Pregroupoids and their enveloping groupoids

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ABSTRACT. We prove that the forgetful functor from groupoids to pregroupoids has a left adjoint, with the front adjunction injective. Thus we get an *enveloping groupoid* for any pregroupoid. We prove that the category of torsors is equivalent to that of pregroupoids. Hence we also get enveloping groupoids for torsors, and for principal fibre bundles.

Introduction

The present note advocates the algebraic notion of *pregroupoid* as a natural context in which to study and compare groupoids, principal fibre bundles, torsors, bitorsors. The aim has been to provide a theory which is *functorial*, and can immediately be interpreted in a wide variety of categories, in particular, in all toposes. Hence, the construction principle "choose a base point" is not used, since it violates not only functorality, but also violates the choice principles available in toposes, where "non-empty" (= inhabited) objects may have no "points" (= global sections). One motivation I had for looking for such a theory, was to have an adequate, purely algebraic, framework for studying connections in principal fibre bundles, cf. [9], in the context of synthetic differential geometry, where topos methods are crucial.

The main construction in this framework is the construction of an "enveloping groupoid" X^+ of a pregroupoid X. It in fact provides a left adjoint for the forgetful functor from groupoids to pregroupoids, and the unit for the adjunction is injective, whence the choice of the adjective "enveloping". In particular, the functor $X \mapsto X^+$ is faithful. – As a special case, the construction provides an enveloping groupoid of any principal fibre bundle.

The enveloping groupoid construction can be described without the notion of pregroupoid; this was in fact done in [9] (for the case of principal fibre bundles).

An essential ingredient in the construction of X^+ is the Ehresmann "edge groupoid" XX^{-1} of a principal fibre bundle X. This edge groupoid construction was carried out in the context of pregroupoids in [7] (but in a less equational manner). The functor $X \mapsto XX^{-1}$, however, unlike $X \mapsto X^+$, is not an adjoint, and is not faithful.

I want to acknowledge a heated but fruitful e-mail exchange with Ieke Moerdijk in the Summer and Fall of 2002, on some of the topics of the present paper and [9].

1 Equational theory of pregroupoids

We consider a groupoid $\mathbf{G} = G_1 \rightrightarrows G_0$. For any two subsets $A \subseteq G_0$ and $B \subseteq G_0$, we let $\mathbf{G}(A, B)$ denote the set of arrows $\in G_1$ whose domain is in A and whose codomain is in B. If $A = B$, this carries structure of groupoid, the full subgroupoid on A, which thus here is denoted $\mathbf{G}(A, A)$.

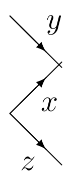
There are evident book-keeping maps

$$d_0 : \mathbf{G}(A, B) \to A \ \text{ and } \ d_1 : \mathbf{G}(A, B) \to B.$$

In $\mathbf{G}$, we compose from left to right, denoting composition by $\circ$. Then composition in $\mathbf{G}$, together with the book-keeping maps, provide $X = \mathbf{G}(A, B)$ with a certain partially defined algebraic structure: a ternary operation denoted $yx^{-1}z$, defined whenever $d_1(x) = d_1(y)$ and $d_0(x) = d_0(z)$, (and then $d_0(yx^{-1}z) = d_0(y)$ and $d_1(yx^{-1}z) = d_1(z)$). Namely

$$yx^{-1}z := y \circ x^{-1} \circ z.$$

The reader may find the following display useful. The vertices are elements of A and B, respectively (with A-objects in the left hand column, B-objects in the right hand column).



The following equations trivially hold for this ternary operation (whenever the expressions are defined):

$$xx^{-1}z = z \tag{1}$$

$$yx^{-1}x = y \tag{2}$$

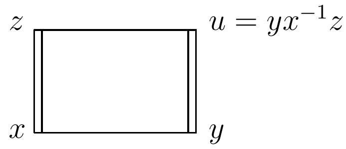
("unit laws"),

$$vy^{-1}(yx^{-1}z) = vx^{-1}z \tag{3}$$

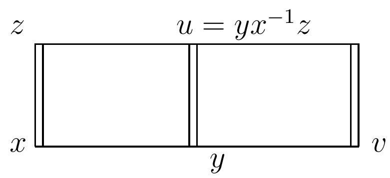
$$(yx^{-1}z)z^{-1}w = yx^{-1}w, \tag{4}$$

("concatenation laws"). The reason for the latter name is motivated by the following diagrammatic device (also used in [7]):

We indicate the assertion that $u = yx^{-1}z$ by a diagram



Here, the single lines connect elements in X which have same codomain, double lines connect elements with same domain. Quadrangles that arise in this way, $u = yx^{-1}z$, we shall call *good* quadrangles, and (3) (resp. (4)) then expresses that good quadrangles may be concatenated horizontally (resp. vertically). The display of (3) in terms of quadrangles in fact is



We proceed to make some purely equational deductions from (1), (2), (3), (4), which we take as axioms for the notion of pregroupoid. To be specific, we pose

Definition 1 *A* pregroupoid *("on A, B") is an inhabited set X equipped with surjections $\alpha : X \to A$, $\beta : X \to B$ and with a partially defined ternary operation, denoted $yx^{-1}z$, defined whenever $\beta(x) = \beta(y)$ and $\alpha(x) = \alpha(z)$; and then $\alpha(yx^{-1}z) = \alpha(y)$ and $\beta(yx^{-1}z) = \beta(z)$; and the equations (1), (2), (3), (4) are supposed to hold.*

(In [7], essentially the same notion was considered, but from a less equational viewpoint).

Since the "primitive" operation $yx^{-1}z$ has three variables, equations quickly are equations in five or more variables, and therefore it is convenient to denote the variables $x_1, x_2, \ldots$. So the basic operation is $x_1 x_2^{-1} x_3$ (note that x_1 corresponds to y, x_2 to x). In fact, to make the notation even more lightweight, we may drop the "x" in x_1, x_2, .. and instead just use the symbols $1, 2, \ldots$. So for instance, (3) is

$$41^{-1}(12^{-1}3) = 42^{-1}3. \tag{5}$$

– The first equational consequence of the axioms is an "associative law":

$$(12^{-1}3)4^{-1}5 = 12^{-1}(34^{-1}5) \tag{6}$$

(provided the book-keeping makes the expressions meaningful, i.e. provided $\beta(1) = \beta(2), \alpha(2) = \alpha(3), \beta(3) = \beta(4), \alpha(4) = \alpha(5)$).

For, $(12^{-1}3)3^{-1}(34^{-1}5)$ equals $(12^{-1}3)4^{-1}5$, by (3), and equals $12^{-1}(34^{-1}5)$ by (4).

Next, we have

$$21^{-1}(12^{-1}3) = 3. \tag{7}$$

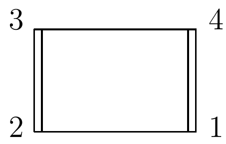
For, if we put $4 = 2$ in (5), we get $21^{-1}(12^{-1}3) = 22^{-1}3$ which is 3, by (1). Similarly, from (4) and (2), we get

$$(12^{-1}3)3^{-1}2 = 1. \tag{8}$$

In the diagrammatic form of quadrangles, as above, these two equations express the following symmetry property for good quadrangles, which we shall use without further comment in the "graphical" calculations that follow.

Proposition 1 *The mirror image og a good quadrangle in a horizontal line, or in a vertical line, is again a good quadrangle. (So the "Four-Group" acts on the set of good quadrangles.)*

Proof. Assume we have a good quadrangle

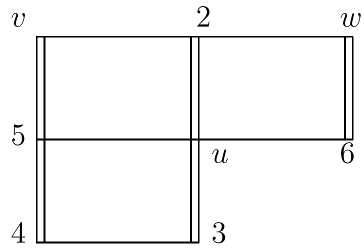


so $4 = 12^{-1}3$. The fact that the reflection in a horizontal line is a good quadrangle is the assertion that $1 = 43^{-1}2$, or, by the assumption on 4, that $1 = (12^{-1}3)3^{-1}2$, which is just (8). The assertion about reflection in a vertical line similarly follows from (7).

We shall use the graphical calculus with good quadrangles to establish the following equation

$$6(34^{-1}5)^{-1}2 = 65^{-1}(43^{-1}2); \tag{9}$$

again the book-keeping conditions are assumed to make the expression meaningful; these conditions are stated in diagrammatic form in the diagram



Assume that all the three displayed quadrangles are good. (These three quadrangles are constructed out of the data of the entries $2, 3, 4, 5, 6$, by first constructing u, and then v and w.) So $u = 34^{-1}5$, and also $v = 43^{-1}2$ (concatenate the two left hand quadrangles). So by concatenating the two top quadrangles, we get

$$w = 65^{-1}v = 65^{-1}(43^{-1}2);$$

on the other hand, the upper right quadrangle expresses that

$$w = 6u^{-1}2 = 6(34^{-1}5)^{-1}2;$$

comparing, we get (9).

Let us call a pair (x,z) with $\alpha(x) = \alpha(z)$ a *vertical* pair. The (horizontal) concatenation property for good quadrangles, together with (1) and one of the symmetries mentioned in the Proposition, imply that we get an *equivalence* relation $\cong_{\mathsf{v}}$ on the set of vertical pairs, namely

$$(x,z) \cong_{\mathsf{v}} (y,u) \text{ iff } u = yx^{-1}z,$$

(geometrically: (x,z) and (y,u) form the vertical sides of a good quadrangle). Note that if $(x,z) \cong_{\mathsf{v}} (y,u)$, then $\beta(x) = \beta(y)$ and $\beta(z) = \beta(u)$. The equivalence class of (x,z) is denoted $x^{-1}z$. The set of such equivalence classes is denoted $X^{-1}X$.

Similarly, let us call a pair (x,y) with $\beta(x) = \beta(y)$ a *horizontal* pair. The (vertical) concatenation property for good quadrangles, together with (2) and one of the symmetries mentioned in the Proposition, imply that we get an *equivalence* relation $\cong_{\mathsf{h}}$ on the set of horizontal pairs, namely

$$(x,y) \cong_{\mathsf{h}} (z,u) \text{ iff } u = yx^{-1}z,$$

(geometrically: (x,y) and (z,u) form the horizontal sides of a good quadrangle). Note that if $(x,y) \cong_{\mathsf{h}} (z,u)$, then $\alpha(x) = \alpha(z)$ and $\alpha(y) = \alpha(u)$. The equivalence class of (x,y) is denoted yx^{-1}. The set of such equivalence classes is denoted XX^{-1}.

We proceed to derive some equations that involve "fractions" $yx^{-1} \in XX^{-1}$ and $x^{-1}z \in X^{-1}X$. Among these is

$$(12^{-1}3)4^{-1} = 1(43^{-1}2)^{-1}. \tag{10}$$

By definition of the equivalence relation that defines XX^{-1}, this means

$$1 = (12^{-1}3)4^{-1}(43^{-1}2).$$

But for the right hand side here, we have that it equals $(12^{-1}3)3^{-1}2$ (using (3)), which in turn by (4) equals $12^{-1}2$, which is 1, by (2). – Similarly, one proves

$$(43^{-1}2)^{-1}5 = 2^{-1}(34^{-1}5) \tag{11}$$

2 Enveloping groupoid of a pregroupoid

Since the notion of pregroupoid is purely algebraic (except for the surjectivity requirement for α and β), it is clear how to organize pregroupoids into a category (it will be upgraded into a 2-category in Section 4): if

$$A \xleftarrow{\alpha} X \xrightarrow{\beta} B$$

and

$$A' \xleftarrow{\alpha'} X' \xrightarrow{\beta'} B'$$

are pregroupoids, a morphism ξ from the first to the second consists of maps $\xi_0 : A \to A'$, $\xi_1 : B \to B'$ and $\xi : X \to X'$ commuting with the structural maps and preserving the ternary operation. It is usually harmless to omit the subscripts and just write ξ for all three maps in question.

In the following, "groupoid" means "inhabited groupoid". There is an evident functor

$$\text{groupoids} \;\to\; \text{pregroupoids}$$

taking the groupoid $\mathbf{G} = G_1 \rightrightarrows G_0$ to the pregroupoid $\mathbf{G}(G_0, G_0)$, (so the ternary operation $yx^{-1}z$ is given by $y \circ x^{-1} \circ z$).

Theorem 1 *This functor has a left adjoint; the front adjunction is injective.*

Proof/construction. Given a pregroupoid $X = (A \leftarrow X \to B)$, we construct a groupoid X^+ whose set of objects is the disjoint sum of A and B. Thus the set of arrows will be a disjoint sum of four sets,

$$X^+(A, A), X^+(B, B), X^+(A, B), X^+(B, A).$$

We first describe these sets with their book-keeping maps d_0 and d_1, and then we describe the composition $\circ$:

$$\begin{array}{lll} X^+(A,A) := XX^{-1}; & d_0(yx^{-1}) := \alpha(y), & d_1(yx^{-1}) := \alpha(x) \\ X^+(B,B) := X^{-1}X; & d_0(x^{-1}z) := \beta(x), & d_1(x^{-1}z) := \beta(z) \\ X^+(A,B) := X; & d_0(x) := \alpha(x), & d_1(x) := \beta(x); \end{array}$$

and finally $X^+(B, A)$ is to be another copy of X, which we will denote X^{-1}. An element x of X will be denoted x^{-1} when considered in this copy of X. Thus we put

$$X^+(B, A) := X^{-1}; \;\; d_0(x^{-1}) := \beta(x), \;\; d_1(x^{-1}) := \alpha(x) \; .$$

The fact that the book-keeping maps for yx^{-1} and $x^{-1}z$ are well defined follows from the remarks at the end of Section 1.

Here is the description of the composition in the form of a multiplication table. We compose from left to right; the type of the left hand factor left hand factor is listed in the column on the left, the type of the right hand factor in the row on the top.

	$A \to A$	$A \to B$	$B \to A$	$B \to B$
$A \to A$	$12^{-1} \circ 34^{-1}$ $:= (12^{-1}3)4^{-1}$	$12^{-1} \circ 3$ $:= 12^{-1}3$		
$A \to B$			$1 \circ 2^{-1}$ $:= 12^{-1}$	$1 \circ 2^{-1}3$ $:= 12^{-1}3$
$B \to A$	$3^{-1} \circ 21^{-1}$ $:= (12^{-1}3)^{-1}$	$2^{-1} \circ 3$ $:= 2^{-1}3$		
$B \to B$			$3^{-1}2 \circ 1^{-1}$ $:= (12^{-1}3)^{-1}$	$2^{-1}3 \circ 4^{-1}5$ $:= 2^{-1}(34^{-1}5)$

We proceed to check that the operation $\circ$ thus defined is associative. There are 16 cases to be considered, namely one for each 4-letter word in the letters A and B. All these cases follow directly from the defining equations in the table, together with equtions already derived for the ternary operation $yx^{-1}z$. The cases $AAAA$, $AAAB$, $AABB$, $ABBB$ and $BBBB$ were in effect dealt with, from a different viewpoint, in [7]. In terms of equations, the case $AAAB$, for instance, is proved as follows:

$$(12^{-1} \circ 34^{-1}) \circ 5 = (12^{-1}3)4^{-1} \circ 5 = (12^{-1}3)4^{-1}5,$$

using two definitions from the table, and

$$12^{-1} \circ (34^{-1} \circ 5) = 12^{-1} \circ (34^{-1}5) = 12^{-1}(34^{-1}5),$$

likewise using two definitions from the table; but these two expressions are equal by virtue of (6).

The cases not dealt with in [7] are those eleven 4-letter words that involve the phrase BA. We proceed with these (eleven) cases.

$$AABA: \quad 12^{-1} \circ 3 \circ 4^{-1} = \begin{cases} (12^{-1}3) \circ 4^{-1} & = (12^{-1}3)4^{-1} \\ 12^{-1} \circ 34^{-1} & = (12^{-1}3)4^{-1} \end{cases}$$

where here, and in the following similar calculations, the first line indicates the bracketing $(x \circ y) \circ z$, the second the bracketing $x \circ (y \circ z)$.

$$ABAA: \quad 1 \circ 2^{-1} \circ 34^{-1} = \begin{cases} 12^{-1} \circ 34^{-1} & = (12^{-1}3)4^{-1} \\ 1 \circ (43^{-1}2)^{-1} & = 1(43^{-1}2)^{-1} \end{cases}$$

and these are equal by (10).

$$BAAA: \quad 2^{-1} \circ 34^{-1} \circ 56^{-1} = \begin{cases} (43^{-1}2)^{-1} \circ 56^{-1} & = (65^{-1}(43^{-1}2))^{-1} \\ 2^{-1} \circ (34^{-1}5)6^{-1} & = (6(34^{-1}5)^{-1}2)^{-1} \end{cases}$$

and these are equal by (9).

$$ABAB: \quad 1 \circ 2^{-1} \circ 3 = \begin{cases} 12^{-1} \circ 3 & = 12^{-1}3 \\ 1 \circ 2^{-1}3 & = 12^{-1}3 \end{cases}$$

(This is the crucial case for the comparison of the pregroupoid X and its enveloping groupoid X^+ !)

$$ABBA: \quad 1 \circ 2^{-1}3 \circ 4^{-1} = \begin{cases} (12^{-1}3) \circ 4^{-1} & = (12^{-1}3)4^{-1} \\ 1 \circ (43^{-1}2)^{-1} & = 1(43^{-1}2)^{-1} \end{cases}$$

and these are equal by (10).

$$BAAB: \quad 2^{-1} \circ 34^{-1} \circ 5 = \begin{cases} (43^{-1}2)^{-1} \circ 5 & = (43^{-1}2)^{-1}5 \\ 2^{-1} \circ (34^{-1}5) & = 2^{-1}(34^{-1}5) \end{cases}$$

and these are equal by (11).

$$BABA: \quad 2^{-1} \circ 3 \circ 4^{-1} = \begin{cases} 2^{-1}3 \circ 4^{-1} & = (43^{-1}2)^{-1} \\ 2^{-1} \circ 34^{-1} & = (43^{-1}2)^{-1} \end{cases}$$

$$BBAA: \quad 2^{-1}3 \circ 4^{-1} \circ 56^{-1} = \begin{cases} (43^{-1}2)^{-1} \circ 56^{-1} & = (65^{-1}(43^{-1}2))^{-1} \\ 2^{-1}3 \circ (65^{-1}4)^{-1} & = ((65^{-1}4)3^{-1}2)^{-1} \end{cases}$$

and these are equal by (6).

$$BABB: \quad 2^{-1} \circ 3 \circ 4^{-1}5 = \begin{cases} 2^{-1}3 \circ 4^{-1}5 & = 2^{-1}(34^{-1}5) \\ 2^{-1} \circ (34^{-1}5) & = 2^{-1}(34^{-1}5) \end{cases}$$

$$BBAB: \quad 2^{-1}3 \circ 4^{-1} \circ 5 = \begin{cases} (43^{-1}2)^{-1} \circ 5 & = (43^{-1}2)^{-1}5 \\ 2^{-1}3 \circ 4^{-1}5 & = 2^{-1}(34^{-1}5) \end{cases}$$

and these are equal by (11).

$$BBBA: \quad 2^{-1}3 \circ 4^{-1}5 \circ 6^{-1} = \begin{cases} 2^{-1}(34^{-1}5) \circ 6^{-1} & = (6(34^{-1}5)^{-1}2)^{-1} \\ 2^{-1}3 \circ (65^{-1}4)^{-1} & = ((65^{-1}4)3^{-1}2)^{-1} \end{cases}$$

and these are equal by (9) and (6).

These calculations prove that the composition $\circ$ is associative. To prove the existence of units for X^+, we use that $\alpha : X \to A$ and $\beta : X \to B$ were assumed surjective. For $a \in A \subseteq A + B$, pick an element $x \in X$ with $\alpha(x) = a$. Then xx^{-1} will serve as a unit for the object a; for, we have

$$xx^{-1} \circ uv^{-1} = (xx^{-1}u)v^{-1} = uv^{-1},$$

using (1). Also,

$$yz^{-1} \circ xx^{-1} = (yz^{-1}x)x^{-1} = y(xx^{-1}z)^{-1},$$

by (10) (put 3 = 4). Now use (1) to get yz^{-1} back. Finally $xx^{-1} \circ u = xx^{-1}u = u$, by (1) again.

Similarly, one proves that $x^{-1}x$ is a unit for the object $b = \beta(x) \in B \subseteq A + B$.

Inverses are also almost tautologically present: yx^{-1} has xy^{-1} as an inverse; for

$$yx^{-1} \circ xy^{-1} = (yx^{-1}x)y^{-1} = yy^{-1},$$

using (2). Similarly, $z^{-1}x$ and $x^{-1}z$ are mutually inverse; and finally, x as an arrow $a \to b$, has $x^{-1} : b \to a$ as inverse.

This proves that X^+ is a groupoid.

The set $X^+(A, B)$ is by construction of X^+ just the given pregroupoid X, so that we have an injective mapping η from X to the (set of arrows of) X^+. Since $y \circ x^{-1} \circ z = yx^{-1}z$ (cf. the case $ABAB$ in the above proof), η is a morphism of pregroupoids into the underlying pregroupoid of X^+, so it just remains to check its universal property. A pregroupoid homomorphism from $A \leftarrow X \to B$ into the underlying pregroupoid of a groupoid $\mathbf{G} = G_1 \rightrightarrows G_0$, consists of $\phi_0 : A \to G_0$, $\phi_1 : B \to G_0$, and a map $\phi : X \to G_1$. The maps ϕ_0 and ϕ_1 together define a map $A + B \to G_0$, which is the object part of the desired functor $\overline{\phi} : X^+ \to \mathbf{G}$. The value of $\overline{\phi}$ on arrows is forced to be $\overline{\phi}(x) = x$ (since we require $\overline{\phi}$ composed with η to give ϕ), and then the remaining three cases are also forced if we want $\overline{\phi}$ to be a functor:

$$\overline{\phi}(x^{-1}) := (\phi(x))^{-1}, \overline{\phi}(xz^{-1}) := \phi(x) \circ \phi(z)^{-1}, \overline{\phi}(yx^{-1}) := \phi(y) \circ \phi(x)^{-1},$$

and it is clear from the defining formulas (from the table) that $\overline{\phi}$ preserves composition, and also clearly identities. This proves the Theorem.

Because X *embeds* into the groupoid X^+, we propose the name *enveloping groupoid of* X for it. It is analogous to the enveloping associative algebra of a Lie algebra in the sense that all equations concerning the ternary operation $yx^{-1}z$ can be checked under the assumption that $yx^{-1}z$ is actually $y \circ x^{-1} \circ z$ for the (associative) composition $\circ$ of a groupoid. We have utilized this principle in the calculations concerning connections in principal fibre bundles, cf. [9], (where the name *comprehensive groupoid* was used for what we here call enveloping groupoid).

The set of objects of the enveloping groupoid X^+ contains the sets A and B as subsets. With respect to these subsets it has a certain property, namely it is *A-B-transitive*; by this we mean that to every $a \in A$ there exists a $b \in B$ and an arrrow $a \to b$; and to every $b \in B$, there exists an $a \in A$ and an arrow $a \to b$. The first assertion is an immediate consequence of the surjectivity of $\alpha : X \to A$, the second of the surjectivity of β.

3 Pregroupoids and torsors

If $\mathbf{G}$ is a groupoid with object set A, and $\alpha : X \to A$ is a map, there is a well known notion of (left) *action* of $\mathbf{G}$ on X: if g is an arrow in $\mathbf{G}$ and $x \in X$ satisfies $\alpha(x) = d_1(g)$, then $g \cdot x \in X$ is defined and $\alpha(g \cdot x) = d_0(g)$. Unit and associative laws are assumed. Then X becomes the set of objects of a groupoid, the *action* groupoid of the action; the arrows are pairs (g, x) with $d_1(g) = \alpha(x)$.

There is an evident category of left groupoid actions: an object is a pair consisting of a groupoid $\mathbf{G} = (G_1 \rightrightarrows A)$ and a map $\alpha : X \to A$ on which $\mathbf{G}$ acts; and a morphism $(\mathbf{G}, X \to A) \to (\mathbf{G}', X' \to A')$ is a pair consisting of a functor $\mathbf{G} \to \mathbf{G}'$ and a map $X \to X'$, which is compatible with the structural maps $X \to A$ and $X' \to A'$, and with the actions.

The category of right groupoid actions is defined similarly.

Finally, there is a category of bi-actions: an object consists a span of maps

$$A \xleftarrow{\ \alpha\ } X \xrightarrow{\ \beta\ } B$$

and a pair of groupoids $\mathbf{G}$ and $\mathbf{H}$ acting on the left and right on $X \to A$ and $X \to B$, respectively, and so that the two actions commute with each other

(A being the object set of $\mathbf{G}$, B the object set of $\mathbf{H}$).

The category of left *torsors* is a full subcategory of the category of left groupoid actions. We take the notion of torsor in the generality which was given to it by Duskin [2]. We say that the an action by $\mathbf{G}$ on $X \to A$ makes X into a $\mathbf{G}$-*torsor* if $X \to A$ is surjective, and the action groupoid is an equivalence relation. Also, X, or equivalently A, is assumed to be inhabited ("non-empty"). So for $x, y \in X$, there is *at most* one $g \in G_1$ with $g \cdot x = y$; such g, then, may be denoted yx^{-1}.

Consider a left $\mathbf{G}$-torsor structure on $\alpha : X \to A$. Since the action groupoid is an equivalence relation on X, we may consider its quotient $\beta : X \to B$. We say that $X \to A$ is a left $\mathbf{G}$-torsor *over* B, or that B is the *orbit set* of the left $\mathbf{G}$-torsor. We may write $B = X/\mathbf{G}$.

The category of right torsors is defined similarly as a full subcategory of the category of right actions. Finally, a bitorsor is a bi-action on a span $A \leftarrow X \to B$, which is a left torsor and a right torsor, and so that the structural map $X \to B$ is a quotient map for the action groupoid (equivalence relation) for the left action, and $X \to A$ is a quotient map for the action groupoid for the right action. The category of bitorsors is then defined as the full subcategory of the category of biactions consisting of bitorsors.

We denote the three categories thus described

$$lTORS,\ rTORS, \text{ and } lrTORS,$$

respectively.

If $\mathbf{K}$ is a groupoid, and A and B are inhabited subsets of its object-set, then the set $X := \mathbf{K}(A, B)$ carries a left action by the groupoid $\mathbf{K}(A, A)$, by precomposition, (with $\alpha = d_0 : X \to A$ as book-keeping map). If $\mathbf{K}$ is A-B-transitive in the sense described at the end of the previous section, this $\mathbf{G}(A, A)$-action is in fact a torsor. For, $\alpha : X \to A$ is surjective, by the A-B-transitivity assumption, and to any $x, y \in \mathbf{K}(A, B)$ there exists at most one $g \in \mathbf{K}(A, A)$ with $g \cdot x = y$. There exists such g precisely when x and y have same codomain (assumed to be in B), and so B is the orbit set of this torsor X (surjectivity of $X \to B$ follows again by the A-B-transitivity).

Similarly, $\mathbf{K}(A, B)$ carries a right action, by post-composition, by the groupoid $\mathbf{K}(B, B)$, and is in fact a right torsor, under assumption of A-B-transitivity. It is in fact a $\mathbf{K}(A, A)$-$\mathbf{K}(B, B)$-bitorsor. For, the actions commute, by the associativity of composition in $\mathbf{K}$.

In [7], we sketched (Example p. 199) how a torsor gives rise to a pre-groupoid. We shall extend this result by showing that this construction is in fact a description of an equivalence between the category of pregroupoids (as described in Section 2), and each of the three categories $lTORS$, $rTORS$ and $lrTORS$.

We first recall the passage (functor) from left torsors to pregroupoids. Let $\mathbf{G}$ be a groupoid, acting on the left on $\alpha : X \to A$ (A being the set of objects of $\mathbf{G}$), and assume that it makes X into a torsor with orbit set B (with quotient map denoted $\beta : X \to B$). The action is denoted by a dot.

If now $\beta(x) = \beta(y)$, they are in the same orbit for the action, and since the action is free (by the torsor condition), there is precisely one $g \in \mathbf{G}$ with $g \cdot x = y$. This g may therefore be denoted by a proper name, we call it yx^{-1}, so that $yx^{-1} \in \mathbf{G}$ is characterized by

$$yx^{-1} \cdot x = y.$$

We note that $d_1(yx^{-1}) = \alpha(x)$, and $d_0(yx^{-1}) = \alpha(y)$. If now $z \in X$ has $\alpha(z) = \alpha(x)$, $yx^{-1} \cdot z$ makes sense, and $\alpha(yx^{-1} \cdot z) = d_0(yx^{-1}) = \alpha(y)$. Also, by construction, z and $yx^{-1} \cdot z$ are in the same orbit for the action, so $\beta(z) = \beta(yx^{-1} \cdot z)$. Thus, if we put

$$yx^{-1}z := yx^{-1} \cdot z,$$

we have defined a ternary operation on X with the correct book-keeping for a pregroupoid. We proceed to check the four equations (1)-(4). First, xx^{-1} is the identity arrow at $\alpha(x)$, by the unitary law of the $\mathbf{G}$-action. So $xx^{-1} \cdot z = z$, by the unitary law of the $\mathbf{G}$-action, hence $xx^{-1}z = z$, proving (1). The equation (2), on the other hand, is just the defining equation for yx^{-1}.

Next consider $x, y, z, v \in X$ with $\beta(x) = \beta(y) = \beta(v)$ and $\alpha(x) = \alpha(z)$ (the reader may want to refer to the graphic display – a double quadrangle – of precisely these book-keeping conditions, in Section 1 above). Then

$$vy^{-1}(yx^{-1}z) = vy^{-1} \cdot (yx^{-1} \cdot z) = (vy^{-1} \circ yx^{-1}) \cdot z,$$

using the defining equations (twice) for the first equality sign, and the assiociative law for the $\mathbf{G}$-action for the second (composition in $\mathbf{G}$ denoted by $\circ$). On the other hand

$$vx^{-1}z = vx^{-1} \cdot z$$

by definition, so (3) will follow by proving

$$vy^{-1} \circ yx^{-1} = vx^{-1}.$$

By the freeness of the action, it suffices to prove

$$(vy^{-1} \circ yx^{-1}) \cdot x = vx^{-1} \cdot x.$$

The left hand side is by the associative law of the action the same as $vy^{-1} \cdot (yx^{-1} \cdot x)$, which is v, by two applications of the defining equations for the "fractions" of the form yx^{-1}; the right hand side is also v, by one such defining equation. This proves (3).

Finally, consider $x, y, z, w \in X$ with $\beta(x) = \beta(y)$ and $\alpha(x) = \alpha(z) = \alpha(w)$. First, we claim that $(yx^{-1}z)z^{-1} = yx^{-1}$; since the action is free, it suffices to see that $(yx^{-1}z)z^{-1} \cdot z = yx^{-1} \cdot z$. But both sides of this equation are equal to $yx^{-1}z$. So

$$(yx^{-1}z)z^{-1}w = ((yx^{-1}z)z^{-1}) \cdot w = yx^{-1} \cdot w = yx^{-1}w,$$

proving (4). Note that no equational assumptions (unitary or associative law of action) were used in the proof of (2) and (4).

It is clear that a morphism of left torsors, in the sense explained above, gives rise to a morphism of pregroupoids, so that the construction described is actually a functor

$$c_{\mathsf{l}} : lTORS \to \text{pregroupoids}. \tag{12}$$

(For a left torsor $X \to A$ with orbit set $X \to B$, the pregroupoid constructed is a pregroupoid on A, B.)

Similarly, we have a functor $c_{\mathsf{r}} : rTORS \to$ pregroupoids.

On the other hand, the envelope construction provides a functor

$$\text{pregroupoids} \to \text{groupoids},$$

$X \mapsto X^+$. If $A \leftarrow X \to B$ is a pregroupoid, then X^+ is a groupoid with object set $A + B$, and so we may form the $X^+(A, A)$-$X^+(B, B)$-bitorsor $X^+(A, B)$ by the recipe in the beginning of the present Section. So the construction of enveloping groupoid gives also gives rise to a functor

$$\text{pregroupoids} \xrightarrow{env} lrTORS.$$

Finally, there are the two obvious forgetful functors

$$lrTORS \xrightarrow{U_{\mathsf{r}}} lTORS \qquad lrTORS \xrightarrow{U_{\mathsf{l}}} rTORS$$

forgetting the right and left action, respectively.

We collect the functors described here together in the diagram

$$\begin{tikzcd}
lrTORS \arrow[r, "U_{\mathsf{l}}"] \arrow[d, "U_{\mathsf{r}}"'] & rTORS \arrow[d, "c_{\mathsf{r}}"] & \\
lTORS \arrow[r, "c_{\mathsf{l}}"'] & \text{pregroupoids} \arrow[r, "env"'] & lrTORS.
\end{tikzcd}$$

Theorem 2 *All functors exhibited here are equivalences. Any endofunctor composed of functors (=cyclic composite) exhibited here is isomorphic to the relevant identity functor. The square commutes on the nose. The cyclic composite (13) (below) is, on the nose, the identity functor on the category of pregroupoids.*

In the next Section, we shall describe yet another category equivalent to these, namely the category of (groupoid-) fibrations over the groupoid **I** (=the generic invertible arrow). – Note that among the four categories proved equivalent in the Theorem, the category of pregroupoids is the most "compact", in the sense of involving least data; this is why it is possible to have certain *strict* equalities between functors with values in the category of pregroupoids.

Proof. We first prove that the square commutes. Let $A \leftarrow X \rightarrow B$ be a bitorsor for left and right actions of the groupoids $\mathbf{G} = G \rightrightarrows A$ and $\mathbf{H} = H \rightrightarrows B$, respectively. The two pregroupoids constructed by the two functors $c_{\mathsf{l}} \circ U_{\mathsf{r}}$ and $c_{\mathsf{r}} \circ U_{\mathsf{l}}$ both have $A \rightarrow X \rightarrow B$ for its underlying sets, so it suffices to see that the two ternary operations on X agree. Consider x, y, z satisfying the relevant book-keeping conditions for formation of the two possible $yx^{-1}z$. So there are (unique) arrows $g \in G$ and $h \in H$ so that $g \cdot x = y$ and $x \cdot h = z$. Then $g \cdot z = (yx^{-1}z)_{\mathsf{l}}$ and $y \cdot h = (yx^{-1}z)_{\mathsf{r}}$ (with $(yx^{-1}z)_{\mathsf{l}}$, resp. $(yx^{-1}z)_{\mathsf{r}}$, denoting the ternary operation coming from the left, respectively right, torsor structure). We then have

$$(yx^{-1}z)_{\mathsf{l}} = g \cdot z = g \cdot (x \cdot h) = (g \cdot x) \cdot h = y \cdot h = (yx^{-1}z)_{\mathsf{r}},$$

using for the middle equality sign that the two actions commute with each other.

We next prove that the composite

$$\text{pregroupoids} \xrightarrow{env} lrTORS \xrightarrow{U_{\mathsf{r}}} lTORS \xrightarrow{c_{\mathsf{l}}} \text{pregroupoids} \tag{13}$$

is the identity functor (on the nose). Starting with a pregroupoid $A \leftarrow X \rightarrow B$, the composite of the two first functors here gives the left $X^+(A, A)$-torsor $X^+(A, B) = X$, with ternary operation given in terms of the composition $\circ$ in X^+. But the composition of arrows from A to A with arrows from A to B in X^+ are precisely defined by the ternary operation in X, cf. the entry with address $(1, 2)$ in the table which defines the composition $\circ$.

Next, we prove that the composite

$$lTORS \xrightarrow{c_{\mathsf{l}}} \text{pregroupoids} \xrightarrow{env} lrTORS \xrightarrow{U_{\mathsf{r}}} lTORS \tag{14}$$

is isomorphic to the identity functor on $lTORS$ (a similar statement holds for $rTORS$). Given a left $\mathbf{G}$-torsor $A \leftarrow X$, with $X \rightarrow B$ as orbit set, the underlying object of the "new" torsor is again $A \leftarrow X$, so we just have to provide isomorphisms between the acting groupoids, in this case $\mathbf{G} \cong X^+(A, A)$ (compatible with the actions). The object sets of both $\mathbf{G}$ and $X^+(A, A)$ are A. The isomorphism on the arrow sets is given by sending $g : a \rightarrow a'$ into yx^{-1} where $y = g \cdot x$, $x \in X$ any element of X over the codomain of the arrow g. The passage the other way takes a "fraction" yx^{-1} to the unique arrow g with $g \cdot x = y$ (as also anticipated by the notation yx^{-1} which we used for this g in the discussion of torsors).

The same argument, applied twice (once on the left and once on the right) proves that the composite

$$lrTORS \xrightarrow{U_{\mathsf{r}}} lTORS \xrightarrow{c_{\mathsf{l}}} \text{pregroupoids} \xrightarrow{env} lrTORS$$

is isomorphic to the identity functor on $lrTORS$. So the three functors displayed here provides a "cycle" of three arrows, with all three cyclic composites isomorphic to the identity functor of the respective vertex. So all three of them are equivalences. Similarly, also c_{r} and U_{l} are equivalences. This proves the Theorem.

Corollary 1 *There is an adjoint equivalence*

$$lTORS \xrightarrow{ad} rTORS$$

Namely, take ad to be the composite

$$lTORS \xrightarrow{c_l} \text{pregroupoids} \xrightarrow{env} lrTORS \xrightarrow{U_l} rTORS$$

The quasi-inverse is constructed similarly (replace all l's by r's and conversely). It may also be denoted ad. – This ad-equivalence is classical, and plays (at least in the case of bitorsors over groups) an important role in Giraud's book, [4] III.1, where the notation ad also appears.

A torsor (right, say) $(\mathbf{G}, X \to B)$, where $B = 1$ (and thus $\mathbf{G}$ is just a *group*) is usually called a *principal* $\mathbf{G}$*-bundle over* A, where A is the orbit set. The construction of the adjoint groupoid $ad(X)$, or gauge groupoid XX^{-1}, which is a groupoid with A as object set, is classical, due to Ehresmann, [3]. In our context, it appears as a full subgroupoid of X^+, which in this case is a groupoid with $A + B = A + 1$ as object set. Note that the functor $X \mapsto X^+$ is faithful (since X^+ contains X as a subset), whereas $X \to XX^{-1}$ is not. A description of the enveloping (=comprehensive) groupoid X^+ for the case of principal bundles was given in [9]. The construction there was carried out without the notion of pregroupoid; but then the naturality and symmetry of the construction is not so visible.

4 Fibrations over I

We discuss fibered categories $\mathbf{E} \to \mathbf{B}$, see e.g. [4] for this notion. It is well known that the fibres of such a fibration are groupoids if and only if all arrows in $\mathbf{E}$ are cartesian. For fixed base category $\mathbf{B}$, we thus get the category of such "fibrations-in-groupoids" over $\mathbf{B}$. If $\mathbf{B}$ happens to be itself a groupoid, then the total category $\mathbf{E}$ of a fibration-in-groupoids is also a groupoid. We let $\mathbf{I}$ denote the groupoid containing the "generic invertible arrow", in other words, $\mathbf{I}$ has two objects a_0 and b_0, and besides the two identity arrows, it has one arrow $i : a_0 \to b_0$ and one arrow $i^{-1} : b_0 \to a_0$, and no other arrows. It can be described in very many ways; for instance, it is the enveloping groupoid of the terminal pregroupoid $\mathbf{1} = 1 \leftarrow 1 \to 1$.

The following result is an application of the enveloping groupoid.

Theorem 3 *The category of bitorsors (hence also the category of pregroupoids, by Theorem 2) is equivalent to the category of fibrations-in-groupoids over* $\mathbf{I}$ *with inhabited total category.*

Proof. Given a fibration $\gamma : \mathbf{X} \to \mathbf{I}$. Let A be the set of objects a in $\mathbf{X}$ with $\gamma(a) = a_0$, and B the set of objects in $\mathbf{X}$ with $\gamma(b) = b_0$. If X is inhabited, then so are both A and B. Then $\mathbf{X}$ is A-B transitive, in the sense of Section 2 (end). For, given $b \in B$, take a (cartesian) arrow over i with codomain b; it will be an arrow from an object in A to b. Similarly for a given object $a \in A$ (utilize i^{-1}). Therefore, $\mathbf{X}(A, B)$ is a $\mathbf{X}(A, A)$-$\mathbf{X}(B, B)$-bitorsor. Conversely, given a $\mathbf{G}$-$\mathbf{H}$ bitorsor X (where the object sets of $\mathbf{G}$ and $\mathbf{H}$ are A and B, respectively). Consider it as a pregroupoid X on A, B via the functor from bitorsors to pregroupoids, described in Theorem 2. Its enveloping groupoid X^+ is a groupoid with object set $A + B$. We get a functor $\gamma : X^+ \to \mathbf{I}$, easily described ad hoc (mapping each $a \in A$ to a_0 etc.); alternatively apply the (left adjoint) functor $(-)^+$: pregroupoids $\to$ groupoids to the unique pregroupoid morphism $X \to \mathbf{1}$.

We note that if $\mathbf{X} \to \mathbf{I}$ is an inhabited fibration, then the inclusion of either of the two fibres, i.e. the "end" groupoids $\mathbf{X}(A, A)$ and $\mathbf{X}(B, B)$, is an equivalence of categories. For, they are clearly full and faithful, and essential surjectivity follows from the A-B transitivity.

However, the *functor* which to a fibration $\mathbf{X} \to \mathbf{I}$ associates the groupoid in either end, say $\mathbf{X}(A, A)$ is not an equivalence; it is not even faithful.

Now the category of inhabited fibrations-in-groupoids over $\mathbf{I}$ is in an evident way a 2-category; the 2-cells are just natural transformations. Thus, the components of the 2-cells (natural transformations) are vertical arrows. In particular, 2-cells are invertible. Since the inclusions of each of the two end-groupoids (or edge groupoids) are equivalences, it follows that a 2-cell between two functors over $\mathbf{I}$ is completely given by its components on the objects in the a_0-end, or by its components on the objects in the b_0-end. In particular, enriching the category of principal bundles (over varying groups) into a 2-category only amounts to considering the category of groups as a 2-category in the standard way (2-cells being given by "conjugation by an element in the codomain group").

From the equivalence of the Theorem follows that there is a 2-dimensional structure (with all 2-cells invertible) on the category of pregroupoids, and the rest of this section just consists in making this 2-dimensional structure explicit.

So consider two inhabited fibrations $\mathbf{X} \to \mathbf{I}$ and $\mathbf{X}' \to \mathbf{I}$, and two functors f and $g : \mathbf{X} \to \mathbf{X}'$ over $\mathbf{I}$, and let $\tau : f \to g$ be a natural transformation. Denote by A and B the set of objects in $\mathbf{X}$ over a_0 and b_0, respectively, and similarly A' and B' in $\mathbf{X}'$. The functor f, being a functor over $\mathbf{I}$, induces maps $A \to A'$ and $B \to B'$, these maps are also just denoted f. Similarly the maps $A \to A'$ and $B \to B'$ induced by g are denoted g. Finally, let X be the pregroupoid $\mathbf{X}(A, B)$ on A, B, and similarly for X' on A', B'.

Consider for $a \in A$ the arrow $\tau_{\mathsf{a}} : f(a) \to g(a)$ in $\mathbf{X}'(A, A)$. For any $u : g(a) \to b$ (b an object $\in B'$; such u exist by A'- B'-transitivity of $\mathbf{X}'$), let $t(a, u) : f(a) \to b$ denote $\tau_{\mathsf{a}} \circ u$. Then of course $\tau_{\mathsf{a}} = t(a, u) \circ u^{-1}$. Note that $u \in X'$. Similarly, for $b \in B$, $\tau_{\mathsf{b}} : f(b) \to g(b)$ may be written $\tau_{\mathsf{b}} = v^{-1} \circ s(b, v)$, where $s(b, v)$ is $v \circ \tau_{\mathsf{b}}$. Note that $v \in X'$.

The reader may find the following display helpful:

$$\begin{tikzcd}
f(a) \arrow[d, "\tau_{\mathsf{a}}"'] \arrow[dr, "t(a,u)"] & \\
g(a) \arrow[r, "u"'] & b
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a \arrow[r, "v"] \arrow[dr, "s(b,v)"'] & f(b) \arrow[d, "\tau_{\mathsf{b}}"] \\
& g(b)
\end{tikzcd}$$

Now t and s are (partially defined) maps which satisfy three equations, and together, encode the information of the 2-cell τ in pure pregroupoid terms. Precisely, $t(a, u)$ is defined whenever $\alpha'(u) = g(a)$ (α' denoting domain formation $X' \to A'$); and then $\alpha'(t(a, u)) = f(a)$, $\beta'(t(a, u)) = \beta'(u)$, where $\beta' : X' \to B'$ is codomain formation. Similarly, $s(b, v)$ is defined whenever $\beta'(v) = f(b)$, and then $\beta'(s(b, v)) = g(b), \alpha'(s(b, v)) = \alpha'(v)$. The following equations hold (assuming that the book-keeping conditions make them meaningful); we omit the sign $\circ$ for composition in $\mathbf{X}$ and $\mathbf{X}'$:

$$t(a, g(x)) = s(b, f(x)) \tag{15}$$
$$t(a, u)v^{-1}w = t(a, uv^{-1}w) \tag{16}$$
$$wu^{-1}s(b, v) = s(b, wu^{-1}v) \tag{17}$$

The equation (15) just follows from the naturality of τ with respect to $x : a \to b$. For, consider the commutative naturality square (expressing naturality of

τ with respect to $x : a \to b$)

$$\begin{tikzcd} f(a) \arrow[r, "f(x)"] \arrow[d, "\tau_{\mathsf{a}}"'] \arrow[dr] & f(b) \arrow[d, "\tau_{\mathsf{b}}"] \\ g(a) \arrow[r, "g(x)"'] & g(b) \end{tikzcd} \tag{18}$$

The diagonal makes both triangles commute, and their commutativity express that the diagonal is, respectively, $t(a, g(x))$ and $s(b, f(x))$, which thus are equal.

For the equation (16), both sides are equal $\tau_{\mathsf{a}} \circ u \circ v^{-1} \circ w$, and for (17), both sides are equal to $w \circ u^{-1} \circ v \circ \tau_{\mathsf{b}}$.

We now show that the data of such t and s, satisfying the three equations, come from a unique natural transformation τ.

To define τ_{a} for $a \in A$, pick by A-B-transitivity an arrow $u : g(a) \to b$ and put

$$\tau_{\mathsf{a}} := t(a, u) \circ u^{-1}.$$

That this is independent of the choice of u is an immediate consequence of (16). Similarly,

$$\tau_{\mathsf{b}} := v^{-1} \circ s(b, v)$$

for some $v : a \to f(b)$; this is independent of choice of v by (17). It remains to check naturality of the τ thus constructed. Now, τ is natural with respect to arrows $a \to b$ (for $a \in A, b \in B$); this follows from (15), by chosing $u := g(x), v := f(x)$ in the defining equations for τ_{a} and τ_{b}, respectively (contemplate (18), now with the two expressions in (15) as diagonal).

But arrows of the form $a \to b$ (for $a \in A, b \in B$) generate $\mathbf{X}$ as a groupoid, so therefore, naturality of τ with respect to such arrows implies naturality with respect to all arrows in $\mathbf{X}$.

5 Examples

Let A and B be two smooth manifolds of dimensions n and k, say, with $n \geq k$, and consider a geometric distribution D on A of codimension equal to the dimension k of B. Let X be the set of all 1-jets of maps from A to B

with D as kernel. Precisely, for each $a \in A$, consider the set X_{a} of 1-jets at a of maps $f : A \to B$ such that the kernel of $df_{\mathsf{a}} : T_{\mathsf{a}}(A) \to T_{\mathsf{f(a)}}(B)$ is the linear subspace $D_{\mathsf{a}} \subseteq T_{\mathsf{a}}(A)$. For dimension reasons, then, df_{a} is surjective. Let X be the disjoint union of all the X_{a}'s. Then X is born with a map $\alpha : X \to A$, but is also comes with a map $\beta : X \to B$, namely to the 1-jet of f at a, asssociate $f(a) \in B$. (Actually X is a submanifold of the standard jet manifold $J^1(A, B)$ of 1-jets of maps from A to B.)

We shall equip this $A \leftarrow X \to B$ with a ternary operation making it into a pregroupoid. So let x, y and z be 1-jets with D as kernel, in the sense explained, represented by functions f, g and h. Assume $\alpha(x) = \alpha(z), = a$, say, and $\beta(x) = \beta(y), = b$, say. Since $d_{\mathsf{a}}f$ and $d_{\mathsf{a}}h$ are surjective linear maps with the same kernel D_{a}, there is a unique bijective linear map $\kappa : T_{\mathsf{f(a)}}B \to T_{\mathsf{h(a)}}B$ with $d_{\mathsf{a}}f \circ \kappa = d_{\mathsf{a}}h$ (composing from left to right). By the Inverse Function Theorem, there is locally around $f(a)$ a smooth map k with $d_{\mathsf{f(a)}}k = \kappa$. We put $yx^{-1}z$ equal to the 1-jet af $a' = \alpha(y)$ of the composite $g \circ k$. This makes sense, since $g(a') = f(a)$ by the book-keeping assumption $\beta(y) = \beta(x)$.

The verification of the four equations is straightforward. – The edge groupoids of this pregroupoid are the following: $X^{-1}X$ is the groupoid of all invertible 1-jets $b \to b'$ from B to itself; XX^{-1} is the groupoid of 1-jets $a \to a'$ from A to itself which "take D into D", i.e. 1-jets at a of functions F such that $dF_{\mathsf{a}} : T_{\mathsf{a}}(A) \to T_{\mathsf{a}^0}(A)$ maps D_{a} into D_{a^0}.

This latter groupoid also occurs as edge groupoid of a principal $GL(k)$ bundle Y over M, namely the bundle of surjective linear maps $T_{\mathsf{a}}A \to \mathbf{R}^{\mathsf{k}}$ with D as kernel. But note that there is no *natural* way of mapping the pregroupoid X to the pregroupoid Y; in fact, such a map would amount to a framing of the tangent bundle of B.

Pregroupoids $A \leftarrow X \to B$ with both A and B equal to the 1-point set were considered in [5] under the name *pregroups*. The two edge groupoids are in this case just groups; in fact two groups which are isomorphic, but not canonically isomorphic, unless they are abelian. *Picking* an element in X will provide a specific isomorphism between the two edge groups.

My contention is that the notion of pregroup is simpler than that of group. In fact, in some cases, it precedes the notion of group in the process of understanding. How many ways can you put three pigeons into three pigeon holes ? Without knowing anything about neither pigeons nor mathematics, most people will, after a moments reflection, be able to answer "six". This *number* of ways (or this *set* of ways, as mathematicians prefer to say) carries canonically the structure of pregroup (being a set of bijections from one set to another), but does not carry structure of group. The two edge groupoids are of course both "the" symmetric group S_3, namely the group of permutations of the given three pigeons, respectively of the three given pigeon holes. What is a permutation of three pigeons ? "Put the white pigeon in the place where the grey pigeon was, and put the grey pigeon in the place where ...". Not a very natural thing to do, and in any case is equivalent to describing permutations in terms of the *places* the pigeons were occupying, before and after the permutation. These places may as well be called "pigeon holes", and then we are precisely describing the elements of the pigeon-permutation group in terms of fractions yx^{-1} made out of the pregroup.

A more mathematical version of this comment is the following: What is "the" symmetric group in three letters ? What is the sense of the definite article "the"? The group of permutations of the three letters A, a, and α is not the *same* as the group of permutations of the three letters b, c, d; these groups are not even *canonically isomorphic* (which in mathematics is sufficient justification for using the definite article). For, an isomorphism between them depends on *choosing* a bijection between the two three-letter sets; a different choice may change the constructed isomorphism by a conjugation. This means that "the symmetric group in three letters" is well defined only in the category of groups and conjugation classes of group homomorphisms, i.e., "*the* symmetric group in three letters" is an object in the category of "liens", or "bands", in the termonology of [4] resp. [2].

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