

New methods for old spaces: synthetic differential geometry

Anders Kock

Introduction

The synthetic method consists in consideration of a class of objects in terms of their (often axiomatically assumed) mutual relationship, say incidence relations, - disregarding what the objects are “made up of”. For geometry, this method goes back to the time of Euclid. With the advent of category theory, it became possible to make the notion of “relationship” more precise, in terms of the *maps* in some category \mathcal{E} of “spaces”, in some broad sense of the word.

A synthetic theory might be presented by giving axioms for some good category \mathcal{E} , possibly with some added structure. Thus, the list of axioms could begin: “let \mathcal{E} be a good category, and let R be a commutative ring object (to be thought of as the number line) in \mathcal{E} ...”.

When is a category \mathcal{E} suitable for playing the role of a place, or scene, where a theory, say axiomatic, of “spaces” and their geometry can be developed?

Experience since the 1950s has showed that many *toposes* \mathcal{E} are suitable. Thus for instance, the topos of simplicial sets was shown to have the capacity for studying homotopy theory, without recourse to the notion of topological space.

A crucial point is that a topos is in many respects like the category of sets; in fact, the understanding of the category of sets category is distilled out of our experience with spaces in a broad sense of the word. In fact, many texts in synthetic differential geometry talk about the assumed \mathcal{E} *as if* it were the category of sets, just making sure not to use the law of excluded middle, which holds in the category of *abstract* sets (discrete spaces), but fails for most other categories of spaces. This is related to the contradiction between the discrete and the continuum, see Section 6 below.

– So much for the question “why topos?” But why “ring”? Because we may hopefully use that ring for introducing *coordinates* and thereby supplement, or even replace, geometric reasoning with algebraic calculation. Such ring (a “number line”) is however not the central geometric notion in differential geometry:

I would like to advance the thesis that the notion of *neighbour* elements is a more basic notion in differential geometry; that central differential geometric concepts can be formulated in terms of that relation; and that this allows one to present such concept in terms of exact *pictures*. Some such concepts and pictures are presented in Section 1. There is no notion of “limiting positions” involved. (A “line type” ring R will enter later, and the axiomatics to be presented will involve such ring.)

So the following Section is a kind of wishful thinking, or dream. But, as we shall partly substantiate, they are dreams that come true. The guideline for this is the theory developed in Algebraic Geometry by Kähler, Grothendieck, and many others: the first neighbourhood of the diagonal of a scheme, or the (first) prolongation space of a smooth manifold ([23] p. 52). From these sources, models for the axiomatics are drawn (and they are briefly recalled in Section 4).

We use the abbreviation ‘SDG’ for ‘synthetic differential geometry’. It is not a mathematical *field*, but a *method*. Not really a *new* one, but category theory has, over several decades by now, contributed, by making the synthetic method more explicit; see also Section 7.

1 Some differential geometry in terms of the neighbour relation

The spaces M considered in differential geometry come equipped with a reflexive symmetric relation \sim_M , the (first order) neighbour relation, often mentioned in the heuristic part of classical texts, but rarely made precise, neither how it is defined, nor how one reasons with it

One aim in SDG is to make precise how one reasons with the neighbour relation, (and this is done axiomatically); it is not a main aim to describe how it is constructed in concrete contexts.

The relation \sim_M is *not* transitive; see the Remark after Corollary 3.1 why transitivity is incompatible with the axiomatics to be presented.

The subobject $M_{(1)} \subseteq M \times M$ defining the relation \sim_M , is called the (*first*) *neighbourhood of the diagonal* of M . This terminology is borrowed from the

theory of schemes in algebraic geometry, or in differential geometry (see e.g. [23], who call it the *(first) prolongation space* of M).

To state some notions of differential-geometric nature, we shall talk about the category \mathcal{E} as if it were the category of sets. If the space M is understood from the context, we write \sim instead of \sim_M . There are also higher order neighbour relations \sim_2, \sim_3 , on M , with $(x \sim y) \Rightarrow (x \sim_2 y) \Rightarrow (x \sim_3 y), \dots$. The neighbour relations \sim_k will not be transitive, but $x \sim_k y$ and $y \sim_l z$ will imply $x \sim_{k+l} z$. For $x \in M$, we call $\{y \in M \mid y \sim_k x\} \subseteq M$ the k th order *monad* around x , and we denote it $\mathfrak{M}_k(x)$. In the axiomatics to be presented, it represents the notion of k -jet at x .

The higher neighbour relations will not be discussed in the present text, but see e.g. [19].

The spaces one considers live in some category \mathcal{E} of spaces; maps in \mathcal{E} preserve the neighbour relations (which is a “continuity” property). But the neighbour relation on a product space $M \times N$ will be more restrictive than the product relation: we will not in general have that $m \sim m'$ and $n \sim n'$ implies that $(m, n) \sim (m', n')$; see the Remark at the end of Section 3.

The first order monad $\mathfrak{M}_1(x)$ will also be denoted $\mathfrak{M}(x)$.

We present some differential geometric notions that may be expressed in terms of the first order neighbour relation \sim . The argument that they comprise the classical notions with the corresponding names, may, for most of them, be found in [19].

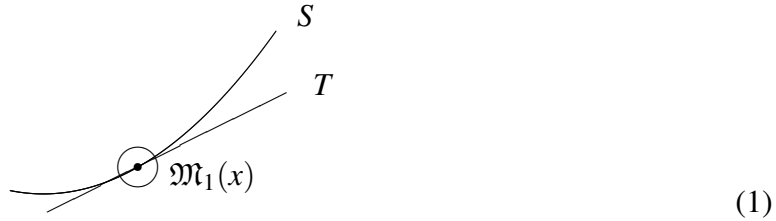
1.1 Tangency

The fundamental notion of *tangency* or (first order) *contact* is expressed as follows:

If S and T are subobjects of M , and if $x \in S \cap T$, we say that S and T are *tangent to each other*, or have *(first order) contact* at x , if $S \cap \mathfrak{M}_1(x) = T \cap \mathfrak{M}_1(x)$. One also says that S and T *touch* at x . To “touch at x ” is clearly an equivalence relation on the set of subobjects of M containing x .

Pictures (not exact, though) can conveniently be drawn for this notion: (the bullet indicates x , and the interior of the circle indicates $\mathfrak{M}_1(x)$; ignore the fact

that T looks like a line:¹

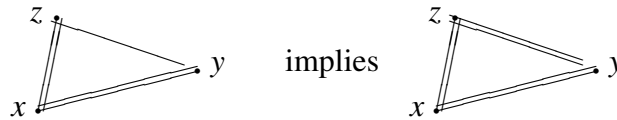


1.2 Geometric distributions

A (geometric) *distribution* on M is a reflexive symmetric relation \approx refining \sim (i.e. $x \approx y$ implies $x \sim y$). It is called *involutive* if it satisfies, for all x, y, z in M :

$$(x \approx y) \wedge (x \approx z) \wedge (y \sim z) \quad \text{implies} \quad y \approx z.$$

A relevant picture is the following; single lines indicate the neighbour relation \sim , double lines indicate the assumed “strong” neighbour relation \approx .



For instance, if $f : M \rightarrow N$ is any map between spaces, the relation on \approx on M defined by $x \approx y$ iff $(x \sim y) \wedge (f(x) = f(y))$ is a distribution, in fact an involutive one.

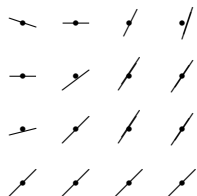
An *integral subset* of a distribution \approx is a subset $F \subseteq M$ such that on F , the relations \sim and \approx agree. An important integration Theorem in differential geometry is Frobenius’ Theorem, whose conclusion is that for an *involutive* distribution, there exist maximal connected integral subsets (leaves).

Such integration results can usually not be *proved* in the context of SDG (even the very formulation may require some further primitive concepts), since they in a more serious way depend on limits and on completeness of the real number system. Sometimes, SDG can *reduce* one integration result to another; this is also an old endeavour in classical differential geometry, e.g. Lie has many results

¹The notion of *line* is an invention of the age of civilization, whereas the notion of *touching* is known already from pre-civilized stone ages and before.

about which differential equations can be solve *by quadrature*, i.e. by reduction to existence of anti-derivatives.

Example. The following is meant as a sketch of an (involutive) distribution in the plane. Consider



In this picture, the “line segments” are the \approx -monads $\mathfrak{M}_{\approx}(x) := \{y \mid y \approx x\}$ around (some of) the points x (drawn as dots) of M . But note that the notion of “line” has not yet entered in our vocabulary, let alone coordinate systems like $R \times R$; when such things are present, an ordinary first order differential equation

$$y' = F(x, y),$$

as in the Calculus Books, gives rise to such a picture, known as the “direction field” of the equation: through each point $(x, y) \in R \times R$, one draws a “little” line segment $S(x, y)$ with slope $F(x, y)$.

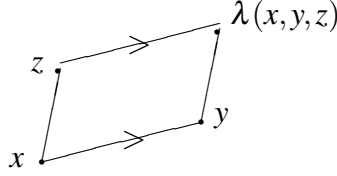
The “integral subsets” of a distribution of this kind are essentially (the graphs of) the solutions of the differential equation.

I cannot draw a good picture of a non-involutive distribution: paper is 2-dimensional. But in three dimensions: consider the scales of a ripe pine cone, and extrapolate radially.

One might experiment with the following as a definition: a space is (at most) 2-dimensional if every distribution on it is involutive.

1.3 Affine connections

An *affine connection* on a space M is a law λ which completes any configuration (x, y, z) consisting of three points x , y , and z with $x \sim y$, $x \sim z$ by a fourth point $\lambda(x, y, z)$ with $y \sim \lambda(x, y, z)$ and $z \sim \lambda(x, y, z)$:



(2)

expressing “infinitesimal parallel transport of z along \overline{xy} ”, or “constructing an infinitesimal parallelogram”. (We assume $\lambda(x, x, z) = z$, and $\lambda(x, y, x) = y$.) The connecting lines indicate the assumed neighbour relations. We use different signature for the edges xy and xz , since we do not assume the symmetry condition $\lambda(x, y, z) = \lambda(x, z, y)$. If symmetry holds, λ is called a *symmetric* or *torsion free* connection.

A *geodesic* for a given torsion free affine connection on M is a subset $S \subseteq M$ which is stable under λ in the sense that if $x \sim y$ and $x \sim z$ with x, y and z in S , then $\lambda(x, y, z) \in S$.

The *curvature* of such a connection may be described combinatorially by asking the question: what happens if we transport $z \sim x_0$ around a circuit from x_0 to x_1 , then from x_1 to x_2 , and finally from x_2 back to x_0 ? This makes sense whenever $x_0 \sim x_1 \sim x_2$ and $x_0 \sim x_2$ (the latter requirement is not automatic: the relation \sim is not transitive). The result of such circuit transport gives a new point $z' \sim x_0$; thus the “infinitesimal 2-simplex” (x_0, x_1, x_2) provides an automorphism $z \mapsto z'$, denoted $R(x_0, x_1, x_2)$, of the pointed set $\mathfrak{M}(x_0)$; this is the curvature of λ , more precisely, the curvature of λ is the law which to an infinitesimal 2-simplex (x_0, x_1, x_2) associates the described automorphism of $\mathfrak{M}(x_0)$. If this automorphism is the identity map for all infinitesimal 2-simplices, the connection is called *flat*. (Any affine connection on a 1-dimensional space M is flat. One may even experiment with this as a definition of “ M is of dimension (at most) 1”.)

1.4 Differential forms

Differential forms are, in analytic differential geometry, certain functions taking values in a *ring* R of quantities (or in a *module* over R), but are in the present context (equivalent to) a special case of a more primitive, non-quantitative, kind of thing: Thus, in SDG, one may, for any group G , define “(combinatorial) G -valued k -form on a space M ” to mean a “function ω , which takes as input infinitesimal k -simplices ($k + 1$ -tuples of mutual neighbour points in M), and returns as output

elements in G ". One imposes the normalization condition that $\omega(x_0, \dots, x_k) = e$ whenever two of the x_i s are equal (where e denotes the neutral element of G). A G -valued 0-form on M is then just a function $f : M \rightarrow G$; it has a "coboundary" df , which is a G -valued 1-form, defined by $df(x_0, x_1) := f(x_0)^{-1} \cdot f(x_1)$. A G -valued 1-form ω on M has a coboundary $d\omega$, which is a G -valued 2-form defined by

$$d\omega(x_0, x_1, x_2) := \omega(x_0, x_1) \cdot \omega(x_1, x_2) \cdot \omega(x_2, x_0).$$

The 1-form ω is *closed* if $d\omega$ is constant e . The 1-form df is always closed.

The group G carries a canonical closed G -valued 1-form, namely df , where $f : G \rightarrow G$ is the identity function. This is the Maurer-Cartan form of G .

If there is given data identifying all the $\mathfrak{M}(x)$ of a given manifold M with each other, then the curvature of an affine connection λ may be seen as a 2-form with values in the automorphism group of the pointed set $\mathfrak{M}(x_0)$ (for some, hence any, $x_0 \in M$). (Alternatively, one gets a 2-form "with local coefficients"; then no identification data is needed.)

For most G , G -valued differential forms are *alternating*: interchanging two of the input entries implies inversion of the value of the form. In particular

$$\omega(x_0, x_1)^{-1} = \omega(x_1, x_0).$$

Such 1-form ω on M then defines a geometric distribution on M by saying $x \approx y$ iff $\omega(x, y) = e$. If ω is closed, the \approx which is defined by ω is involutive.

There is a relationship between combinatorial group valued 1-forms, on the one hand, and the general notion of connection in a fibre bundle, or in a groupoid, on the other. This we expound in Section 4.4 below.

In case the value group G is commutative (additively written), there are, for good M and G , a natural bijection between combinatorial forms, and the standard multilinear alternating forms on $T(M)$, the tangent bundle of M , see [12] I.18.

1.5 Characteristics and envelopes

Assume that a space T parametrizes a family $\{S_t \mid t \in T\}$ of subsets S_t of a space M . Then for $t_0 \in T$, the *characteristic* C_{t_0} (at the parameter value t_0) is the intersection of all the neighbour sets of S_{t_0} , i.e.

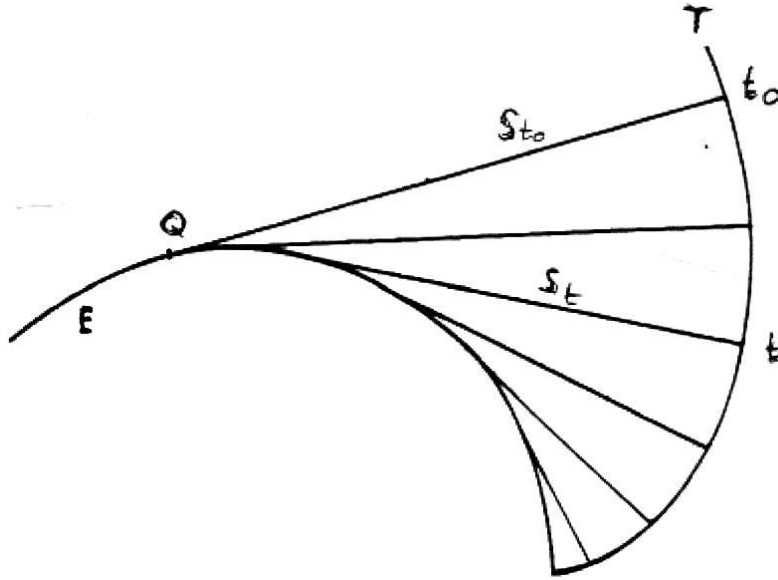
$$C_{t_0} := \bigcap_{t \sim t_0} S_t,$$

and the *envelope* E of the family is

$$E := \bigcup_{t_0 \in T} C_{t_0} = \bigcup_{t_0 \in T} \bigcap_{t \sim t_0} S_t.$$

Under non-singularity assumptions, E is the *disjoint* union of the characteristics, i.e. there is a function $\tau : E \rightarrow T$ associating to a point Q of E the parameter value t such that $Q \in C_t$. So $Q \in \bigcap_{t' \sim \tau(Q)} S_{t'}$.

Here is a picture of a family of subsets (lines) in a Euclidean plane (here, the family of normals at the points of the curve T); the characteristics are singleton sets. The characteristic for the parameter value t_0 is the singleton subset $\{Q\}$.



(picture adapted from [F. Borceux, *Invitation à la géométrie*, Ciaco 1986].)

We would like to prove that S_{t_0} is *tangent to E at Q* , i.e. that $E \cap \mathfrak{M}(Q) = S_{t_0} \cap \mathfrak{M}(Q)$. We can prove at least the inclusion $E \cap \mathfrak{M}(Q) \subseteq S_{t_0} \cap \mathfrak{M}(Q)$. For let $Q' \sim Q$ and $Q' \in E$. Then $\tau(Q') \in T$ is defined, and since τ , as any function, preserves \sim , the assumption $Q \sim Q'$ implies $t_0 = \tau(Q) \sim \tau(Q')$. Now $Q' \in \bigcap_{t' \sim \tau(Q')} S_{t'}$, in particular $Q' \in S_{t_0}$. Therefore

$$E \cap \mathfrak{M}(Q) \subseteq S_{t_0} \cap \mathfrak{M}(Q).$$

(To pass from the proved inclusion to the desired equality would involve a dimension argument like: “the two sets have the same dimension (here: dimension 1), so the inclusion implies equality”. We don’t have such argument available at this primitive stage.)

The primary notion here is that of *characteristic*; the envelope is derived from the characteristics. In the literature, based on analytic geometry, the characteristic C_{t_0} is sometimes, with some regret or reservation, defined as “the limit of the sets

$S_{t_0} \cap S_t$ as t tends to t_0 ". In [5], Courant (talking about a family of surfaces S_t in 3-space, where the intersection of any two of them therefore, in non-degenerate cases, is a curve) thus writes about a characteristic curve, say C_{t_0} , for the family: "This curve is often referred to in a non-rigorous but intuitive way as the intersection of "neighbouring" surfaces of the family" (p. 169) (offering instead: "If we let h tend to zero, the curve of intersection will approach a definite limiting position" (p. 180). What is the topology on the set of subsets which will justify the limit notion?)

We shall see (Subsection 3.2) that the axiomatics for SDG makes the "limit" intersection curve rigorous by replacing the dubious *limit* with the simultaneous intersection of *all* neighbouring curves, now with "neighbouring S_t " in the strict sense of $t \sim t_0$. Thus, the "non-rigorous but intuitive" description in Courant's text now gets the status: *rigorous and intuitive*.

2 Neighbours in the context of Euclidean geometry

In this Section, we move from the Stone Age into the era of Civilization, and assume that some classical Euclidean geometry (plane, say) is available in a space E (with a given neighbour relation \sim). In particular, there are given subsets called points and lines; they are the *affine* subspaces of E (without yet assuming the existence of a "number" line $R \subseteq E$, i.e. a line equipped with a commutative ring structure).

Then we can be more explicit about our wishes for the compatibilities between the Euclidean notions and the combinatorics of the neighbour relation. We refrain from calling these wishes for "Axioms", since they (lead to and) are subsumed in a more complete comprehensive axiom scheme later on; so we call these wishes for "Principles".

There are also some incompatibilities, essentially because in Euclid, the law of excluded middle is explicitly used. Thus, in Euclid, a curve, say a circle, has *exactly one* point in common with any of its tangents, so that the picture (1) (with S as part of a circle) is an illusion for Euclid; already the contemporary Greek philosopher Protagoras is said to have ridiculed Euclidian geometry for insisting on the "only one point"-idea, which seemed to him to go against experience. In Euclidean geometry, $\mathfrak{M}(x)$ is always just the one-point set $\{x\}$. Certainly, the following principle is incompatible with such a small $\mathfrak{M}(x)$:

Principle. Given two affine subspaces l_1 and l_2 and a point $x \in l_1 \cap l_2$. Then

$$l_1 \cap \mathfrak{M}(x) = l_2 \cap \mathfrak{M}(x) \text{ implies } l_1 = l_2.$$

To reduce terminology, let us assume that E is just a model of *plane* Euclidean geometry, so l_1 and l_2 are lines. A subspace $C \subset E$ is called a *curve* if for each $x \in C$, there exists a line l such that $l \cap \mathfrak{M}(x) = C \cap \mathfrak{M}(x)$; such a line is unique, by the Principle. This line then deserves the name: the *tangent* of C in $x \in C$. In the picture (1), if T is a line (as the picture suggests), then the picture says that this line is the tangent to the curve S at x .

For any curve C , the family of its tangents T_x ($x \in C$) is a parametrized family, parametrized by the points of curve C .

Proposition 2.1 Any curve is contained in the envelope of its family of tangents.

Proof. Let T_x denote the tangent to the curve C at $x \in C$. Then for any $x' \in C$ with $x' \sim x$, we have $x \in T_{x'}$, so $x \in \bigcap_{x' \sim x} T_{x'}$, which is the characteristic, at parameter value x , for the family of the tangents. So x belongs to the union of the characteristics, i.e. to the envelope.

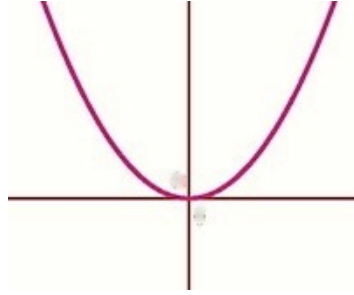
The characteristic $\bigcap_{x' \sim x} T_{x'}$ may be strictly larger than $\{x\}$. (This applies e.g. to the graphs of the family of functions $x \mapsto (x - t)^3$.) Under non-singularity assumptions, we will have equality, and hence also the envelope will be equal to the curve C . (Analytically, the non singularity means the invertibility of a second derivative.)

Let M and N be spaces (objects in \mathcal{E}). It will not in general be the case that $x \sim x'$ and $y \sim y'$ in N implies $(x, y) \sim (x', y')$ in $M \times N$ (although the converse implication will hold, since the projections, like any other map, preserve the assumed neighbour relations \sim). But if $f : M \rightarrow N$ is a map, then we also have that

$$x \sim x' \text{ in } M \text{ iff } (x, f(x)) \sim (x', f(x')) \text{ in } M \times N; \quad (3)$$

for, the map $M \rightarrow M \times N$ given by $x \mapsto (x, f(x))$ preserves, like any map, the neighbour relation.

In the classical geometry of conics, consider a parabola. Then the tangent line at the apex of it is perpendicular to the axis of the parabola:



When coordinates are introduced in the plane, by making “the” geometric line R into a commutative ring, we may consider in particular the parabola P given as the graph of $y = x^2$. The axis of P is the y -axis, and the tangent line at the apex of P is the x -axis X . It follows from (3) that $x \sim 0$ implies $(x, x^2) \sim (0, 0)$. Since $\mathfrak{M}(0, 0) \cap P = \mathfrak{M}(0, 0) \cap X$, we conclude that $(x, x^2) \in X$, which is to say $x^2 = 0$. Thus

$$x \sim 0 \text{ implies } x^2 = 0, \quad (4)$$

or, writing $D \subseteq R$ for $\{x \in R \mid x^2 = 0\}$, this says that $\mathfrak{M}(0) \subseteq D$.

Next consider $(x, y) \sim (0, 0) \in R^2$. Since the projections $R^2 \rightarrow R$ preserve \sim , we conclude $x \sim 0$ and $y \sim 0$, so $(x, y) \in D \times D$, i.e. $x^2 = y^2 = 0$; but we can say more, namely that $x \cdot y = 0$. For, the addition map $R \times R \rightarrow R$ preserves, like any map, the neighbour relation, so $(x, y) \sim (0, 0)$ implies $(x + y)^2 = 0$. But $(x + y)^2 = x^2 + y^2 + 2x \cdot y$. The two first terms we already know are 0, hence so is $2x \cdot y$, and since 2 is invertible, we conclude $x \cdot y = 0$; thus

$$(x, y) \sim (0, 0) \text{ implies } x^2 = y^2 = x \cdot y = 0. \quad (5)$$

We embark in the following Section on a more serious investigation of how synthetic notions like \sim can be conveniently coordinatized, by suitable axiomatization of properties of the ring R .

3 Coordinate geometry, and the axiomatics

It is not the intention of SDG to avoid using the wonderful tool of coordinates. So we now embark on the interplay between an assumed neighbour relation on the geometric objects, and an assumed basic geometric line R with a commutative ring structure.

The reason we did not start there, is to stress that the “arithmetization” in terms of R is a *tool*, not the *subject matter*, of geometry. This also applies in differential geometry, which has some important aspects without any R (as illustrated by the

material in Section 1 and partly in Section 2); so in particular, it has a life without the ring \mathbb{R} of real numbers, who sometimes thinks of himself as being the owner and boss of the company.

The scene of SDG in its present form is thus a category \mathcal{E} (whose object we call *spaces* or *sets*), together with a commutative ring object R in it. But \mathcal{E} is not the category of *discrete* sets, so some of the logical laws valid for the category of discrete sets, like the law of excluded middle, cannot be used. In differential geometry, whose maps are *smooth* maps, the law of excluded middle does anyway not apply; it would immediately lead out of the smooth world, like when one attempts to construct the absolute value function $x \mapsto |x|$ on the number line.

Nevertheless, we shall talk about the objects and maps of \mathcal{E} as if they were sets; just recall that they are not *discrete* sets². This is a basic technique in modern mathematics, more or less explicitly used in many other contexts. We shall not say more about it here. Basic concepts for making the technique explicit are Cartesian closed categories, or even better, locally Cartesian closed categories, in particular toposes (when talking about “families” of objects, as in the discussion above on envelopes). (There is some explicit description of the technique relevant for SDG in Part II of [12] and in Appendix A2 in [19].)

The axioms concern R ; the category \mathcal{E} should just have sufficiently good properties. The maximal thing wanted is that \mathcal{E} is a topos, but less will usually do. Thus, to get hold of an object like the unit circle $\{(x, y) \in R^2 \mid x^2 + y^2 = 1\}$, one needs only that \mathcal{E} has finite limits; the circle then is a subobject of $R \times R$ given as the equalizer of two particular maps $R \times R \rightarrow R$. (In fact, the term “equalizer” came from such equational conditions as $x^2 + y^2 = 1$.)

For simplicity, we therefore in the following assume that \mathcal{E} is a topos; and that R is a commutative \mathbb{Q} -algebra. The intuition and terminology is: R is the number line; and also: R is the ring of scalars.

3.1 The axiomatics

The axiom for such data, which is at the basis of the form of SDG considered here, is an axiom-*scheme*³, with one axiom for each Weil algebra; a Weil algebra is a finite dimensional commutative algebra (over \mathbb{Q} , for the present purpose), where the nilpotent elements form an ideal of codimension 1. The name ‘Weil algebra’ is used because they were introduced in the “Points proches”-paper by A. Weil, [35],

²See the discussion in Section 6

³often referred to as the general KL axiom, for “Kock-Lawvere”, cf. e.g. [31] or [24]

whose aim was related to the one we present here. The simplest non-trivial Weil algebra is the “ring of dual numbers” $\mathbb{Q}[\varepsilon] = \mathbb{Q}[X]/(X^2)$, which is 2-dimensional over \mathbb{Q} .

Concerning R , we have already seen in (4) that $\mathfrak{M}(0) \subseteq \{x \in R \mid x^2 = 0\}$. The latter object, we call D , as at the end of Section 2. To relate the combinatorics of \sim with the algebra of R , we postulate the converse inclusion $D \subseteq \mathfrak{M}(0)$. It then follows that $x \sim y$ in R iff $(y - x)^2 = 0$.

The simplest instantiation of the axiom scheme concerns D . (It can be seen as the instantiation of the axiom scheme for the Weil algebra $\mathbb{Q}[\varepsilon]$.)

Axiom 1. *Every map $f : D \rightarrow R$ is of the form $d \mapsto a + d \cdot b$ for unique a and b in R .*

This has to be true *with parameters*, thus if $f : I \times D \rightarrow R$ is an I -parametrized family of maps $D \rightarrow R$, then the a and b asserted by the axiom are likewise I -parametrized points of R , i.e. they are maps $I \rightarrow R$. In a Cartesian closed category \mathcal{E} , the “true with parameters” follows from a more succinct property, namely the property that the map $R \times R \rightarrow R^D$, given by $(a, b) \mapsto [d \mapsto a + d \cdot b]$, is invertible. Thus, the axiomatics for SDG is simpler to state under the assumption that the category \mathcal{E} is Cartesian closed (although the idea and logic of parametrized families can also be made precise, even without Cartesian closedness).

Cartesian closedness of \mathcal{E} is an aspect of talking about the objects of \mathcal{E} as if they were sets.

We leave to the reader to prove (using D as a space of parameters)

Corollary 3.1 *Every map $f : D \times D \rightarrow R$ is of the form $(d_1, d_2) \mapsto a + d_1 \cdot b_1 + d_2 \cdot b_2 + d_1 \cdot d_2 \cdot c$, for unique a, b_1, b_2 , and c in R .*

In rough terms, since $R^D \cong R^2$, it follows that $(R^D)^D \cong (R^2)^D \cong (R^D)^2 \cong (R^2)^2 \cong R^4$. In itself, the Corollary also appear as an instantiation of the axiom scheme, namely for the four-dimensional Weil algebra $\mathbb{Q}[\varepsilon_1, \varepsilon_2] = \mathbb{Q}[X_1, X_2]/(X_1^2, X_2^2)$.

Remark. The relation \sim defined in terms of D cannot be transitive. For, transitivity is easily seen to be equivalent to D being stable under addition, and hence (using that 2 is invertible) that $d_1 \in D$ and $d_2 \in D$ implies $d_1 \cdot d_2 = 0$. But this contradicts the uniqueness of the coefficient c in the above Corollary. So Axiom 1 implies that \sim is not transitive.

We shall not be explicit how one goes from a (finite presentation of) a Weil algebra to the corresponding Axiom (see [12] I.16). The reader may guess the pattern from the examples given.

From the uniqueness assertion in Axiom 1 one derives

Principle of cancelling universally quantified d 's: let $r, s \in R$. Then:

If $d \cdot r = d \cdot s$ for all $d \in D$, then $r = s$.

In the classical treatment, any individual $x \neq 0$ in R is cancellable, i.e. it has the property that it detects equality; $x \cdot r = x \cdot s$ implies $r = s$; for, in the classical treatment, R is a field, so $x \neq 0$ implies that x is invertible. On the other hand, in SDG, no individual $d \in D$ can be cancellable; for, any such d is nilpotent. This, for some intuition, means that it is very small, “infinitesimal”. So none of these small elements individually have the strength that they can detect equality; but when the small elements join hands, they can. Collective strength, of all the small together, replaces the strength of any individual.

Another consequence of the Axiom 1 is that the beginnings of differential *calculus* become available: given $f : R \rightarrow R$, one applies, for each $x \in R$, the axiom to the function $d \mapsto f(x + d)$; so one gets for each x that there are unique a and b such that $f(x + d) = a + d \cdot b$ for all $d \in D$. The a and b depend on the x chosen, so write them $a(x)$ and $b(x)$, respectively. By setting $d = 0$, we conclude $a(x) = f(x)$; but $b(x)$ deserves a new name, we call it $f'(x)$, so for all $d \in D$, we have the exact “Taylor expansion”

$$f(x + d) = f(x) + d \cdot f'(x) \text{ for all } d \text{ with } d^2 = 0. \quad (6)$$

And this property characterizes $f'(x)$, by the principle of cancelling universally quantified ds .

Since such Taylor expansion holds also with parameters, one also gets partial derivatives for functions in several variables, by considering the variables, except one, as parameters. See (7) below for an example.

Remark. For differential *calculus*, there are other synthetic/axiomatic theories available: e.g. the “Fermat”-axiom (suggested by Reyes), see e.g. [31] VII.2.3; the axiomatics of “differential categories” (cf. [2], [4], and references therein); and the “topological differential calculus” (cf. [1], and references therein).

The Corollary 3.1 could be seen as an instantiation of the axiom scheme; a more interesting instantiation (Axiom 2 below) of the axiom scheme comes about by considering the Weil algebra $\mathbb{Q}[\varepsilon_1, \varepsilon_2]/(\varepsilon_1 \cdot \varepsilon_2)$ ($= \mathbb{Q}[X_1, X_2]/(X_1^2, X_2^2, X_1 \cdot X_2)$), (which is 3-dimensional), namely Axiom 2 below: Let $D(2) \subseteq R^2$ be given as

$$\{(d_1, d_2) \in R^2 \mid d_1^2 = d_2^2 = d_1 \cdot d_2 = 0\}.$$

(Clearly, $D(2) \subseteq D \times D$. Note that $D(2)$ is defined by the equations occurring in (5).) Then

Axiom 2. *Every map $f : D(2) \rightarrow R$ is of the form $(d_1, d_2) \mapsto a + d_1 \cdot b_1 + d_2 \cdot b_2$ for unique a, b_1 and b_2 in R .*

One may have deduced Axiom 2 from the Corollary above *provided* one knew that any function $D(2) \rightarrow R$ may be extended to a function $D \times D \rightarrow R$. But this is not automatic - rather, this is guaranteed by the Axiom 2.

Of course, there are similar axioms for $n = 3, 4, \dots$, using

$$D(n) := \{(d_1, \dots, d_n) \in R^n \mid d_i \cdot d_j = 0 \text{ for all } i, j = 1, \dots, n\}.$$

In short form, the general Axiom 2 says: *Any map $D(n) \rightarrow R$ extends uniquely to an affine map $R^n \rightarrow R$.*

Another typical instantiation of the axiom scheme gives the following Axiom (which we shall not use here): Let $D_2 := \{x \in R \mid x^3 = 0\}$. Then *every function $f : D_2 \rightarrow R$ is of the form $x \mapsto a_0 + a_1 \cdot x + a_2 \cdot x^2$* , or: every $f : D_2 \rightarrow R$ extends uniquely to a polynomial function $R \rightarrow R$ of degree ≤ 2 . This axiom corresponds to the 3-dimensional Weil algebra $\mathbb{Q}[X]/(X^3)$. More generally, let $D_k(n) := \{(x_1, \dots, x_n) \in R^n \mid \text{all products of } k+1 \text{ of the } x_i\text{'s is } 0\}$. Then *every function $D_k(n) \rightarrow R$ extends uniquely to a polynomial function $R^n \rightarrow R$ of degree $\leq k$* . – The polynomial functions occurring here are the Taylor polynomials at $0 \in R$ (resp. at $(0, \dots, 0) \in R^n$) of f .

As a final example of an instantiation of the axiom scheme, let $D_L \subseteq R^2$ be given by

$$D_L := \{(x_1, x_2) \in R^2 \mid x_1^2 = x_2^2 \text{ and } x_1 \cdot x_2 = 0\}.$$

Then the following axiom is likewise an instantiation of the axiom scheme: *every function $f : D_L \rightarrow R$ is of the form $f(x) = a + b_1 \cdot x_1 + b_2 \cdot x_2 + c \cdot (x_1^2 + x_2^2)$* . The c occurring here can then be seen as (one fourth of) the Laplacian $\Delta(f)$ of f at $(0, 0)$. Note that $D(2) \subseteq D_L \subseteq D_2(2)$. The space D_L corresponds to a certain 4-dimensional Weil algebra.

3.2 Envelopes again

This subsection is to “justify” in classical terms the correctness of our description of envelopes. For simplicity, we consider a 1-parameter family of (unparameterized) surfaces S_t in R^3 . We assume that there is some smooth function $F(x, y, z, t)$ such that the t th surface S_t is given as the zero set of $F(-, -, -, t)$. We then prove

that the classical analytic “discriminant” description of the characteristics and the envelope agrees with the synthetic one we have given; but note that our description is coordinate free, so in particular, it follows that the constructed envelope is independent of the analytic representation. To say that (x, y, z) belongs to the t_0 characteristic is by the synthetic definition to say that $F(x, y, z, t_0 + d) = 0$ for all $d \in D$ (the neighbours t of t_0 are of the form $t_0 + d$). Equivalently, by Taylor expansion,

$$F(x, y, z, t_0) + d \cdot \partial F / \partial t(x, y, z, t_0) = 0, \quad (7)$$

for all $d \in D$. By the principle of cancellation of universally quantified ds , this is equivalent to the conjunction of the two equations

$$F(x, y, z, t_0) = 0 \quad \text{and} \quad \partial F / \partial t(x, y, z, t_0) = 0,$$

which is how the t_0 characteristic is described analytically in, say, [5].

3.3 Defining \sim in terms of R ?

We have already postulated that $x \sim y$ in R means $y - x \in D$, or $(y - x)^2 = 0$. A (first order) neighbour relation \sim on any object $M \in R$ can be defined by

$$x \sim y \text{ in } M \text{ iff } \alpha(x) \sim \alpha(y) \in D \text{ for all } \alpha : M \rightarrow R. \quad (8)$$

So \sim is, for all objects M , defined in terms of the scalar valued functions on M . Trivially, any map $M' \rightarrow M$ preserves \sim . This is the “contravariant” or “weak” way of defining \sim . There is also a “covariant” or “strong” way of defining it, see [19] p. 31. For good spaces, like R^n , they coincide. (The weak determination is not adequate in algebraic geometry, since projective space, and other important geometric objects, only admit constant scalar valued functions. So one must here replace the consideration of scalar-valued functions by *locally defined* scalar valued functions, and for this, one needs some notion of “local”, which we shall not go into here.)

For the weak determination of \sim , we can identify the monad $\mathfrak{M}(\underline{0})$ around the origin in R^n :

Proposition 3.2 *We have $\mathfrak{M}(\underline{0}) = D(n)$.*

(For $n = 1$, this was postulated.) Let us prove it for $n = 2$. We have already seen in (5) that $\mathfrak{M}(\underline{0}) \subseteq D(2)$. For the converse, we have to consider an arbitrary map $\alpha : R^2 \rightarrow R$ and prove that $(d_1, d_2) \in D(2)$ implies $\alpha(d_1, d_2)^2 = 0$. By Axiom 2,

$\alpha(d_1, d_2) = a + b_1 \cdot d_1 + b_2 \cdot d_2$, and so $\alpha(d_1, d_2) - \alpha(0, 0) = b_1 \cdot d_1 + b_2 \cdot d_2$, which has square 0 since $d_1^2 = d_2^2 = d_1 \cdot d_2 = 0$.

Remark. Since $D(2)$ is strictly smaller than $D \times D$, we therefore also have that $(d_1, d_2) \sim (0, 0)$ is stronger than the conjunction of $d_1 \sim 0$ and $d_2 \sim 0$.

The Principle in the beginning of Section 2 may now be proved algebraically: we may assume that coordinates are chosen so that the considered common point $x \in l_1 \cap l_2$ is $(0, 0)$, and that l_1 and l_2 are graphs of functions $x \mapsto b_1 \cdot x$ and $x \mapsto b_2 \cdot x$. The argument given at the end of Section 2 then applies; more algebraically, since $\mathfrak{M}(0, 0) = D(2)$, it follows from the principle of cancelling universally quantified ds , as follows:

We must prove that $b_1 = b_2$. For $d \in D$, we have $(d, b_1 \cdot d) \in \mathfrak{M}(x) = D(2)$. So by assumption, for all $d \in D$, we have $(d, b_1 \cdot d) \in \mathfrak{M}(x) \cap l_1 = \mathfrak{M}(x) \cap l_2 \subseteq l_2$, so for all $d \in D$, we have $b_1 \cdot d = b_2 \cdot d$; cancelling the universally quantified d then gives $b_1 = b_2$.

3.4 Contravariant and covariant hierarchy

The polynomial function $x : R \rightarrow R$ vanishes at 0; one also says that it vanishes to *first* order at 0, and that x^2 vanishes to *second* order at 0, etc.; more generally, $f : R \rightarrow R$ vanishes to second order at 0 if it may be written $f(x) = x^2 \cdot g(x)$ for some function $g : R \rightarrow R$. Similarly for k th order vanishing. It generalizes to “order of vanishing” of $f : M \rightarrow R$ at a point $a \in M$. Note that k th order vanishing is a *weaker* condition than $(k + 1)$ st order vanishing. This (essentially classical) hierarchy of scalar valued functions (quantities) is to be compared with the hierarchy of neighbours, applicable to points of spaces M , where k th order neighbour is a *stronger* condition than $(k + 1)$ th order neighbour. The neighbour relations are covariant notions, applicable to *points (elements)* of spaces (the assumed neighbour relations \sim_1, \sim_2, \dots , are preserved by mappings, and thus are *covariant*); the order-of-vanishing is a contravariant notion, applicable to *quantities* on M , i.e. to R -valued functions $M \rightarrow R$.

The notions are related as follows, for a and b in M : $a \sim_k b$ iff for any quantity $f : M \rightarrow R$ vanishing to k th order at a , we have $f(b) = 0$; this is, for $k = 1$, just a reformulation of (8). Recall that $a \sim_k b$ on R is defined in terms of $(a - b)^{k+1} = 0$, i.e. in terms of order of *nilpotency*.

A classical formulation is that we can “ignore” quantities of higher order, in comparing a and b : “*Dabei sehen wir von unendlich kleinen Grössen höhere Ordnung ab.*” (“Ignoring hereby infinitely small quantities of higher order.”), [29]

p. 523. In rigorous mathematics, one cannot “ignore” anything except 0. But one can certainly consider nilpotent elements in rings. Thus, an explicit theory of infinitesimals came in through the back door, namely from algebraic geometry:

3.5 Wisdom from algebraic geometry

The development leading to the modern formulations of SDG began in French algebraic geometry in the mid 20th Century by Grothendieck and his collaborators, with the notion (and category!) of *schemes*, as a generalization of the notion of algebraic varieties (over a field k , say).

In particular, the category \mathcal{E}_k of affine schemes over k is the by definition the dual of the category \mathcal{A}_k of commutative k -algebras, suitably size-restricted, say: of finite presentation. The algebras are allowed to have nilpotent elements. Such algebra A is seen as the ring of scalar valued functions on the scheme M (geometric object, “space”) which it defines. One writes $M = \text{Spec}(A)$. (The “scalars” R is the scheme represented by $k[X]$.) Then the algebra $A \otimes A$ defines the space $M \times M$. If I is the kernel of the multiplication map $A \otimes A \rightarrow A$, then $(A \otimes A)/I \cong A$. Consider the ideal $I^2 \subseteq I$. The k -algebra $(A \otimes A)/I^2$ gives $M_{(1)}$, the first neighbourhood of the diagonal of M . So $M_{(1)} := \text{Spec}((A \otimes A)/I^2)$. The quotient map $(A \otimes A)/I^2 \rightarrow (A \otimes A)/I \cong A$ defines, in the category of schemes, the diagonal $M \rightarrow M_{(1)}$.

Note that $I/I^2 \subseteq (A \otimes A)/I^2$ consists of elements of square 0. It is in fact the module of *Kähler differentials of A* ; $(A \otimes A)/I^2$ is the ring of scalar valued functions on $M_{(1)}$, and the submodule I/I^2 consists of those functions that vanish on the diagonal $M \subseteq M_{(1)}$, i.e. the combinatorial scalar valued 1-forms, in the sense of Subsection 1.4. (Kähler introduced these differentials already in the 1930s.)

The simplest scheme which is not a variety is D , the affine scheme given by $k[\varepsilon]$, the ring of dual numbers over k . The underlying variety of D has just one global point, since $k[\varepsilon]$ has only one prime ideal, namely (ε) . Geometrically, D is a “thickened” version of its unique global point. Mumford ([32] p. 338) describes D as “a sort of disembodied tangent vector”, meaning that a map $D \rightarrow X$ may be identified with a tangent vector to X , for any scheme X .

The relationship between the infinitesimal objects like D , and the neighbourhoods of diagonals may be exemplified by the isomorphism

$$R_{(1)} \cong R \times D,$$

given by $(x, y) \mapsto (x, y - x)$, for $x \sim y$ in R .

The crucial step in the formation of contemporary SDG was when Lawvere in 1967 combined this consideration of a “tangent vector representor” D with the idea of Cartesian closed category \mathcal{E} . Thus, for any object X in \mathcal{E} , X^D is then the *object* (space) in \mathcal{E} of all tangent vectors to X , in other words, it is the (total space of the) tangent bundle $T(X) \rightarrow X$.

To put this relationship into axiomatic form is most conveniently done by assuming a ring object R , and describing D in terms of R , (as is done in Section 3). There is a more radical approach, advocated by Lawvere in [27], where the ring R is to be constructed out of an infinitesimal object T (“an instant of time”) (ultimately then proved to be isomorphic to D); see also [4].

4 Models of the axiomatics

For an axiomatic theory, models are useful, but not crucial. Euclidean geometry has been useful for more than two thousand years. When exactly was a model for it presented? Did it have to wait for the real numbers, or at least some subfields of it? Models are useful, - they may guide the intuition, and prevent inner contradictions. This also applies to SDG. The models for SDG come in two main groups: arising from algebraic geometry, and from classical differential geometry over \mathbb{R} , respectively (and in fact, SDG serves to make explicit what the two groups have in common).

4.1 Algebraic models

The category \mathcal{E}_k of affine schemes over a commutative ring k (i.e., the dual of the category of (finitely presented, say) commutative k -algebras) is a model⁴, with $k[X]$ as R . \mathcal{E}_k is not quite Cartesian closed, but at least $k[\mathcal{E}]$ and other Weil algebras are exponentiable. The set valued presheaves $\hat{\mathcal{E}}_k$ on \mathcal{E}_k is a full fledged topos model (with R represented by $k[X]$). The topos $\hat{\mathcal{E}}_k$ is of course the same as the category of covariant functors from commutative k -algebras to sets, and R is in this set up just the forgetful functor, since $k[X]$ is the free k -algebra in one generator.

Many of the subtoposes of $\hat{\mathcal{E}}_k$ are likewise models; passing to suitable subtoposes, one may force R to have further properties; one may for instance force R to become a local ring; the subtopos forcing this is also known as the Zariski Topos. These toposes are explicitly the main categories studied in [7].

⁴If 2 is not invertible in k , there are things that work differently.

4.2 Analytic models based on \mathbb{R}

There is of course a special interest in models (\mathcal{E}, R) which contain the category Mf of smooth manifolds as a full subcategory, in a way which preserve known constructions and concepts from classical differential geometry. So there should be a full and faithful functor $i : Mf \rightarrow \mathcal{E}$, with $i(\mathbb{R}) = R$. Also *transversal* pull-backs should be preserved, and $i(T(M))$ should be $i(M)^D$. The properties of such a functor i has been axiomatized by Dubuc [8]) under the name of “well adapted model for SDG”; see also [11]. The book [31] is mainly devoted to the construction and study of such models.

The earliest well-adapted model (constructed by Dubuc [8]) is one now known as the “Cahiers topos”. It can be proved to contain the category of convenient vector spaces (with smooth maps between them) as a full subcategory, in a way which preserves the Cartesian closed structure, cf. [15], [21].

A main tool in the construction of analytic models is to take heed of the wisdom of algebraic geometry, but replacing the algebraic theory \mathbb{T} of commutative rings with the richer algebraic theory \mathbb{T}_∞ , whose n -ary operations are not only the real polynomial functions, but *all* the smooth maps $\mathbb{R}^n \rightarrow \mathbb{R}$. It contains the theory of commutative rings as a subtheory, since a polynomial in n variables defines a smooth function in n variables. The theory \mathbb{T}_∞ , and its importance for the project of categorical dynamics, was already in Lawvere’s seminal 1967 lectures.

Note that any smooth manifold M gives rise to an algebra for this theory, namely $C^\infty(M)$, the ring of smooth \mathbb{R} -valued functions on M . We may think M as a “reduced” affine scheme corresponding to the ring $C^\infty(M)$, and then mimic the construction (described above) of set valued presheaves on affine schemes, and subtoposes thereof. But note that also $\mathbb{R}[\varepsilon]$ (and all other Weil algebras over \mathbb{R}) are algebras for \mathbb{T}_∞ , and define (non-reduced) affine schemes.

Modules of Kähler differentials for algebras for \mathbb{T}_∞ were studied in [9]

(If one takes just the category of smooth manifolds (with open coverings) as site of definition for a topos, one gets a topos already considered in SGA4, under the name of “the smooth topos”; it contains the category of diffeological spaces as a full subcategory, but lacks the infinitesimal objects like D . These categories are models for the Fermat-Reyes axiomatics. See [12] Exercise III.8.1).

4.3 New spaces

Except for the “infinitesimal” spaces like $D_k(n)$, the present account does not do justice to the *new* spaces which have emerged through the development of SDG.

In particular, it has not capitalized on the unproblematic way in which function spaces exist in this context, by Cartesian closedness of \mathcal{E} . These function spaces opens the door to a synthetic treatment of calculus of variations, continuum mechanics, infinite dimensional Lie groups, . . . For such spaces, the neighbour relation (which has been my main focus here) is more problematic, however, and is not well exploited. Instead, one uses the (classical) method of encoding the infinitesimal information of a space X in terms of its tangent bundle $T(X) = X^D$, rather than in terms of $X_{(1)}$ (first neighbourhood of the diagonal). Notably Nishimura has pushed the SDG-based theory far in this direction, cf. e.g. [34].

Another type of new spaces come from the observation that the functor $(-)^D$ in many of the models has a right adjoint, $(-)^{1/D}$ (Lawvere’s notation, “fractional exponent”); the spaces $M^{1/D}$ are reminiscent of Eilenberg-Mac Lane spaces. There is some discussion of them in [12] I.20, in [22], and in [25].

Finally, the notion of *jets*, and the jet bundles, as considered by Ehresmann in the 1950s form, on the one hand, one of the sources for SDG as presented here; on the other hand, the SDG method makes the consideration of jets and jet bundles simpler, since SDG makes the notion of jet *representable*, in the sense that a k -jet at $x \in M$, with values in N , is here simply a map $\mathfrak{M}_k(x) \rightarrow N$, rather than an equivalence class of maps $x \in U \rightarrow N$. Ehresmann’s theory of differentiable groupoids as carrier of a general theory of connections receives from this also some simpler formulations in these terms:

4.4 Connections in fibre bundles and groupoids

For the present purpose, a *fibre bundle* over a space M is just a map $\pi : E \rightarrow M$. (When it comes to proving things, one will need good exactness properties of π , like being an effective descent map, or being locally a projection $F \times M \rightarrow M$.) Then a combinatorial connection in the bundle $E \rightarrow M$ is an action ∇ of $M_{(1)}$ on E , in the sense that $(x, y) \in M_{(1)}$ and $e \in E_x$ define an element $\nabla(x, y)(e)$ in E_y . (Here, $E_x := \pi^{-1}(x)$ and similarly for E_y . One requires the normalization condition $\nabla(x, x)(e) = e$. (For good spaces, it then follows that $\nabla(y, x)\nabla(x, y)(e) = e$.) The notion of affine connection λ considered above is a special case: the bundle $E \rightarrow M$ is in this case the first projection $M_{(1)} \rightarrow M$, and $\nabla(x, y)(z) = \lambda(x, y, z)$. If $E \rightarrow M$ is a vector bundle, say, a *linear* connection is a connection ∇ where the map $\nabla(x, y)(-) : E_x \rightarrow E_y$ is linear for all $x \sim y$. For good spaces M , linear connections in the tangent bundle $T(M) \rightarrow M$ contain exactly the same information as affine connections λ on M .

There is also a notion of connection ∇ in a *groupoid* $\Phi \rightrightarrows M$. (This is closely related to the notion of *principal connection* in a principal fibre bundle $P \rightarrow M$; in fact, such P defines, according to C. Ehresmann, a groupoid $PP^{-1} \rightrightarrows M$, and a principal connection in $P \rightarrow M$ is then the same data as a groupoid connection in $PP^{-1} \rightrightarrows M$.) Recall that a groupoid $\Phi \rightrightarrows M$ carries a reflexive symmetric structure: the reflexive structure picks out for every $x \in M$ the identity arrow at x , and the symmetric structure associates to an arrow $f : x \rightarrow y$ its inverse $f^{-1} : y \rightarrow x$. Then a connection in $\Phi \rightrightarrows M$ is simply a map $M_{(1)} \rightarrow \Phi$ preserving (the two projections to M and) the reflexive and symmetric structure,

$$\nabla(x, x) = \text{id}_x \quad \text{and} \quad \nabla(y, x) = \nabla(x, y)^{-1}$$

for all $x \sim y$.

Given a bundle $E \rightarrow M$ in \mathcal{E} . If \mathcal{E} is locally Cartesian closed, one may form the groupoid $\Phi \rightrightarrows M$ where the arrows $x \rightarrow y$ are the invertible maps $f : E_x \rightarrow E_y$. Then a connection on $E \rightarrow M$, in the bundle sense, is equivalent to a connection, in the groupoid sense, of this groupoid $\Phi \rightrightarrows M$. If $E \rightarrow M$ is a vector bundle, there is a subgroupoid of $\Phi \rightrightarrows M$ consisting of the *linear* isomorphisms $E_x \rightarrow E_y$ (this groupoid deserves the name $GL(E)$). Similarly if $E \rightarrow M$ is a group bundle, or has some other fibrewise structure.

The groupoid formulation of the notion of connection is well suited to formulate algebraic properties, like curvature. We may observe that the curvature, as described in Subsection 1.3 for affine connections λ , is purely groupoid theoretical. Thus if x, y, z form an infinitesimal 2-simplex in M , it makes sense to ask whether $\nabla(x, y)$ followed by $\nabla(y, z)$ equals $\nabla(x, z)$, or better: consider the arrow $R(x, y, z) : x \rightarrow x$ given as the composite (composing from left to right)

$$R(x, y, z) := \nabla(x, y) \cdot \nabla(y, z) \cdot \nabla(z, x) \in \Phi(x, x).$$

This is the *curvature* of ∇ , more precisely, the curvature R is a combinatorial 2-form with values in the group bundle $\text{gauge}(\Phi)$ of vertex groups $\Phi(x, x)$ of Φ . Now the connection ∇ in Φ gives rise to a connection $ad\nabla$ in the group bundle $\text{gauge}(\Phi)$: $ad\nabla(x, y)$ is the (group-) isomorphism $\Phi(x, x) \rightarrow \Phi(y, y)$ consisting in conjugation by $\nabla(x, y) : x \rightarrow y$. This conjugation we write $(-)^{\nabla(x, y)}$. In terms of this, we have an identity, which deserves the name the *Bianchi* identity for the connection ∇ ; namely for any infinitesimal 3-simplex (x, y, z, u) , we have

$$\text{id}_x = R(yzu)^{\nabla(y, x)} \cdot R(xyu) \cdot R(xuz) \cdot (R(xzy)), \quad (9)$$

verbally, *the covariant derivative of the gauge(Φ) valued 2-form R , with respect to the connection $ad\nabla$ in the group bundle, is “zero”*, i.e. takes only identity arrows as values.

The proof of (9) is trivial, in the sense that it is a case of Ph. Halls 14-letter identity, which holds for any six elements in a group, or for the six arrows of a tetrahedron-shaped diagram in a groupoid; here, the six arrows are the $\nabla(x,y)$, $\nabla(x,z)$, \dots , $\nabla(z,u)$ in Φ . See [19], and [16] for how this implies the classical Bianchi identity for linear connections in vector bundles.

5 The role of analysis

5.1 Analysis in geometry?

The phrase *analytic* geometry may be used in the wide sense: using coordinates and calculations. In this sense, SDG as presented here quickly becomes analytic (e.g. the basic axiomatics is formulated in such terms, as expounded in Section 3). But the more common use of the phrase “analytic” is that *limit* processes and topology are utilized.

Ultimately, topology and limits in real analysis have their origin in the strict order relation $<$ on \mathbb{R} . Then the partial order \leq is defined by $x \leq y$ iff $\neg(y < x)$. The elements in $\mathbb{R}_{>0}$ are invertible. In SDG, it is also natural to have an order $<$ on R , given primitively, or in terms of the algebraic structure of R . (In well adapted models $i : Mf \rightarrow \mathcal{E}$, the relation $<$ is definable in terms of the inclusion of the smooth manifold $\mathbb{R}_{>0}$ into \mathbb{R} , which by the embedding i defines a subobject $R_{>0} \subset R$, out of which a strict order $<$ can be defined.) Nilpotent elements d in a non-trivial ring cannot be invertible. It follows, for any nilpotent d that $d \leq 0$, and hence also $-d \leq 0$ (since also $-d$ is nilpotent). So $0 \leq d \leq 0$. So if \leq were a partial order (not just a preorder), this would imply that any nilpotent d is 0, which is incompatible with SDG. Thus, in SDG, \leq is only a preorder, not a partial order. For preordered sets, the notion of supremum is not well defined; one needs partial order.

This is one reason why limit processes are not used in SDG, at the present stage.

Topology comes in play when formulating statements about *local* existence of, say, solutions to particular differential equations. ‘Local’ refers to some topology on a given object, and in SDG, there may be several natural choices. The finest topology on an object (space) X is, in the context of SDG, the one where the

open subsets are those $U \subseteq X$ which are closed under the neighbour relation \sim . For instance, a local solution f (for this fine topology) for a differential equation $f' = F(x, y)$ amounts to a formal power series solution, and is therefore cheap. More serious existence statements are when stronger topologies, like the “intrinsic Zariski topology”, are involved : an subset $U \subseteq X$ is open if it is of the form $f^{-1}(P)$, where $P \subseteq R$ consists of the invertible elements; or if P consists in the strictly “positive” elements - which then in turn have to be described or assumed.

SDG does not *prove* basic integration results, and even the formulation of such results does not come for free.

The most basic integration result is the (essentially unique) existence of anti-derivatives: for $f : R \rightarrow R$, there exists $F : R \rightarrow R$ with $F' = f$. In an axiomatic development, this has to be taken as an *axiom*, – one that actually can be proved to hold in all the significant topos models (\mathcal{E}, R) for SDG. Similarly for many other basic results, like a suitable version of the intermediate value theorem.⁵ Thus, full fledged analysis in axiomatic terms, incorporating SDG, quickly becomes overloaded with axioms, and is better developed as a *descriptive* theory, describing what actually *holds* in *specific* models (\mathcal{E}, R) . This is the approach of [31] which significantly has the title “Models for Smooth Infinitesimal Analysis” (although also a full-fledged axiomatic theory is attempted in loc.cit., Chapter VII). Note that the term “smooth”, in so far as SDG is concerned, is a void term, since unlimited differentiability is automatic in this context; and “smooth implies continuous” (equivalently, “all maps are continuous”) is a *Theorem* in the good well-adapted models, see e.g. Theorem III.3.5 in [31].

I prefer not to think of SDG as a monolithic global theory, but as a *method* to be used locally, in situations where it provides insight and simplification of a notion, of a construction, or of an argument. The assumptions, or axioms that are needed, may be taken from the valuable treasure chest of real analysis.

Thus, the very construction of well adapted models $Mf \rightarrow \mathcal{E}$ depends on the theory \mathbb{T}_∞ whose n -ary operations are the smooth functions $\mathbb{R}^n \rightarrow \mathbb{R}$, so that e.g. the exponential function $\exp : \mathbb{R} \rightarrow \mathbb{R}$, or the trigonometric functions, are “imported” from the treasure chest (here, from Euler, much prior to the rigorous formulation of limit processes). In the context of SDG, it is possible to introduce existence of, say, these particular transcendental functions axiomatically, by functional equations, or by differential equations. This is what the Calculus Books in essence do.

⁵Significantly, the version valid in significant SDG models applies to functions f with a *transversality* condition, like $f' > 0$, – like in constructive analysis.

5.2 Non-standard analysis ?

Non standard analysis (NSA) is another theory where the notion of infinitesimals has an explicit and well defined status. Therefore, one sometimes asks whether there is some relationship between SDG and NSA. There is very little relationship; NSA is a descriptive, not an axiomatic, theory, dealing (at least in so far as differential geometry goes) with the real number field \mathbb{R} , and crucially capitalizing on its Cauchy completeness, since it is crucial that *every (bounded) non-standard real number $\in \mathbb{R}^*$ has a unique standard part*. This is another expression of the completeness of the real number system. In this sense, NSA is a reformulation, with a richer vocabulary, of standard real analysis, and can, as such, cope with things defined in terms of limits, like definite integrals in terms of Riemann sums, say; SDG cannot do this, at best, it can introduce some integration by axioms, cf. the remark on the Frobenius integration Theorem in Subsection 1.2.

In NSA, one has a neighbour notion for elements in \mathbb{R}^* ; it is an *equivalence* relation, and the equivalence classes are called *monads* – a term which SDG has imported; but in SDG it is crucial that the neighbour relations are not transitive, and come in a hierarchy: first order, second order, \dots , (hence first order, second order, \dots , *k*th order monads $\mathfrak{M}_k(x)$, \dots), and this comes closer to important aspects of mathematical practice, where notably the first order neighbour relation takes most of the work on its shoulders, and has been the sole concern in this note. (The second order monads in SDG play a role when discussing e.g. dynamic or metric notions; thus a (pseudo-) Riemannian metric may be defined in terms of a R -valued functions $f(x, y)$ defined for $x \sim_2 y$, and with $f(x, y) = 0$ if $x \sim_1 y$; see [19].)

NSA can also be axiomatized, but this amounts essentially to axiomatizing a further structure (an endo-functor) on the *category* of (discrete) sets [20], or a further primitive predicate in axiomatic (Zermelo Fraenkel) set theory [33].

6 The continuum and the discrete

An important historical problem in (the philosophy of) mathematics is the problem of understanding the nature of the continuum, and its relationship to the discrete. Is the continuum just a discrete set of points? (and motion therefore impossible, according to Parmenides). In contrast, in Euclidean geometry, *line* (line segment) was a primitive notion, and was not just the set of points in it. (And *time* was not a set of instants.) Even a contemporary geometer like Coxeter makes the distinction

between a line and the “range of points” on it, cf. [6] p. 20.

The principal side of the contradiction between continuum and discrete was, historically, the continuum. With the full arithmetization of the continuum, in the hands of, say, Dedekind, with the construction of the real number system \mathbb{R} , the continuum was reduced to a set of points, and the cohesion of the continuum was reduced to a topology on this point set. For mainstream differential geometry synthetic axiomatic considerations became, in principle, redundant. Everything is reducible to real analysis.

Synthetic differential geometry refuses to take this one-sided reductionist view. (For one reason. \mathbb{R} , as a point set (set of global points), has no non-trivial nilpotent elements d .) Rather, SDG learns from (and possibly contributes to) *analyzing* the relationship between the continuum and the discrete. Such analysis typically has the form of a functor $\gamma_* : \mathcal{E} \rightarrow \mathcal{S}$, with \mathcal{S} some category of discrete sets, and with \mathcal{E} some category of spaces with some kind of cohesion⁶. Preferably, both \mathcal{E} and \mathcal{S} are toposes, and γ_* a geometric morphism, associating to a space $X \in \mathcal{E}$ its set of (global) points. The left adjoint γ^* of γ_* is a full embedding, so that discrete spaces form a full subcategory of \mathcal{E} . An example of such \mathcal{E} - \mathcal{S} -pair is with \mathcal{E} the topos of simplicial sets, with $\gamma_*(X)$ the set of 0-simplices (= global points) of X . This example is relevant to algebraic topology (cf. e.g. [30]), not to differential geometry, but it illustrates a phenomenon which is crucial also for SDG: namely that there are non-trivial objects with only one global point (e.g. in the topos of simplicial sets: the simplicial n -sphere $\Delta(n)/\dot{\Delta}(n)$) - just like D in SDG has 0 as the only global point.

A well-adapted model \mathcal{E} of SDG contains not only the the category of discrete manifolds (sets) as a full subcategory, but even the category of all smooth manifolds, in particular \mathbb{R} . By the fullness, \mathbb{R} , when seen in \mathcal{E} (and there denoted R) does not acquire any new global points (unlike the \mathbb{R}^* of NSA). But it does acquire new subobjects, - e.g. $D \subseteq R$. When we talk about general elements $d \in D$, we are therefore not talking about *global* points $1 \rightarrow D \subseteq R$.

A space is an object in a category of spaces (Grothendieck, Lawvere). So what “is” the space \mathbb{R} ? It depends on the category in which it is considered. In SDG, one considers \mathbb{R} in certain (“well adapted”) toposes \mathcal{E} ; \mathbb{R} does not change, it is the ambient category which changes.

⁶a situation axiomatized in Lawvere’s [26] “Mengen” vs. “Kardinalen”.

7 Looking back

The discovery, by Huygens in the 17th Century, of the notion of envelopes and their relatives, (leading to a theory of waves, isochrones, . . .), was coined in geometric terms, without essential reference to analytic considerations. When differential calculus, as we know it today, was developed, analytic methods became more dominant. A main treatise like Monge's in 1795 was entitled "*L'application de l'analyse à la géométrie*". But this treatise of Monge's goes also in the other direction: it forcefully uses geometric and synthetic reasoning for explaining the analytic theory of first order PDEs of Lagrange, - a thread taken up later by Sophus Lie; this comprises in particular the theory of *characteristics* of such PDEs, the curves, out of which the solutions of the PDE can be built. (They are built up from characteristics in the sense of subsection 1.5, namely intersections of families of surface elements.) Lie's 1896 book [29] on contact geometry has a chapter called *Die Theorie der partiellen Differentialgleichungen als Teil der Theorie der Flächenelemente*. (Flächenelement = surface element; these are subsets, like $\mathfrak{M}_{\approx}(x)$ of the 1st order monads $\mathfrak{M}(x)$ used in Subsection 1.2 above.)

In one of Lie's early articles on the theory of differential equations, he wrote:

"The reason why I have postponed for so long these investigations, which are basic to my other work in this field, is essentially the following. I found these theories originally by synthetic considerations. But I soon realized that, as expedient [zweckmässig] the synthetic method is for discovery, as difficult it is to give a clear exposition on synthetic investigations, which deal with objects that till now have almost exclusively been considered analytically. After long vacillations, I have decided to use a half synthetic, half analytic form. I hope my work will serve to bring justification to the synthetic method besides the analytical one."

(From Lie's "Allgemeine Theorie der partiellen Differentialgleichungen erster Ordnung", Math. Ann. 9 (1876); my translation.)

In spite of Lie's call for a synthetic language and logic, the differential geometry in the 20th Century became more and more analytic, and removed from the geometric intuition - at the time of Einstein, the "débauche of indices", and rules for how the coordinates transform, later on more abstract and coordinate free, but still somewhat un-geometric - as it must be when explicit infinitesimals (neighbour points) have to be avoided.

The editors of the present volume asked me to address the question about the "advantages of SDG over other approaches . . .". First of all, the neighbour notion, and synthetic reasoning and concept formation with it, is not an invention of

present day SDG; it has been, and is, used again and again by engineers, physicists, by Sophus Lie (cf. the above quotation), David Hilbert [10], and (at least secretly) also by later mathematicians. However, explicit rules for such concept-formation, construction and reasoning have not been well formulated, and SDG is an attempt to provide such rules, so that the concepts, constructions and reasoning can be clearly communicated, and tested for rigour. What is the advantage of communication and rigour? It is not a question of “advantage”, but a question of necessity.

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Anders Kock,
 Dept. of Math.
 University of Aarhus, Denmark
 kock (at) math.au.dk