'} E(X \times E(Y)) \xrightarrow{E(t'')} EE(X \times Y) \xrightarrow{\mu} E(X \times Y),$$

and similarly we get a monoidal structure $\tilde{\psi}$ when we interchange the role of t' and t''. The "set-theoretic" description of ψ is then, not surprisingly, (with $f \in R^{X \times Y}$)

$$\psi(\phi_1, \phi_2)(f) = \phi_1(x \mapsto \phi_2(y \mapsto f(x, y)))$$

or

$$\psi(\phi_1, \phi_2)(f) = \iint f(x, y)\, d\phi_2(y)\, d\phi_1(x), \tag{1.2}$$

whereas

$$\tilde{\psi}(\phi_1, \phi_2)(f) = \iint f(x, y)\, d\phi_1(x)\, d\phi_2(y).$$

In general (even for commutative R), $\psi(\phi_1, \phi_2) \neq \tilde{\psi}(\phi_1, \phi_2)$, but cases of equality will be encountered below for special X, Y.

For completeness, let us also describe the combinator λ, "the cotensorial strength", [5], of E

$$E(X^Y) \xrightarrow{\lambda} E(X)^Y;$$

we have

$$\lambda(\phi)(y)(f) = \quad \phi(X^Y \xrightarrow{\mathrm{ev}_y} X \xrightarrow{f} R)$$

where $\phi \in E(X^Y), y \in Y, f \in R^X$; or equivalently, $\lambda(\phi)$ is given by

$$y \mapsto \Big[f \mapsto \phi(\alpha \mapsto f(\alpha(y))) \Big]$$

where $\alpha \in X^Y$.

These considerations and descriptions apply whether or not R is commutative, and could be applied for any algebraic category over $\mathcal{E}$. For instance, if R is a commutative ring (respectively a $\mathbb{T}_\infty$-algebra, cf. [8]) in $\mathcal{E}$, we may consider R-$\underline{\mathrm{Alg}}$, the category of commutative R-algebras (respectively the category of $\mathbb{T}_\infty$-algebras in $\mathcal{E}$ under R). Then we have the restricted double dualization monad G,

$$G(X) := \underline{Hom}_{R-\mathrm{Alg}}(R^X, R)$$

(G for "Gelfand"), which is a submonad of E.

The description of the combinators $\eta, \mu, t', t'', \psi, \tilde{\psi}$ and λ are the same as for E, in fact these descriptions work for any restricted double dualization monad in any cartesian closed category.

2 Some calculations in certain presheaf categories

Let $\mathcal{C}$ be a small category of commutative rings, more precisely, $\mathcal{C}$ should be equipped with a functor $\Box$ to the category $\underline{\mathrm{Rng}}$ of commutative rings. We assume $\mathcal{C}$ has finite coproducts $\hat{\otimes}$ and an initial object k. Let $\mathcal{E}$ be the functor category $\underline{\mathrm{Set}}^{\mathcal{C}}$, and let $R \in \underline{\mathrm{Set}}^{\mathcal{C}} = \mathcal{E}$ be the composite "forgetful" functor $\mathcal{C} \xrightarrow{\Box} \underline{\mathrm{Rng}} \to \underline{\mathrm{Set}}$. So R is a commutative ring object in $\mathcal{E}$. If $A \in \mathcal{C}$, we denote by $\overline{A}$ the representable functor $\mathcal{C}(A, -) \in \mathcal{E}$.

Then we have

Proposition 2.1. *An element of $R^{\overline{A}}$ at stage $\overline{B}$ is the same as an element of $A \hat{\otimes} B$.*

Proof. Via the conversions

$$\begin{array}{ccc} \overline{B} & \longrightarrow & R^{\overline{A}} \\ \hline \overline{A} \times \overline{B} & \longrightarrow & R \\ \hline \overline{A \hat{\otimes} B} & \longrightarrow & R \\ \hline & \in & A \hat{\otimes} B \end{array}$$

Proposition 2.2. *An element of $\underline{Hom}_R(R^{\overline{A}}, R^{\overline{C}})$ at stage $\overline{B}$ is the same as an R-linear map $R^{\overline{A}} \to R^{\overline{C \hat{\otimes} B}}$.*

Proof. Via the conversions

$$\begin{array}{ccl}
\overline{B} & \longrightarrow & \underline{Hom}_R(R^{\overline{A}}, R^{\overline{C}}) \\
\hline
\overline{B} \times R^{\overline{A}} & \longrightarrow & R^{\overline{C}} \quad R\text{-lin. in 2nd variable} \\
\hline
R^{\overline{A}} & \longrightarrow & (R^{\overline{C}})^{\overline{B}} = R^{\overline{C\hat{\otimes}B}} \quad R\text{-linear.}
\end{array}$$

Proposition 2.3 *An R-linear map $R^{\overline{A}} \to R^{\overline{B}}$ is the same as a natural transformation $\phi : R(A\hat{\otimes}-) \to R(B\hat{\otimes}-)$ such that each component of ϕ*

$$A\hat{\otimes}C \xrightarrow{\phi_C} B\hat{\otimes}C$$

is C-linear w.r. to the natural C-algebra structures. (Recall that $R : \mathcal{C} \to Set$ is just a forgetful functor, so may be omitted from notation).

Proof. A map $R^{\overline{A}} \to R^{\overline{B}}$ is a family $\phi_C : R^{\overline{A}}(C) \to R^{\overline{B}}(C)$, natural in C. But $R^{\overline{A}}(C) = R(A\hat{\otimes}C)$, by Proposition 2.1 and Yoneda's lemma. The given map is R-linear iff ϕ_C is C-linear $\forall C$.

Corollary 2.4 *An element of $E(\overline{A})$ at stage $\overline{B}$ is the same as a natural transformation $\phi : R(A\hat{\otimes}-) \to R(B\hat{\otimes}-)$ such that each component ϕ_C is C-linear.*

Proof. Via the conversions

$$\begin{array}{ccl}
\overline{B} & \longrightarrow & E(\overline{A}) \\
\hline
\overline{B} & \longrightarrow & \underline{Hom}_R(R^{\overline{A}}, R) \\
\hline
R^{\overline{A}} & \longrightarrow & R^{\overline{B}} \quad R\text{-linear;}
\end{array}$$

Now apply Proposition 2.3.

Assume that $\Box : \mathcal{C} \to$ Rng factors through k-$\underline{\text{alg}}$ (= category of commutative k-algebras, $k \in$ Rng, and that $\Box : \mathcal{C} \to k$-$\underline{\text{alg}}$ preserves finite coproducts, so $\hat{\otimes} = \otimes_k$. Then a natural transformation ϕ like in Proposition 2.3 or Corollary 2.4 is completely given by its component at k: $\phi_k : A \otimes_k k \to B \otimes_k k$, which we shall identify with a k-linear map $\phi_k : A \to B$. For now

$$\phi_k \otimes_k C : A \otimes_k C \to B \otimes_k C$$

makes sense and is easily proved equal to ϕ_C, by the following argument: let $\gamma \in A \otimes_k C$. Write

$$\gamma = \sum a_i \otimes c_i \quad a_i \in A \quad c_i \in C.$$

Then

$$\phi_C(\gamma) = \phi_C(\sum a_i \otimes c_i) = \sum \phi_C(a_i \otimes 1) \cdot c_i \quad (C\text{-linearity of } \phi_C)$$

$$= \sum \phi_k(a_i) \cdot c_i \quad (\text{naturality of } \phi)$$

$$= (\phi_k \otimes_k C)(\sum a_i \otimes c_i).$$

Thus, $\otimes_k$ has what probably should be called a *nuclearity property*, and we get

Proposition 2.5 *For $\mathcal{C}$ as above (where $\hat{\otimes} = \otimes_k$), an element of $E(\overline{A})$ at stage $\overline{B}$ is the same thing as a k-linear map $A \to B$.*

More generally, an element of $\underline{Hom}_R(R^{\overline{A}}, R^{\overline{C}})$ at stage $\overline{B}$ is the same as a k-linear map $A \to C \otimes_k B$.

We quote below (Theorem 2.7) a result about $E(\overline{A})$ with a $\otimes$ which is not $\otimes_k$.

We next consider the monad G where a much simpler result holds. Again, we assume that $\Box : \mathcal{C} \to \underline{\text{Rng}}$ factors through k-$\underline{\text{alg}}$, $\Box : \mathcal{C} \to k$-$\underline{\text{alg}}$.

Proposition 2.6. *If $\Box : \mathcal{C} \to k$-$\underline{alg}$ is full and faithful and $k \in \mathcal{C}$, then $\overline{A} \to G(\overline{A})$ is an isomorphism for any $A \in \mathcal{C}$.*

Proof. We have the conversions

$$(2.1) \qquad \begin{array}{ccl} \overline{B} & \longrightarrow & G(\overline{A}) \\ \hline \overline{B} & \longrightarrow & \underline{Hom}_{R\text{-Alg}}(R^{\overline{A}}, R) \\ \hline R^{\overline{A}} & \longrightarrow & R^{\overline{B}} \quad R\text{-algebra homomorphism;} \end{array}$$

this latter data is equivalent to a family

$$A \widehat{\otimes} C \xrightarrow{\phi_C} B \widehat{\otimes} C$$

($C \in \mathcal{C}$) natural in C and with each component a C-algebra homomorphism (C-$\underline{\text{alg}} = C \downarrow k$-$\underline{\text{alg}}$). This implies, for trivial categorical reasons, that $\phi_C = \phi_k \widehat{\otimes} 1$, where ϕ_k is a k-algebra homomorphism. Thus the list of conversions continues with

A	$\xrightarrow{\phi_k}$	B	k-algebra homomorphism
A	$\longrightarrow$	B	morphism in $\mathcal{C}$
$\overline{B}$	$\longrightarrow$	$\overline{A}$	(Yoneda)

We leave it to the reader to "internalize" further to get, under the assumptions of the Proposition:

$$\underline{Hom}_{R\text{-Alg}}(R^{\overline{A}}, R^{\overline{B}}) \cong \overline{A}^{\overline{B}}. \tag{2.2}$$

Since this proof was purely categorical, it applies for *any* algebraic theory $\mathbb{T}$, with $\mathcal{C}$ a full subcategory of $\mathbb{T}$-Alg, having finite coproducts, and with R the forgetful functor $\mathcal{C} \to \underline{\text{Set}}$. Then R-Alg denotes the category $R \downarrow \mathbb{T}\text{-alg}(\mathcal{E})$ where $\mathcal{E} = \underline{\text{Set}}^{\mathcal{C}}$. (One should think of Proposition 2.6 as a kind of *finiteness property* of affine objects $\overline{A}$, since, for $\mathcal{E} = \underline{\text{Set}}$, $\mathbb{T}$ = theory of boolean algebras, $R = 2$, $X \to G(X)$ is an iso iff X has no non-principal ultrafilters.)

Using the notation of [8], I.12, we have

$$\overline{A} = \text{Spec}_R(A)$$

and hence

$$\overline{A}^{\overline{B}} = \text{Spec}_R(A)^{\overline{B}} = \text{Spec}_{R^{\overline{B}}}(A),$$

so that (2.2) gives

$$\underline{Hom}_{R\text{-Alg}}(R^{\overline{A}}, R^{\overline{B}}) \cong \text{Spec}_{R^{\overline{B}}}(A) \tag{2.3}$$

for any $A, B \in \mathcal{C}$.

A stronger conclusion holds if the inclusion $\mathcal{C} \hookrightarrow \mathbb{T}$-Alg preserves finite coproducts; then, in (2.3), $R^{\overline{B}}$ can be replaced by an arbitrary R-algebra object C in $\mathcal{E}$:

Theorem 2.7. *Under these assumptions,*

$$\underline{Hom}_{R\text{-}Alg}(R^{\overline{A}}, C) \cong \text{Spec}_C(A)$$

for any R-algebra C in $\mathcal{E}$.

Proof. See [7] or [8] III Theorem 1.2 (for a slightly more general result, and a more precise statement).

Problem. To what extent does this result hold for $\mathcal{C} = \mathcal{B} \hookrightarrow \mathbb{T}\text{-}\underline{\text{alg}}$, the category used for constructing the Dubuc-Topos $\mathcal{G}$ (see e.g. [8] p. 230 (p. 165 in 2nd ed.)). It holds, by [8] III Theorem 1.2 when A is a Weil algebra. But does it hold for $A = C^\infty(\mathbb{R}^n)$, say (which would imply that $R^{R^n} \in \mathcal{G}$ is the free $\mathbb{T}_\infty$-algebra in n generators in $\mathcal{G}$).

For the case $\mathbb{T} = \mathbb{T}_\infty$, $\mathcal{C}$ = category of finitely generated $\mathbb{T}_\infty$-algebras, the results about G hold. Proposition 2.5 about E fails, but at least a special case holds when the word "k-linear" is replaced by "*continuous* $\mathbb{R}$-linear".

Theorem 2.8. *(Quê-Reyes [9].) Let $\mathcal{C}$ be as above, and let M be a manifold. Then a global element of $E(\overline{C^\infty(M)})$ is the same as an $\mathbb{R}$-linear continuous $C^\infty(M) \to \mathbb{R}$ (i.e. a distribution-with-compact-support on M).*

Here $C^\infty(M)$ is equipped with the standard Whitney [[Frechet]] topology. – We have quoted this theorem to show that the synthetic distribution notion, E, does, in certain models, contain notions from "classical" functional analysis.

Problem. It is not clear to me to what extent the analogous result holds when $\mathcal{C} = \mathcal{B}$, the category which defines the Dubuc model $\mathcal{G}$.

3 Tensor products of distributions

The general theory of strong (= $\mathcal{E}$-enriched) monads gave, as explained in §1, two monoidal structures ψ and $\widetilde{\psi}$ on the functor part of the restricted double dualization monad E,

$$E(X) \times E(Y) \underset{\widetilde{\psi}}{\overset{\psi}{\rightrightarrows}} E(X \times Y).$$

If we think of $E(X)$ and $E(Y)$ as the object of distributions on X and Y, respectively (which Theorem 2.8 partially justifies), then ψ and $\widetilde{\psi}$ are to be thought of as tensor product formation of distributions, as (1.2) indicates. We can be more specific about what ψ and $\widetilde{\psi}$ do in the case considered in Proposition 2.5 (i.e. where $\widehat{\otimes} = \otimes_k$), and where X and Y are affine:

Proposition 3.1 *The maps ψ and $\widetilde{\psi} : E(\overline{A_1}) \times E(\overline{A_2}) \to E(\overline{A_1} \times \overline{A_2})$ are equal; the value on the pair*

$$\overline{B} \xrightarrow{s_1} E(\overline{A_1}) \qquad \overline{B} \xrightarrow{s_2} E(\overline{A_2})$$

is, under the identification of such elements with k-linear maps $A_i \to B$ ($i = 1, 2$), equal to the k-linear map

$$A_1 \otimes_k A_2 \xrightarrow{s_1 \otimes s_2} B \otimes B \xrightarrow{\mu} B \tag{3.1}$$

where μ is the multiplication.

Instead of giving the proof, we give the diagram which is the scene of the proof; we give it in the general case where $\widehat{\otimes}$ is not assumed to be $\otimes_k$, so that the data of $S_1 : \overline{B} \to E(\overline{A}_1)$ means an $S_1^C : C \otimes A_1 \to C \otimes B$ for all C, of Corollary 2.4. The diagram then is (writing $\otimes$ for $\widehat{\otimes}$)

$$\begin{array}{ccccccccc}
B \otimes A_2 \otimes A_1 \cong B \otimes A_1 \otimes A_2 & \xrightarrow{S_2^{B\otimes A_1}} & B \otimes A_1 \otimes B & \cong & (B \otimes B) \otimes A_1 & \xrightarrow{\mu_B \otimes A_1} & B \otimes A_1 \\
\downarrow S_1^{B\otimes A_2} & & & & \downarrow S_1^{B\otimes B} & & \downarrow S_1^{B} \\
B \otimes A_2 \otimes B \cong B \otimes B \otimes A_2 & \xrightarrow{S_2^{B\otimes B}} & B \otimes B \otimes B & \xrightarrow{1 \otimes \tau} & B \otimes B \otimes B & \xrightarrow{\mu \otimes 1} & B \otimes B \\
\downarrow \mu \otimes 1 & & \downarrow \mu \otimes 1 & & & & \downarrow \mu \\
B \otimes A_2 & \xrightarrow[S_2^B]{} & B \otimes B & & \xrightarrow[\mu]{} & & B;
\end{array}$$

here, τ is the twist map. The upper left hand square does not in general commute, but in case $\otimes = \otimes_k$ it takes $b \otimes a_2 \otimes a_1$ into, respectively, $b \otimes S_2(a_2) \otimes S_1(a_1)$ and $b \otimes S_1(a_1) \otimes S_2(a_2)$ (where $S_i = S_i^k$), and thus the total diagram commutes, in this case.

4 Representable and corepresentable R-modules

The present § (whose content is essentially from [10]) deals with the case considered for Proposition 2.5, i.e. when $\mathcal{C}$ is a category of k-algebras, such that the forgetful functor $\Box : \mathcal{C} \to k$-alg preserves finite coproducts (so $\otimes_k$ is coproduct in $\mathcal{C}$). As in §2, we let $\mathcal{E} = \underline{\mathrm{Set}}^{\mathcal{C}}$, and $R =$ the forgetful functor; it is a ring object on $\mathcal{E}$.

Let V be a k-module. We then get an R-module object $\widetilde{V} \in \mathcal{E}$, namely

$$\widetilde{V}(C) := V \otimes_k C$$

whose R-module structure comes about because $V \otimes_k C$ carries a natural C-module structure. Such R-modules are called *corepresentable* ([10])("corepresented by V"). Also, we get another R-module object $V^v \in \mathcal{E}$, namely

$$V^v(C) := \hom_k(V, C),$$

whose R-module structure comes about because $\hom_k(V, C)$ carries a natural C-module structure. Such R-modules are called *representable* ([10]) ("represented by V").

We have

Proposition 4.1 *Let $A \in \mathcal{C}$. Then $R^{\overline{A}} = \widetilde{A}$ (where, on the right, we consider the underlying k-module of A).*

Proof. Immediate from Proposition 2.1.

Since R is commutative, the category $\underline{Mod}_R$ of R-module objects in $\mathcal{E}$ is closed (symmetric closed, in fact), so that $\underline{Hom}_R(U, V)$ is an R-module object whenever U and V are. In particular, denote by U^* the "dual" R-module of U, meaning $\underline{Hom}_R(U, R)$.

We have

Proposition 4.2 *Let V be k-module. Then $V^v = (\widetilde{V})^*$.*

Proof. We have the conversions

$\overline{B}$	$\longrightarrow$	$\tilde{V}^* = \underline{Hom}_R(\tilde{V}, R)$
$\tilde{V}$	$\longrightarrow$	$R^{\overline{B}}$ R-linear
$\tilde{V}(C)$	$\longrightarrow$	$(R^{\overline{B}})(C)$ nat. in C, C-linear
$V \otimes_k C$	$\longrightarrow$	$B \otimes_k C$ nat. in C, C-linear

by Proposition 2.1. But the "nuclearity argument" preceding Proposition 2.5 (which works even when A is just a k-module), this data is equivalent to a k-linear $V \to B$, so the string of conversions continues

$$\begin{array}{ccl} V & \longrightarrow & B \quad k\text{-linear} \\ \hline & \in & V^v(B) \\ \hline \overline{B} & \longrightarrow & V^v \end{array}$$

[[proving the Proposition.]]

5 On the Waelbroeck property

Lawvere stressed some years ago the importance of a Theorem of Waelbroeck [11], which states that the vector space of distributions [[with compact support]] on a manifold has the universal property to be the free $\mathbb{R}$-module on the manifold, with respect to a certain subcategory ("b-spaces"). Since $E(X)$ is like the space of distributions [[with compact support]] on X, he was led to ask two, related, questions:

1) which universal propert does $\eta : X \to E(X)$ have?
(or, what is the category of algebras for E)?
2) to what extent is the monad E commutative?

We have some partial results. Let $\mathcal{C}$ be a small full subcategory of the category k-alg of k-algebras, stable under finite colimits and such that, whenever $C \in \mathcal{C}$, the $S^\bullet_k C \in \mathcal{C}$ where $S^\bullet_k C$ is the symmetric k-algebra on (the underlying k-module of) C. We consider again $\mathcal{E} = \underline{\mathrm{Set}}^{\mathcal{C}}$, with R forgetful $\mathcal{C} \to \underline{\mathrm{Set}}$.

Proposition 5.1 *For any $A \in \mathcal{C}$, $E(\overline{A}) = \overline{S^\bullet_k A}$.*

Proof. Via the conversions

$$\begin{array}{ccl} \overline{B} & \longrightarrow & E(\overline{A}) \\ \hline A & \longrightarrow & B \quad k\text{-linear (by Proposition 2.5)} \\ \hline S^\bullet_k A & \longrightarrow & B \quad \text{in } \mathcal{C} \text{ (universal property of } S^\bullet A) \\ \hline \overline{B} & \longrightarrow & \overline{S^\bullet A} \quad \text{Yoneda.} \end{array}$$

Under the same assumptions, we have the following "Little Waelbroeck Theorem":

Theorem 5.2 *Composition with $\eta : \overline{A} \to E(\overline{A})$ mediates an isomorphism*

$$\underline{Hom}_R(E(\overline{A}), R) \cong R^{\overline{A}}.$$

Verbally, *"R perceives $E(\overline{A})$ to be the free R-module on $\overline{A}$".*

Proof. We have the conversions

$$\begin{array}{ccl}
\overline{B} & \xrightarrow{\beta} & \underline{Hom}_R(E(\overline{A}), R) \\ \hline
E(\overline{A}) & \longrightarrow & R^{\overline{B}} \quad R\text{-linear} \\ \hline
\overline{S_k^{\bullet}\overline{A}} & \longrightarrow & R^{\overline{B}} \quad R\text{-linear (by Proposition 3.1)} \\ \hline
S_k^{\bullet}\overline{A} \times \overline{B} & \longrightarrow & R \quad R\text{-linear in } 1^{st} \text{ variable} \\ \hline
S_k^{\bullet}\overline{A} \times \overline{B} & \longrightarrow & R \times \overline{B} \quad R\text{-linear in } \mathcal{E}/B = \mathrm{Set}^{B/\mathcal{C}} \\ \hline
\beta & \in & S_k^{\bullet}A \otimes_k B = S_B^{\bullet}(A \otimes_k B) \quad \text{homogeneous of degree 1}
\end{array}$$

where for the last conversions use (with $\mathcal{C} = B/\mathcal{C}$, $A = A \otimes_k B$, $k = B$).

Lemma 5.3 *An element $\beta \in S_k^{\bullet}(A)$ represents an R-linear map $\overline{S_k^{\bullet}(A)} \to R$ iff it is homogeneous of degree* 1.

Proof. This is well known, cf. [10] p. 93. Thus, the string of conversions continues

$$\begin{array}{ccl} \hline
\beta & \in & A \otimes_k B \\ \hline
\overline{B} & \longrightarrow & R^{\overline{A}} \quad \text{(Proposition 2.1)}
\end{array}$$

We shall say that an R-module V has the *Waelbroeck property* w.r.to the object X if composition with $\eta : X \to E(X)$ mediates an isomorphism

$$\underline{Hom}_R(E(X), V) \cong V^X,$$

i.e. V perceives $E(X)$ to be the free R-module on X.

A purely formal calculation gives that if V has the Waelbroeck property w.r.to X, then so does $\underline{Hom}_R(U, V)$, for any U:

$$\begin{aligned}
\underline{Hom}_R(E(X), \underline{Hom}_R(U, V)) &\cong \underline{Hom}_R(U, \underline{Hom}_R(E(X), V)) \\
&\cong \underline{Hom}_R(U, V^X) \\
&\cong (\underline{Hom}_R(U, V))^X.
\end{aligned}$$

In particular, the theorem has the following

Corollary 5.4 *Any dual module U^* has the Waelbroeck property w.r.to any affine object $\overline{A}$.*

The reader may note that the category of R-modules with the Waelbroeck property w.r.to a given class of objects is closed under inverse limits and cotensors, etc., just like the class of infinitesimally linear objects, so should form a reflexive subcategory of R-Mod. A serious category theoretic study of such kind of "orthogonal subcategories" is still lacking in the present "enriched" case. [[See Kelly's book, Chapter 6, for an account.]]

Theorem 5.5. *([10] II..1.1.2) Any corepresentable R-module is reflexive (canonically isomorphic to its double dual), provided $S^\bullet_k V \in \mathcal{C}$.*

Proof. This is very much like the proof of Theorem 5.2, which is actually a special case (see below). For, if $S^\bullet_k V \in \mathcal{C}$, then

$$V^v(B) = \hom_k(V, B) \cong \hom_{k\text{-Alg}}(S^\bullet_k V, B),$$

so that

$$V^v = \overline{S^\bullet_k V}.$$

Then we have the conversions (the first one by Proposition 4.2):

$\overline{B}$	$\longrightarrow$	$\widetilde{V}^{**} = (V^v)^* = \overline{S^\bullet_k V}^*$
$\overline{S^\bullet_k V}$	$\longrightarrow$	$R^{\overline{B}}$ R-linear
β	$\in$	$S^\bullet_B(V \otimes_k B)$ homogeneous of degree 1
β	$\in$	$V \otimes_k B$
$\overline{B}$	$\longrightarrow$	$\widetilde{V}$

where the second is as in the proof of Theorem 5.2.

Theorem 5.2 is a special case: take $V = A$, considered as a k-module. Then if $\widetilde{A} = \widetilde{A}^{**}$, we have

$$R^{\overline{A}} = \widetilde{A} = \widetilde{A}^{**} = (R^{\overline{A}})^{**} = E(\overline{A})^*,$$

which is the isomorphism of Theorem 5.2.

Problem. For which M and $\mathcal{E}$ is it true that the tangent bundle $TM \to M$ is a *reflexive* R-module object in $\mathcal{E}/M$. (This is certainly so if M is a manifold, and $\mathcal{E}$ is a well-adapted model for synthetic differential geometry.)

6 Symmetric-algebra formation as a monoidal monad

As in the previous §, we consider a small full subcategory of the category $\underline{k\text{-alg}}$ of k-algebras, stable under finite colimits, and such that, whenever $C \in \mathcal{C}$, then $S_k^\bullet C \in \mathcal{C}$.

In particular, we have a comonad $S_k^\bullet$ on $\mathcal{C}$, namely the composite of the forgetful $\mathcal{C} \to k\text{-}\underline{\text{mod}}$ and its left adjoint $S_k^\bullet : k\text{-}\underline{\text{mod}} \to \mathcal{C}$ (where $k\text{-}\underline{\text{mod}}$ is the full subcategory of k-modules, consisting of underlying k-modules of algebras in $\mathcal{C}$).

We now have the situation

$$\begin{array}{ccc} \mathcal{C}^{op} & \overset{y}{\hookrightarrow} & \mathcal{E} = \underline{\text{Set}}^{C} \\ \circlearrowleft & & \circlearrowleft \\ S_k^\bullet & & E \end{array}$$

where the ($\mathcal{E}$-strong) monad E restricts to the monad $S_k^\bullet$ on $\mathcal{C}^{op}$, by Proposition 5.1. By Prop. 3.1, the two monoidal structures ψ, $\widetilde{\psi}$ on the functor E agree for objects of $\mathcal{C}^{op}$, and y preserves the monoidal structure, $y(A \otimes_k B) \cong yA \times yB$, $y(k) = 1$. In [3], we proved that $\psi = \widetilde{\psi}$ implies that the monad is a symmetric monoidal monad (i.e. that μ is a symmetric monoidal transformation). Inspecting the proof there (diagram (3.2)), we see that in order to conclude commutativity of

$$\begin{array}{ccc} T^2A \otimes T^2B & \xrightarrow{\psi} & T(TA \otimes TB) \\ & & \downarrow T(\psi) \\ \mu \times \mu \downarrow & & T^2(A \otimes B) \\ & & \downarrow \mu \\ TA \otimes TB & \xrightarrow[\psi]{} & T(A \otimes B), \end{array} \tag{6.1}$$

we need only that $\psi_{A,BT} = \widetilde{\psi}_{A,BT}$ ($T(\psi_{A,BT})$ and $T(\widetilde{\psi}_{A,BT})$ form the possibly non-commutative upper right hand corner of the diagram [3] (3.2) whose total commutativity yields (6.1)).

Taking $T = E$, we therefore get

Theorem 6.1 *The comonad $S_k^\bullet(-)$ on $\mathcal{C}$ (in particular on k-$\underline{Alg}$) carries a canonical structure of symmetric monoidal monad on $\mathcal{C}^{op}$ (w.r.to the monoidal structure $\otimes_k$ on $\mathcal{C}$).*

[[Labels in this section have been corrected from the incorrect 5.?? to 6.??.]]

7 Problems.

Most of the positive results quoted deal with the "purely algebraic" case, where $\widehat{\otimes} = \otimes_k$, which is not the primary concern of synthetic differential geometry, which is more interested in $\otimes_\infty$, where the proof of the key result Proposition 2.5 fails. The main question is really: is there a notion of "module over a $\mathbb{T}_\infty$-algebra A" such that $\otimes_\infty$ makes sense not just for C-algebras, but for *C-modules*? Perhaps the basic commutative algebra for $\mathbb{T}_\infty$-algebras and their modules should be based not on the topos of sets, but the topos of *bornological* sets, as described by Lawvere (site of definition: countable sets; the family of coverings generated by *finite disjoint* coverings). Any $\mathbb{T}_\infty$-algebra $C^\infty(M)$ has a natural bornology, cf. [1], and hence so does any quotient $C^\infty(M)/I$.

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