Mh. Math. 107, 1—3 (1989)

Monatshefte für
Mathematik
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A Note on Closed Ideals in Rings of Smooth Functions

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(Received 7 November 1988)

Abstract. We prove that if finitely many smooth functions on a manifold M generate a closed ideal in $C^\infty(M)$, then they generate a closed ideal in $C^\infty(M\times N)$.

Let M be a smooth manifold (Hausdorff, with a countable base), and let $C^\infty(M)$ denote the ring of smooth functions $M \to \mathbb{R}$. It is topologized in the standard way so as to become a Frechet space, cf. e.g. [5], 1.46. If $I \subset C^\infty(M)$ is an ideal, and N is another manifold, we denote by $I^* \subset C^\infty(M\times N)$ the ideal generated by $I \subset C^\infty(M) \subset C^\infty(M\times N)$ ($C^\infty(M) \subset C^\infty(M\times N)$ being the subring of smooth functions $M\times N \to \mathbb{R}$ which do not depend on the N-variable).

When E and F are locally convex vector spaces, we let $E \hat{\otimes} F$ denote the completed projective tensor product of Grothendieck [1], [2]. We prove

Proposition A. *Let $I \subset C^\infty(M)$ be a closed ideal. Then the closure of $I^* \subset C^\infty(M\times N) = C^\infty(M) \hat{\otimes} C^\infty(N)$ equals $I \hat{\otimes} C^\infty(N)$ (which topologically is a subspace of $C^\infty(M) \hat{\otimes} C^\infty(N)$).*

(The identification $C^\infty(M\times N) = C^\infty(M) \hat{\otimes} C^\infty(N)$ is well-known, cf. [2] p. 105.)

Proposition B. *Let $I \subset C^\infty(M)$ be a finitely generated closed ideal. Then $I^* \subset C^\infty(M\times N)$ is closed.*

Proof of Proposition A. Since $C^\infty(M)$ is nuclear and I is a closed subspace, I is nuclear ([1], Ch. II §2 No. 2). From Corollary in ([1], Ch. II §3 No. 1, p. 70), we conclude that the map $I \hat{\otimes} C^\infty(N) \to C^\infty(M) \hat{\otimes} C^\infty(N)$, induced by functoriality from the inclusion map $I \to C^\infty(M)$, is a subspace inclusion proving the parenthetical remark in the Proposition.

We also remark that $I \hat{\otimes} C^{\infty}(N)$ is a Frechet space since both factors are ([2] p. 77); so being a subspace, it is a closed subspace ([1], Thm. 1.27, say). Since it contains the subspace $I \otimes C^{\infty}(N) = I^*$, we get that $\overline{I^*} \subseteq I \hat{\otimes} C^{\infty}(N)$. This inclusion cannot be a proper inclusion since $I \otimes C^{\infty}(N)$ is dense in $I \hat{\otimes} C^{\infty}(N)$. □

Proof of Proposition B. Let $I \subset C^{\infty}(M)$ be the ideal generated by $f_1, \ldots, f_k \in C^{\infty}(M)$ and assume that I is closed. We have a continuous linear surjection $(C^{\infty}(M))^k = C^{\infty}(M) \oplus \ldots \oplus C^{\infty}(M) \xrightarrow{q} I$ given by $(g_1, \ldots, g_k) \mapsto \sum g_i f_i$. Consider the map

$$(C^{\infty}(M))^k \hat{\otimes} C^{\infty}(N) \xrightarrow{q \otimes 1} I \hat{\otimes} C^{\infty}(N).$$

Since all spaces considered here are Frechet, it follows from ([2], Corollary of Thm. 5) that $q \otimes 1$ is a surjection. Consider the commutative diagram

$$\begin{array}{ccc}
(C^{\infty}(M))^k \hat{\otimes} C^{\infty}(N) & \xrightarrow{\quad q \otimes 1 \quad} & I \hat{\otimes} C^{\infty}(N) \\
\cong \uparrow & & \\
(C^{\infty}(M) \hat{\otimes} C^{\infty}(N))^k & & \uparrow j \\
\cong \uparrow & & \\
(C^{\infty}(M \times N))^k & \xrightarrow[\quad q' \quad]{} & I^*
\end{array}$$

where q' is given by $(h_1, \ldots, h_k) \mapsto \sum h_i \cdot f_i$. It follows that the inclusion j is a bijection. Since $I \hat{\otimes} C^{\infty}(N)$ by Proposition A is the closure of I^*, it follows that I^* is closed. □

The results proved has applications in the theory of C^{∞}-rings (cf. e.g. [3] or [4]) since we have $C^{\infty}(M) \hat{\otimes} C^{\infty}(N) = C^{\infty}(M) \otimes_{\infty} C^{\infty}(N)$, where $\otimes_{\infty}$ denotes coproduct of C^{∞}-rings. (Proof: both equal $C^{\infty}(M \times N)$.) For instance, if $\bar{\otimes}_{\infty}$ denotes the coproduct of C^{∞}-rings which are defined by a closed ideal, we get from Proposition A that

$$B \hat{\otimes} C = B \bar{\otimes}_{\infty} C$$

for B and C such rings.

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