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# MANIFOLDS IN FORMAL DIFFERENTIAL GEOMETRY

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# MANIFOLDS IN FORMAL DIFFERENTIAL GEOMETRY

A. Kock and G.E. Reyes

This paper is the fourth in a series whose general theme may be described as formal differential geometry (the other three being [6],[7], and [11]). The basic idea, which goes back to Lawvere [8], is to work in a category with a ring object A ("the line") and another object D ("the generic tangent vector", or alternatively "a point with an infinitesimal linear neighbourhood"), by means of which one may interpret directly geometric entities on suitable "manifold" objects M of the category, and their combinatorial relationships, by performing simple operations of the category on M,A, and D. Thus, the tangent bundle of M becomes the exponential object $M^D$, the double tangent bundle becomes $(M^D)^D \cong M^{D\times D}$, etc. To the extent that geometric entities and their relations are considered as primitives, we may view this study as <u>synthetic</u> differential geometry. To the extent that no limit processes are involved in these geometric constructions besides the <u>formal</u> operations of the category, this study may be considered as a generalization of certain aspects of "differential calculus on schemes" appearing in algebraic geometry (see §§4 and 5, and [11]).

We should point out, however, that we do not yet have a model for the axiomatic approach which comprises "classical" differential geometry.

The specific purpose of the present paper is to investigate conditions for the tangent bundle of M to have good fibrewise vector space structure (or rather A-module structure), and to prove that certain classical geometric objects, like Grassmannian manifolds, satisfy these conditions. Part of the conditions are on the ambient category $\underline{E}$ and on a given, basic ring object $A$ in it. In particular, we construct in §4 Grassmannian manifolds "internally" in this situation generalizing [5] slightly. However, for the differential-geometric structure of the constructed manifolds we need further that $A$ is of line type [6], a notion which we recall in §1.

The two main general tools are the notion of etale descent, and of free group actions. These are treated in §2 and 3 respectively. For the concrete models which we exhibit in §5, our etaleness and infinitesimal linearity is closely related to the "classical" notion, as exposed, say in [4].

We benefited from several discussions with Gavin Wraith.

## §1. Infinitesimal linearity

Let $\underline{E}$ be a category with finite inverse limits, and let $A$ be a ring object in $\underline{E}$. We let $D$ be the subobject of $A$

$$D = D(1) = [a \in A \mid a^2 = 0]$$

(formally, $D \rightarrowtail A$ is the equalizier of the squaring map $A \to A$ and the constant-zero map $A \to A$; we shall use set theoretic notation throughout, both for describing subobjects (defined by finite limits), and maps between such). As in [6] we say that $A$ is of line type if $D$ is exponentiable in $\underline{E}$, and the map $\alpha: A \times A \to A^D$, exponential adjoint of $\langle a_o, a_1, d\rangle \mapsto a_o + d \cdot a_1$, is invertible. Throughout the paper, we assume $A$ to be of line type.

More generally, we consider the subobject

$$D(n) \rightarrowtail A^n$$

defined by

$$D(n) = [(a_1,\ldots,a_n) \mid a_i \cdot a_j = 0 \quad \forall\, i,j = 1,\ldots,n].$$

We have $n$ "inclusion maps" $i_r: D(1) \to D(n)$ $(r = 1,\ldots,n)$ given by $i_r(d) = (0,\ldots,0,d,\ldots,0)$ with the $d$ placed in r'th position. We shall henceforth assume that each $D(n)$ is exponentiable.

We say that an object $M$ is infinitesimally linear if for each $n$

$$M^{D(n)} \; \underset{M^{i_n}}{\overset{M^{i_1}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}}}} \; M^{D(1)}$$

makes

$$M^{o}\colon M^{D(n)} \to M$$

into an n-fold product of $M^{o}\colon M^{D(1)} \to M$ in $\underline{E}/M$. (We express this also by saying: $M^{D(n)}$ is an n-fold pull-back of $M^{D(1)}$ over $M$.)

In particular, for $n = 2$, the condition says that

$$\begin{array}{ccc} M^{D(2)} & \xrightarrow{M^{i_2}} & M^{D(1)} \\ {\scriptstyle M^{i_1}}\downarrow & & \downarrow{\scriptstyle M^{o}} \\ M^{D(1)} & \xrightarrow[M^{o}]{} & M \end{array}$$

is a pull-back diagram in $\underline{E}$, In [11] §1 is proved that this implies

<u>Theorem</u>. The object in $\underline{E}/M$

(1.1) $$M^{D(1)} \xrightarrow{M^{o}} M$$

is an abelian group object in $\underline{E}/M$, with addition given by

$$M^{D(1)} \underset{M}{\times} M^{D(1)} = M^{D(2)} \xrightarrow{M^{\Delta}} M^{D(1)}$$

where $\Delta\colon D(1) \longrightarrow D(2)$ is given by $d \longmapsto (d,d)$. (In fact, (1.1) is really an A-module object in $\underline{E}/M$).

So for infinitesimally linear objects $M$, we have a tangent bundle $M^{D}$ with good algebraic properties: it is fibrewise linear. (Our assumptions so far do not imply that $A$ itself is infinitesimally linear, but it is so in all models we know of for our axiomatics).

We note that if an object $M$ is infinitesimally linear, and $J$ is any exponentiable object, then $M^{J}$ is infinitesimally linear. This follows because the functor $(-)^{J}$ preserves limits, in particular those n-ary pull-backs which define the notion of infinitesimally linear objects.

We shall now introduce the auxiliary notion of <u>1-étale</u> map (Def. 1.1 below), whose purpose are that they allow "descent of infinitesimal linearity" which will be the content of §2.

We shall say that an object $J$ of form

$$J = D(n_1) \times \ldots \times D(n_r)$$

is a <u>1-small</u> object. Any 1-small object is <u>pointed</u> in the sense that there exists a canonical map

$$\mathbb{1} \xrightarrow{\langle 0,\ldots,0\rangle} D(n_1) \times \ldots \times D(n_r)$$

(which we just denote $0$). Since we have assumed that each $D(n_i)$ is exponentiable, it follows that each 1-small object $J$ is exponentiable.

<u>Definition 1.1</u>. A map $f\colon M \to N$ in $\underline{E}$ is called <u>1-étale</u> if for any 1-small object $J$, the diagram

(1.2)
$$\begin{array}{ccc} M^J & \xrightarrow{f^J} & N^J \\ {\scriptstyle M^0}\downarrow & & \downarrow{\scriptstyle N^0} \\ M & \xrightarrow[f]{} & N \end{array}$$

is a pull-back. (In the present article, we often write "étale" instead of "1-étale".)

By taking exponential adjoints, 1-étaleness is seen to be equivalent to the condition that any commutative square

(1.3)
$$\begin{array}{ccc} X & \xrightarrow{X\times 0} & X\times J \\ {\scriptstyle m}\downarrow & \overset{t}{\swarrow} & \downarrow{\scriptstyle n} \\ M & \xrightarrow[f]{} & N \end{array}$$

has unique commutative fill-out $t$. (If $f$ is monic one may think of this as expressing: The subobject $M$ of $N$ is stable under 1-small extensions.)

Proposition 1.2. Let $h: J \to K$ be a 0-preserving map between 1-small objects. If $f: M \to N$ is 1-étale, then the diagram

(1.4)
$$\begin{array}{ccc} M^K & \xrightarrow{f^K} & N^K \\ {\scriptstyle M^h}\downarrow & & \downarrow{\scriptstyle N^h} \\ M^J & \xrightarrow[f^J]{} & N^J \end{array}$$

is a pull-back.

Proof. Place the square (1.4) on top of the square (1.2). Then the total rectangle is a pull-back since $f$ satisfies the étaleness condition with respect to $K$. And the lower square is a pull-back since $f$ satisfies the étaleness condition with respect to $J$. By a well know diagram lemma (see e.g. [9], Ex.8(b) p.72), the top square (which is (1.4)) is then also a pull-back.

Proposition 1.3. If the square

$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \cdot & \longrightarrow & \cdot \end{array}$$

is a pull-back and $f$ is 1-étale, then $g$ is 1-étale.

Proof. Easy from the fact that functors $(\ )^J$ commute with pull-backs, and the above mentioned diagram lemma.

Proposition 1.4. Let $f: M \to N$ be any map between infinitesimally linear objects. If the diagram (1.2) is a pull-back for

for $J = D(1)$, then it is a pull-back for all 1-small $J$ (or equivalently: then $f$ is 1-étale).

<u>Proof</u>. We first prove it for $J$ a 1-small objects of form $D(n)$. The argument is by induction in $n$. For $n = 1$, it is the assumption. Assume now that it holds for $J = D(n-1)$. By infinitesimal linearity of $M$

$$M^{D(n)} = M^{D(n-1)} \times_M M^{D(1)} ,$$

and similarly for $N$. Now it is a pure diagrammatic argument that if

$$\begin{array}{ccc} M' & \longrightarrow & N' \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & N \end{array} \qquad\qquad \begin{array}{ccc} M'' & \longrightarrow & N'' \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & N \end{array}$$

are pull-backs, then so is

$$\begin{array}{ccc} M' \times_M M'' & \longrightarrow & N' \times_N N'' \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

Apply this for $M' = M^{D(n-1)}$, $M'' = M^{D(1)}$, $N' = N^{D(n-1)}$, and $N'' = N^{D(1)}$.

To prove the desired conclusions for any 1-small $J$, it suffices to see that if (1.2) is a pull-back for $J = K$ and $J = L$ ($K$ and $L$ arbitrary pointed objects) then it is a pull-back for $J = K \times L$.

Consider the commutative diagram

$$(1.5)\qquad \begin{array}{ccc} (M^L)^K & \xrightarrow{(f^L)^K} & (N^L)^K \\ \downarrow & & \downarrow \\ M^K & \xrightarrow{f^K} & N^K \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & N \end{array} \quad ;$$

here the lower square is a pull-back by the assumption on $K$. The upper square is obtained by applying the functor $(-)^K$ to the square 1.2 with $J = L$. Since $(\ )^K$ preserves pull-backs, the upper square is a pull-back by the assumption on $L$. Thus the total rectangle (1.5) is a pull-back. Under the identification $(M^L)^K \cong M^{K\times L}$ it becomes that diagram which we wanted to prove to be a pull-back.

The notion of <u>subobject of units</u> of a commutative ring object $A$ makes sense in any category with finite lim, namely as (again using set theoretic notation)

$$U(A) = [(x,y) \in A^2 \mid x\cdot y = 1] \ ;$$

The composite

$$(1.5)\qquad U(A) \rightarrowtail A \times A \xrightarrow{proj_1} A$$

is monic, due to the uniqueness of multiplicative inverses in commutative rings, and in this way $U(A)$ may be considered a subobject of $A$.

The specific 1-étale map which we shall use later will ultimately stem from the following

Proposition 1.5. Assume that $A$ is of line type and infinitesimally linear. Then $U(A)$ is infinitesimally linear, and the map $U(A) \rightarrowtail A$ is 1-étale.

Proof. For any exponentiable object $K$, $(-)^K$ preserves limits, so takes ring objects to ring objects; in particular $A^K$ is a ring object. For $K = D(1)$, it is proved in [76i] that $A$ being of line type implies that we have a ring isomorphism

$$A^{D(1)} \cong A[\varepsilon]$$

the right hand side being $A \times A$ made into a ring by using the idea of ring-of-dual-numbers over $A$. To $A^o \colon A^{D(1)} \to A$ corresponds $\beta \colon A \times A \to A$, projection onto first factor. It follows now from infinitesimal linearity that $A^{D(n)} \cong A^{n+1}$ with the $n$ maps

$$A^{D(n)} \cong A^{n+1} \xrightarrow[A^{i_n}]{A^{i_n}} \vdots \; A^{D(1)} = A^2$$

given by

$$\langle a_o, \ldots, a_n \rangle \overset{A^{i_r}}{\longmapsto} \langle a_o, a_{i_r} \rangle$$

Since the forgetful functor from Rings-in-$\underline{E}$ to $\underline{E}$ creates limits, we can also describe the ring structure on $A^{D(n)}$ in terms of $A^{n+1}$, namely

$$A^{D(n)} \cong A^{n+1} = A[\varepsilon_1, \ldots, \varepsilon_n]$$

where the multiplication table for this latter ring is given by $\varepsilon_i \cdot \varepsilon_j = 0 \quad \forall i,j.$

Since $(-)^{D(n)}$ commutes with limits, it commutes with the object-of-units construction $U$:

$$U(A^{D(n)}) = (U(A))^{D(n)} .$$

But it is easy to see from the "multiplication table" of $A[\varepsilon_1,\ldots,\varepsilon_n] = A^{n+1}$ that

$$U(A^{n+1}) = U(A) \times A^n$$

and thus

$$(U(A))^{D(n)} = U(A) \times A^n \begin{array}{c}\longrightarrow\\ \vdots\\ \longrightarrow\end{array} (U(A))^{D(1)} = U(A) \times A$$

with the $n$ displayed maps given by the $n$ projections $A^n \to A$. But this is clearly an n-fold pull-back over $U(A)$. This proves that $U(A)$ is infinitesimally linear. To see that $U(A) \rightarrowtail A$ is 1-étale, it suffices, by Proposition 1.4 to see that

$$\begin{array}{ccc} U(A)^{D(1)} & \longrightarrow & A^{D(1)} \\ \downarrow & & \downarrow \\ U(A) & \longrightarrow & A \end{array}$$

is a pull-back. But again this follows from $A^{D(1)} = A \times A$ and $U(A)^{D(1)} = U(A^{D(1)}) = U(A) \times A$.

## §2. Étale descent of infinitesimal linearity

In this paragraph, we need more exactness assumptions on the ambient category $\underline{E}$. For simplicity, we shall assume that $\underline{E}$ is a regular category in the sense of [2]. Recall that in such a category, regular epimorphisms (= epics that occur as co-equalizers) are stable under pull-back and composition. We shall need

Lemma 2.1. If we have a pull-back diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ q \downarrow & & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

in a regular category $\underline{E}$, with $p$ regular epic and $g$ iso, then $f$ is iso.

Proof. Since both $g$ and $p$ are regular epic, then so is their composite $p \circ g = f \circ q$ Assume therefore that $f \circ q$ is co-equalizer for a pair of maps $r, s \colon Z \to X'$. By pull-back stability of regular epics, $q$ is regular epic, and in particular epic. Then it is easy to see that $f$ is coequalizer for $q \circ r$, $q \circ s$, and thus is regular epic. An easy diagram chase gives that $f$ is monic. But monic and regular epic implies iso.

Theorem 2.2 (Étale descent). Let $p \colon M \twoheadrightarrow N$ be any map which is regular epic and 1-étale. Then

$$M \text{ infinitesimally linear} \Rightarrow N \text{ infinitesimally linear}.$$

Proof. Consider the diagram

$$\begin{array}{ccccc} M^{D(n)} & \xrightarrow{\cong} & \prod_M^n M^{D(1)} & \xrightarrow{\mu} & M \\ {\scriptstyle p^{D(n)}}\downarrow & & \downarrow{\scriptstyle \Pi_p} & & \downarrow{\scriptstyle p} \\ N^{D(n)} & \longrightarrow & \prod_N^n N^{D(1)} & \xrightarrow{\nu} & N \end{array}$$

where $\prod_M^n$ denotes n-fold product in $\underline{E}/M$, i.e., n-fold pull-back over M, with $\mu$ the structural map, and similarly $\prod_N^n$ for product in $\underline{E}/N$. Now the functor "pull-back along p",

$$p^*: \underline{E}/N \to \underline{E}/M$$

preserves products (it has a left adjoint "composing with p"), and $p^*(N^{D(1)} \to N) = (M^{D(1)} \to M)$ by etaleness assumption. These two things imply that the right hand square is a pull-back. The total diagram is a pull-back, again by 1-étaleness of p. From a well known diagram lemma (see e.g. [ML], Ex. 8(6) p.72) we conclude that the left hand square is also a pull-back. The map $\Pi_p$ is regular epic because the right hand square is a pull-back and p is regular epic. From Lemma 2.1 we then conclude that

$$N^{D(n)} \to \prod_N^n N^{D(1)}$$

is iso, which means that N is infinitesimally linear.

There is a dual, but easier, statement about "étale restriction"; for that, we only need that $\underline{E}$ has finite limits:

Proposition 2.3. Let $m: M \rightarrowtail N$ be any map which is monic and 1-étale. Then

N infinitesimally linear ⇒ M infinitesimally linear

Proof. The functor $(\ )^{D(n)}$ preserves monic maps. So the top map in the diagram

$$\begin{array}{ccc}
M^{D(n)} & \overset{m^{D(n)}}{\rightarrowtail} & N^{D(n)} \\
\downarrow \cdots \downarrow & & \downarrow \cdots \downarrow \\
M^{D(1)} & \underset{m^{D(1)}}{\longrightarrow} & N^{D(1)} \\
\downarrow & & \downarrow \\
M & \underset{m}{\rightarrowtail} & N
\end{array}$$

is monic. The right hand column is an n-fold pull-back; each of the n upper squares is a pull-back, by étaleness of $m$ and Proposition 1.2. Now it is ean easy diagram chase to conclude that the left hand column displays $M^{D(n)}$ as an n-fold pull-back of $M^{D(1)}$ over $M$, which is the desired conclusion.

(Note: This Proposition cannot be used to simplify the proof of Prop. 1.5, because there we used infinitesimal linearity as a tool for concluding etaleness.)

## §3. Equivalence relations and free group actions

For our ultimate sufficient conditions for infinitesimal linearity, we need two further assumptions on the category $\underline{E}$ and the ring object $A$ in it. The assumption on $\underline{E}$ is that it is an exact category ([2]), i.e. a regular category where equivalence relations are kernel pairs of their coequalizers ("equivalence relations are effective" (Terminology of SGA4,[1]). The assumption on $A$ is (besides $A$ being of line type and infinitesimally linear) that the 1-small objects are <u>internally projective</u>: an object $J$ in $\underline{E}$ is called internally projective if it is exponentiable, and the functor $(-)^J$ preserves regular epics. (In §4 we shall use "internally projective" in a stronger sense: $(\ )^J$ commutes with finite colimits.)

<u>Proposition 3.1</u>. Assume $\rho_o, \rho_1 : R \rightrightarrows M$ is an equivalence relation with $\rho_o$ and $\rho_1$ 1-étale maps. Then the coequalizer $M \to M/R$ is 1-étale. If further $M$ is infinitesimally linear, then so is $M/R$.

<u>Proof</u>. Let $J$ be 1-small, and consider the diagram

$$\begin{array}{ccccc} R^J & \rightrightarrows & M^J & \longrightarrow & (M/R)^J \\ \downarrow & & \downarrow & & \downarrow \\ R & \rightrightarrows & M & \longrightarrow & M/R \end{array}$$

the horizontal maps in the left hand diagram being $\rho_o^J, \rho_1^J, \rho_o$, and $\rho_1$ respectively. By the étaleness condition on $\rho_o, \rho_1$, each of the left hand squares is a pull-back. The lower row is exact by definition of $M/R$. The upper row is exact because $J$ is internally projective. By a well known diagram lemma for exact categories (see e.g. Barr [2],p. 73 ), we conclude that the right hand square is a pull-back, which proves that $M \to M/R$ is 1-étale. Since $M \to M/R$ is furthermore regular epic, evidently, we conclude by the étale descent Theorem 2.2 that $M/R$ is infinitesimally linear if $M$ is.

One way in which one obtains equivalence relations (not étale relations in general) is by free group actions. Let the group object $G$ act on the object $M$ in an associative and unitary way. The action $a$ is called <u>free</u> if

$$G \times M \underset{proj_2}{\overset{a}{\rightrightarrows}} M \qquad (3.1)$$

is jointly monic. In this case, (3.1) is an equivalence relation. Its coequalizer is denoted $M \to M/G$.

<u>Proposition 3.2</u>. Let $G$ act freely on $M$, and assume that both $M$ and $G$ are infinitesimally linear. Then $M/G$ is infinitesimally linear.

<u>Proof</u>. Consider the diagram

$$\begin{array}{ccccc} M^{D(n)} & \xrightarrow[c]{\cong} & \prod^n_M & M^{D(1)} & \longrightarrow & M \\ p^{D(n)} \downarrow & & \downarrow & \prod_p D(1) & & \downarrow p \\ (M/G)^{D(n)} & \xrightarrow[c']{} & \prod^n_{M/G} & (M/G)^{D(1)} & \longrightarrow & M/G \end{array}$$

with the notation $\prod^n_M$ etc. as in the proof of Theorem 2.2. We want to prove that the comparison map $c'$ is iso. Apart from the operating with exponential objects with set theoretic notation, we shall now also do the exactness argument required in the category of sets. This is justified by Barr's Meta theorem,[2] 6.8. The internal projectivity assumptions first gives that $p^{D(n)}$ is surjective. Now let us first prove that the middle vertical map

$\Pi\, p^{D(1)}$ is surjective. Let $\langle\bar{x}_1,\ldots,\bar{x}_n\rangle \in \overset{n}{\Pi}_{M/G}\,(M/G)^{D(1)}$. Thus $\bar{x}_r\colon D(1) \to M/G$. By the surjectivity of $p^{D(1)}$, each $\bar{x}_r$ can be written $p\circ x_r$ for some $x_r\colon D(1) \to M$. The n-tuple $\langle x_1,\ldots,x_r\rangle \in \overset{n}{\Pi}\, M^{D(1)}$ is not necessarily in $\overset{n}{\Pi}_M\, M^{D(1)}$, but we have, for unique $g_2,\ldots,g_n \in G$

$$x_1(0) = g_r\cdot x_r(0)\rangle \qquad r = 2,\ldots,n$$

since $\bar{x}_r(0) = p(x_r(0))$ is independent of $r$. Now $G$ acts on $M^{D(1)}$, since it acts on $M$. Consider

$$\langle x_1, g_2\cdot x_2,\ldots,g_n\cdot x_n\rangle \in \overset{n}{\Pi}\, M^{D(1)} .$$

This element is seen to belong to the required subobject $\overset{n}{\Pi}_M\, M^{D(1)}$, and to go to $\langle\bar{x}_1,\ldots,\bar{x}_n\rangle$ by the required map. This proves surjectivity of $\Pi\, p^{D(1)}$, and thus since $c$ is iso, and

$$\Pi\, p^{D(1)}\circ c = c'\circ p^{D(n)}$$

and we see that $c'$ is surjective.

We now prove $c'$ monic. Because $D(n)$ is internally projective, we may make the identification

$$(M/G)^{D(n)} = M^{D(n)}/G^{D(n)}$$

and similarly for $D(1)$. We are thus considering the comparison map

$$M^{D(n)}/G^{D(n)} \longrightarrow \overset{n}{\Pi}_{M/G}\, M^{D(1)}/G^{D(1)}$$

To prove this monic is essentially a finite inverse limit argument which we again shall do in the category of sets.

Assume $z$ and $z'$ are elements in $M^{D(n)}$ which represent elements $\bar{z}$ and $\bar{z}'$ in $M^{D(n)}/G^{D(n)}$. Assume $c'(\bar{z}) = c'(\bar{z}')$. This means that for each $r = 1,\ldots,n$

(3.2) $$z \circ i_r \equiv z' \circ i_r \quad \text{mod} \quad G^{D(1)} .$$

($i_r$ denoting, as in §1, the r'th inclusion $D(1) \to D(n)$). By (3.2) we can find (unique) elements $g_r \in G^{D(1)}$ with

(3.3) $$g_r \cdot (z \circ i_r) = z' \circ i_r$$

View $g_r$ as a map $D(1) \to G$, and evaluate (3.3) at $0$. This yields

(3.4) $$g_r(0) \cdot z(0) = z'(0),$$

independent of $r$. Since the action of $G$ on $M$ is free, we conclude from (3.4) that $g_r(0)$ is independent of $r$. Thus

$$\langle g_1,\ldots,g_n \rangle \in \prod_G^n G^{D(1)} \cong G^{D(n)} ,$$

the isomorphism by infinitesimal linearity of $G$. So there is an $x \in G^{D(n)}$ with

(3.5) $$x \circ i_r = g_r \qquad r = 1,\ldots,n.$$

We shall now prove

(3.6) $$x \cdot z = z' \quad : \quad D(n) \to M.$$

Since $M$ is infinitesimally linear, it follows that this can be proved by proving, for each $r = 1,\ldots,n$

$$(x \cdot z) \circ i_r = z' \circ i_r : D(1) \to M$$

We compute on the left hand side

$$\begin{aligned}(x\cdot z)oi_r &= (xoi_r)\cdot(zoi_r)\\ &= g_r\cdot(zoi_r) && \text{by (3.5)}\\ &= z'oi_r && \text{by (3.3).}\end{aligned}$$

This proves (3.6), and thus $\bar{z} = \bar{z}'$, and thus that the comparison map $c'$ is monic.

Since we have also seen it to be regular epic, we conclude that it is iso, which proves $M/G$ infinitesimally linear.

## §4. Manifolds, and Grassmannians in particular

Let $A$ be a ring object of line type and infinitesimally linear in a category $\underline{E}$. If $M$ is any object in $\underline{E}$, and $V$ is an $A$-module object, $\mathrm{proj}_1\colon M\times V \to M$ is an $A$-module object in $\underline{E}/M$.

Definition 4.1. Let $M$ be infinitesimally linear. We say that $M$ is parallelizable if there is a fibrewise $A$-linear isomorphism $\varphi$:

$$\begin{array}{ccc} M^D & \xrightarrow{\varphi} & M\times V \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

for some $A$-module object $V$ in $\underline{E}$. Alternatively, $\varphi$ is an $A$-module morphism in $\underline{E}/M$. $V$ is called the fibre.

We shall only be interested in the case where $V = A^n$ for some $n$.

Definition 4.2. Let N be an arbitrary object. We say that N is a manifold (of dimension n) if there is a 1-étale, regular epic $M \twoheadrightarrow N$ with M infinitesimally linear and parallelizable with fibre $A^n$.

(Heuristically, if $\{N_\alpha \subseteq N \mid \alpha \in a\}$ is an open covering of N by $N_\alpha$'s diffeomorphic to open subjects of $\mathbb{R}^n$, $\coprod N_\alpha \twoheadrightarrow N$ will serve as such $M \twoheadrightarrow N$.) Note that by Theorem 2.2 (étale descent), any manifold N is infinitesimally linear.

Proposition 4.3. ("Invariance of dimension"). Assume that the ring $\hom(\mathbb{1}, A)$ is non-trivial $(0 \neq 1)$, and let N be an n-dimensional manifold with full support (i.e. the unique map $N \to \mathbb{1}$ is regular epic). If N also is m-dimensional, then $n = m$.

Proof. Assume that M (respectively M') is parallelizable n-dimensional (respectively m-dimensional) with $M \to N$, $M' \to N$ 1-etale regular epics. Let $P = M \times_N M'$. Then we have $P^D \cong P \times A^n$ and $P^D \cong P \times A^m$ in $\underline{E}/P$. Going over to $\underline{E}/P$, A remains non-trivial (since $P \to \mathbb{1}$ is epic), and, furthermore, $A^n \cong A^m$ by means of an A-linear isomorphism. Taking global sections, we obtain a linear isomorphism $\Gamma(A^n) \cong \Gamma(A^m)$ where $\Gamma(A)$ is a non-trivial ring (in Set). By standard commutative algebra, $n = m$.

We proceed to show that certain specific geometric objects, namely the Grassmannians, which can be constructed from a ring object $A$ in a sufficiently exact category, are in fact manifolds (of correct dimension), provided $A$ is of line type and each 1-small object $J$ is internally projective in the sense that $(-)^J$ commutes with finite colimits.

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We shall further assume that $\underline{E}$ is an exact category with disjoint and universal coproducts (SGA 4,[1], II.4.5). In particular, it has stable sup in the sense of [10]. Therefore, any first order formula $\varphi$ about elements in $A^r$ which is built from polynomial equations and $\wedge, \vee, \exists$, has an extension

$$[\![ (x_1, \dots, x_r) \mid \varphi ]\!]$$

which is a subobject of $A^r$.

---

Let us remark that all assumptions made hold for the generic ring object $A$ which lives in the topos $\underline{E} = \underline{\text{Sets}}^{\underline{\text{FP Rings}}}$. But they also hold for the generic local ring object $A$ in the Zariski topos [3]. That $A$ is of line type in these cases is proved in [6]. The internal projectivity of 1-small objects is trivial in the former case, but requires a slight argument in the latter (see below, §5).

We proceed to construct the Grassmannian objects. We first define the "Stiefel manifold $V(k,n)$" ; it is defined as that subobject of $A^{kn}$ which is the extension of the following formula about elements in $A^{kn}$ (= $k \times n$-matrices over $\hom(X,A)$):

"At least one $k \times k$ minor is invertible",

(or in case A is local, by the equivalent

"the $k \times k$ minors generate the unit ideal of A".)

The "at least" here is an $\binom{n}{k}$-fold disjunction, namely over the $\binom{n}{k}$ possible $k \times k$ submatrices of a $k \times n$-matrix. Minors (= determinants) are formed purely equationally, and invertibility is defined by an $\exists$ quantifier. Thus the extension exist.

In particular $V(k,k) \rightarrowtail A^{k^2}$ is the object of $k \times k$ matrices with invertible determinant, which is the same as the object $GL(k) = GL(k,A)$ of invertible $k \times k$ matrices. It carries a group object structure, namely matrix multiplication. This group acts on $V(k,n)$ by matrix-multiplication from the left. If $n \geq k$, it is easy to see that the action is free in the sense of §3.

For $n \geq k$, we define $G(k,n)$ to be the object $V(k,n)/GL(k)$ ("Grassmannian object of k-planes in n-space".)

Theorem 4.1. The Grassmannian $G(k,n)$ is a manifold. Its dimension is $k \cdot (n-k)$.

We first look at $V(k,n)$. Being the extension of an $\binom{n}{k}$-fold disjunction, $V(k,n)$ is a union (sup) of $\binom{n}{k}$ subobjects of $A^{kn}$. We consider a typical one of these, $Q_H$ where $H = \{i_1 < \ldots < i_k\}$ is a subset of $\{1,2,\ldots,n\}$; $Q_H$ is the extension of the formula "the $k \times k$ submatrix with column indices from H has invertible determinant". Clearly $Q_H$ is stable under the action (matrix multiplication from the left) of $GL(k)$. Denote by $P_H$ the quotient $Q_H/GL(k)$. It is a subobject of $V^{(k,n)}/GL(k) = G(k,n)$, and since the union of the $Q_H$'s is $V(k,n)$, the union of the $P_H$'s is $G(k,n)$.

It is easy to see that $P_H \cong A^{k\cdot(n-k)}$. For simplcity of notation, consider the case where $H = \{1,\ldots,k\}$. Then the following composite is invertible:

$$A^{k\cdot(n-k)} \to Q_H \twoheadrightarrow Q_H/GL(k) = P_H$$

where the first map has description

$$\underline{\underline{B}} \longmapsto \{\underline{\underline{E}}_k\ \underline{\underline{B}}\}$$

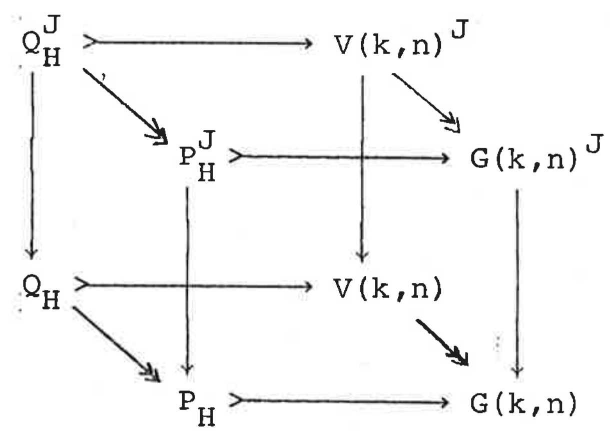
(for $\underline{\underline{B}}$ any $k\times(n-k)$ matrix, and where $\underline{\underline{E}}_k$ is the unit $k\times k$ matrix).

We now want to argue that the inclusion $P_H \rightarrowtail G(k,n)$ is 1-étale. We first note that $Q_H \rightarrowtail V(k,n)$ is etale; for, there exists a pull-back square

$$\begin{array}{ccc} Q_H & \rightarrowtail & V(k,n) \\ \downarrow & & \downarrow d_H \\ U_A & \rightarrowtail & A \end{array}$$

where $d_H$ to a $k\times n$ matrix associates the determinant of the submatrix with column indices from $H$; $U(A) \rightarrowtail A$ is étale by Proposition 1.5, and pulling back an étale map along anything yields an étale map by Proposition 1.3.

Now we can prove étaleness condition for $P_H \rightarrowtail G(k,n)$ with respect to an arbitrary 1-small object $J$. We must prove that the front square in the box



is a pull-back. This is a pure diagram chasing argument using only the facts 1) that the maps indicated by $\twoheadrightarrow$ are regular epic (for the two upper ones this follows from internal projectivity of $J$), 2) that $Q_H \rightarrowtail V(k,n)$ is monic and stable under that equivalence relation which defines $G(k,n)$ out of $V(k,n)$ and 3) the fact that the back square is a pull-back (which is the êtaleness of $Q_H \rightarrowtail V(k,n)$ just proved). The diagram chase argument may, by Barr's Metatheorem ([2], Theorem 6.8) be proved in the category of sets: Let $g \in G(k,n)^J$ and $p \in P_H$ meet each other in $G(k,n)$. Pick representatives $g' \in V(k,n)^J$ and $p' \in Q_H$, respectively. Then $p'$ may not meet $g'$ in $V(k,n)$, but since they meet in $G(k,n)$, they are equivalent. Since $Q_H$ is stable under the equivalence relation, we may replace $p'$ by $p''$ such that $p''$ meets $g'$ in $V(k,n)$. Using that the back square is a pull-back, we get a $q' \in Q_H^J$ whose image in $P_H^J$ will hit $g$ and $p$ under the relevant maps.

Since $P_H \cong A^{k\cdot(n-k)}$, $P_H$ is parallelizable of dimension $k\cdot(n-k)$, and since $P_H \rightarrowtail G(k,n)$ is etale and the $P_H$'s cover $G(k,n)$ the Theorem will be proved when the following two general lemmas have been established.

<u>Lemma 4.2</u>. A finite coproduct of parallelizable objects $X_i$ of dimension $r$ is parallelizable of dimension $r$.

<u>Lemma 4.3</u>. If $\{\xi_i : X_i \to X \mid i \in I\}$ is a finite family of êtale maps, then the induced map $\coprod X_i \to X$ is êtale.

<u>Proof</u> of Lemma 4.2. First, we must prove $\coprod X_i$ infinitesimally linear. Consider

$$(\coprod X_i)^{D(n)} \rightrightarrows (\coprod X_i)^{D(1)} \longrightarrow \coprod X_i$$

By the internal projectivity assumption, $(\coprod X_i)^{D(n)}$ may be rewritten $\coprod(X_i^{D(n)})$, and similarly for $D(1)$. Each

$$X_i^{D(n)} \; \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \; X_i^{D(1)} \longrightarrow X_i$$

is an n-fold pull-back. If we take a coproduct of such, we again get an n-fold pull-back, using that coproducts are assumed disjoint and universal.

Next

$$(\coprod X_i)^D \cong \coprod(X_i^D) \cong \coprod(X_i \times A^r) \cong (\coprod X_i) \times A^r$$

proves the parallelizability.

Proof of Lemma 4.3. Let $J$ be 1-small. We must prove the following diagram to be a pull-back

$$\begin{array}{ccc} (\coprod X_i)^J & \longrightarrow & X^J \\ \downarrow & & \downarrow \\ \coprod X_i & \longrightarrow & X \end{array}$$

but rewriting $(\coprod X_i)^J$ as $\coprod(X_i^J)$ (again by internal projectivity of $J$), this again becomes an easy consequence of coproducts being disjoint and universal.

Note that the argument which gave that $Q_H \rightarrowtail V(k,n)$ is étale also will give that $Q_H \rightarrowtail A^{kn}$ is étale. Specializing to $k = n$, we get that $GL(k) \rightarrowtail A^{kk}$ is étale, and thus by Proposition 2.3 that $GL(k)$ is infinitesimally linear. Using Proposition 3.2, we get another proof that $P_H$ $(= Q_H/GL(k))$ is infinitesimally linear.

## §5. Models for the axioms

We shall prove in this paragraph that "the generic ring", the "generic local ring", as well as the "generic strictly local ring" satisfy all the axioms which have been used in §§1-4. Recall from Hakim [3], III.3 that the functor category (topos) $\underline{\text{Set}}^{\mathcal{R}}$ (where $\mathcal{R}$ = category of finitely presented rings) has a ring object $A: \mathcal{R} \to \underline{\text{Set}}$, namely the forgetful functor, and that this ring object is the generic (commutative) ring, in the sense of classifying toposes; $\underline{\text{Set}}^{\mathcal{R}}$ contains subtoposes ("étale topos" and "Zariski topos")

$$\underline{\text{Et}} \subseteq \mathfrak{Z} \subseteq \underline{\text{Set}}^{\mathcal{R}}. \tag{5.1}$$

The Yoneda embedding $y: \mathcal{R}^{op} \to \underline{\text{Set}}^{\mathcal{R}}$ factors through $\underline{\text{Et}}$, and since $A = y(\mathbb{Z}[X])$, $A$ lives in each of these two subtoposes, and $A \in \mathfrak{Z}$ is the generic local ring object, $A \in \underline{\text{Et}}$ is the generic strictly local ring object. (Hakim's notation is $\mathcal{O}_o, S_o$ for $A$, $\underline{\text{Set}}^{\mathcal{R}}$, $\mathcal{O}_1, S_1$ for $A, \mathfrak{Z}$, and $\mathcal{O}_2, S_2$ for $A, \underline{\text{Et}}$.)

In [6], Theorem 12, it is proved that $A \in \underline{\text{Set}}^{\mathcal{R}}$ is of line type, and from ibid., Remark 13, it follows that $A \in \underline{\text{Et}}$ and $A \in \mathfrak{Z}$ are likewise of line type.

Also, $A$ considered in each of the three toposes in (5.1) is infinitesimally linear. This follows from

Proposition 5.1. Each representable functor $y(B)$ (considered in any of the three toposes of (5.1)) is infinitesimally linear.

Proof. We remark that

$$D(n) = y(\mathbb{Z}[\varepsilon_1, \ldots, \varepsilon_n]) \qquad (\text{with } \varepsilon_i \cdot \varepsilon_j = 0 \ \forall i,j), \tag{5.2}$$

and that

$$\mathbb{Z}[\varepsilon_1,\ldots,\varepsilon_n] \;\vdots\; \mathbb{Z}[\varepsilon] \longrightarrow \mathbb{Z}$$

is an n-fold pull-back in $\mathcal{R}$, which is preserved as such by any functor for form $C \otimes -$ $(C \in \mathcal{R})$. Now the left exactness of

$$(5.2) \qquad y(B)^{D(n)} \;\vdots\; y(B)^{D(1)} \longrightarrow y(B)$$

can be checked by proving that for each $C \in \mathcal{R}$, the functor $[y(C),-]$ takes (5.2) into a left exact sequence in Set. This is easy, using the above remark.

We now prove a property of 1-small object which was used in the axiomatic approach of §§3 and 4 (internal projectivity).

Theorem 5.2. In each of the toposes Et, $\mathfrak{Z}$, and $\text{Set}^{\mathcal{R}}$, 1-small objects $J$ are internally projective (meaning $(-)^J$ commutes with all colimits).

Proof. The case $\text{Set}^{\mathcal{R}}$ is easy. First observe that each $J$ is built by finite limits from $A$, and since the Yoneda embedding preserves finite limits, each 1-small $J$ is representable, i.e. of form $y(B)$ for suitable $B \in \mathcal{R}$. But we have a general categorical fact:

Proposition 5.3. In any functor category $\text{Set}^{\underline{C}^{op}}$ (with $\underline{C}$ small), if $B \in \underline{C}$ has the property that $C \times B$ exist for any $C \in \underline{C}$, then the functor

$$(\ )^{y(B)} : \text{Set}^{\underline{C}^{op}} \longrightarrow \text{Set}^{\underline{C}^{op}}$$

has adjoints on both sides, and in particular preserves limits and colimits.

Proof. For any $F \in \text{Set}^{\underline{C}^{op}}$ it is easy (using Yoneda Lemma) to see that $F^{y(B)}$ is the composite functor

$$\underline{C}^{op} \xrightarrow{-\times B} \underline{C}^{op} \xrightarrow{F} \underline{Set} \ ,$$

so that the endofunctor $(\ )^{y(B)}$ on $\underline{Set}^{\underline{C}^{op}}$ is isomorphic to $\underline{Set}^{\beta}$ where $\beta: \underline{C} \to \underline{C}$ is the functor $-\times B$. But any functor of form $\underline{Set}^{\beta}$ has adjoints on both sides: the Kan extensions along $\beta$.

We next deal with the $\mathfrak{Z}$-case. This is deduced from the $\underline{Set}^{\mathcal{R}}$ case already established, by means of

Lemma 5.4. Let $r: \underline{Set}^{\mathcal{R}} \to \mathfrak{Z}$ denote the sheaf reflection functor. Then for any 1-small object $J$, and any $X \in \underline{Set}^{\mathcal{R}}$

$$r(X^J) \cong r(X)^J$$

From the lemma, we easily get the theorem: denote colimits in $\mathfrak{Z}$ and $\underline{Set}^{\mathcal{R}}$ by $\varinjlim_{\mathfrak{Z}}$ and $\varinjlim$, respectively. Then for any diagram $\{X_i \mid i \in I\}$ in $\mathfrak{Z}$, we have

$$(\varinjlim_{\mathfrak{Z}}(X_i))^J = (r(\varinjlim X_i))^J = r((\varinjlim X_i)^J),$$

using the lemma, and then, using internal projectivity of $J$ in $\underline{Set}^{\mathcal{R}}$, we continue:

$$= r(\varinjlim(X_i^J)) = \varinjlim_{\mathfrak{Z}}(X_i^J).$$

Proof of Lemma 5.4. Since any 1-small $J$ is a product of objects $y(\mathbb{Z}[\varepsilon_1,\ldots,\varepsilon_n])$, it suffices, by iteration, to see Lemma 5.4 for $J = y(\mathbb{Z}[\varepsilon_1,\ldots,\varepsilon_n])$. We have to use an explicite construction of the sheaf reflection functor $r: \underline{Set}^{\mathcal{R}} \to \mathfrak{Z}$, namely the classical $r = \ell \circ \ell$ of SGA 4, ([1] Exposeé II), where

$$\ell(F)(B) = \varinjlim_{R \hookrightarrow B} [R,F],$$

$R$ running over the filtered system of covering cribles of $B$.

This means that an element of

$$\ell(X^{y(\mathbb{Z}[\varepsilon_1,\ldots,\varepsilon_n])})(B)$$

is given by an element in

$$[R, X^{y(\mathbb{Z}[\varepsilon_1,\ldots,\varepsilon_n])}] ,$$

for some $R$, or equivalently, by a compatible family of elements in $[y(B[b_i^{-1}]), X^{y(\mathbb{Z}[\varepsilon_1,\ldots,\varepsilon_n])}]$ where

$$\{B \longrightarrow B[b_i^{-1}] \mid i \in I\}$$

is some co-covering for the Zariski (co-) site structure on $\mathcal{R}$. Such a compatible family, by exponential adjointness and Yoneda Lemma is equivalent to a compatible family of elements in

$$X(B[b_i^{-1}, \varepsilon_1,\ldots,\varepsilon_n]) \qquad i \in I.$$

Similarly, an element of

$$(\ell(X))^{y(\mathbb{Z}[\varepsilon_1,\ldots,\varepsilon_n])}(B) = \ell(X)(B[\varepsilon_1,\ldots,\varepsilon_n])$$

is given by a Zariski co-covering

$$\{B[\varepsilon_1,\ldots,\varepsilon_n] \longrightarrow B[\varepsilon_1,\ldots,\varepsilon_n][b_i'^{-1}] \quad | \quad i \in I'\}$$

and a compatible family of elements in

$$X(B[\varepsilon_1,\ldots, {}_n, b_i'^{-1}])$$

But it is easy to see that there is a 1-1 correspondence between Zariski co-coverings of $B$ and Zariski-co-coverings of $B[\varepsilon_1,\ldots,\varepsilon_n]$, essentially because an element

$$b' = c_o + \varepsilon_1 c_1 + \ldots + \varepsilon_n c_n$$

in $B[\varepsilon_1,\dots,\varepsilon_n]$ is invertible if and only if $c_o$ is invertible in B. In that case one has furthermore

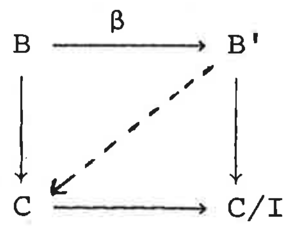
$$B[\varepsilon_1,\dots,\varepsilon_n,b'^{-1}] = B[c_o^{-1},\varepsilon_1,\dots,\varepsilon_n].$$

We conclude $\ell(X^J) = \ell(X)^J$, and since $r = \ell\circ\ell$, we get the result stated in the lemma for r.

We shall finally deal with the case of the topos Ét. The proof of internal projectivity of 1-small objects J in Ét is similar to the one given for the case $\mathfrak{Z}$, provided we can prove the analogue of Lemma 5.4, for the sheaf reflection functor $r: \underline{Set}^{\mathcal{R}} \to \underline{Et}$. As in the proof of Lemma 5.4 for the $\mathfrak{Z}$ case, it is sufficient to see that there is a natural 1-1 correspondence between étale cocoverings of $B[\varepsilon_1,\dots,\varepsilon_n]$ and étale cocoverings of B. Recall (from [3], say), that an étale covering of a $B \in \mathcal{R}$ is a finite family

$$\{\beta_i: B \to B_i \mid I \in I\}$$

such that (i) each prime ideal in B comes from a prime ideal in some $B_i$, (ii) the $\beta_i$ are essentially of finite presentation (iii) the $\beta_i$ are <u>formally étale</u>. We recall the latter notion (see e.g. [4], II.1.6): a morphism of commutative rings $\beta: B \to B'$ is <u>formally étale</u> if whenever C is a commutative ring, with a nilpotent ideal I, then any commutative square (full arrows)




admits a unique commutative fill-in map $B' \to C$ (dotted arrow). (This étaleness-notion is closely related to the one we consider in §1. In fact $y(B') \to y(B)$ will be 1-étale in $\underline{Set}^{\mathcal{R}}$ if

$B \to B'$ is formally étale).

Now, the 1-1 correspondence between étale cocoverings of $B$ and $B[\varepsilon_1,\dots,\varepsilon_n]$ follows immediately from

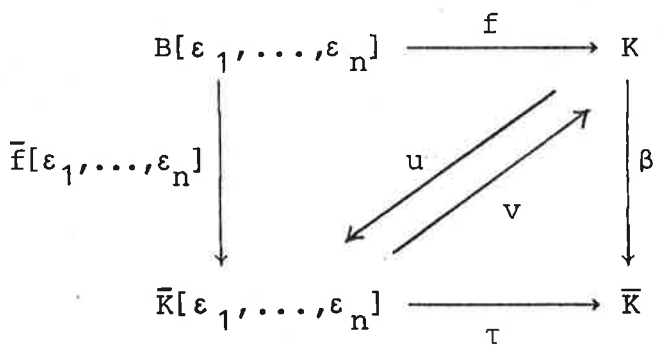
<u>Lemma 5.5</u>. Assume that $B$ is a commutative ring and that $f\colon B[\varepsilon_1,\dots,\varepsilon_n] \to K$ is formally étale. Then

$$K = \bar{K}[\varepsilon_1,\dots,\varepsilon_n]$$

where

$$\bar{K} = K/(f(\varepsilon_1),\dots,f(\varepsilon_n)).$$

<u>Proof</u>. Applying the functor $-\otimes_{B[\varepsilon_1,\dots,\varepsilon_n]} B$ to $f$ yields a map $\bar{f}\colon B \to \bar{K}$ which is again formally étale (formally étale morphism being stable under tensoring up, see e.g. [4] II.1.7). Tensoring up $B \to \bar{K}$ along the zero-section $B \to B[\varepsilon_1,\dots,\varepsilon_n]$ yields $\bar{f}[\varepsilon_1,\dots,\varepsilon_n]\colon B[\varepsilon_1,\dots,\varepsilon_n] \to \bar{K}[\varepsilon_1,\dots,\varepsilon_n]$ which again is formally étale (for the same reason). We then have a commutative square




($\tau$ being the ring map with $\tau(\varepsilon_i) = 0\ \forall i$). Since $f$ is formally étale and $\tau$ has kernel of square zero, we get the map $u$ making the traingles commute. Similarly, using that $\bar{f}[\varepsilon_1,\dots,\varepsilon_n]$ is formally étale and $\beta$ has kernel of square zero (namely $(f(\varepsilon_1),\dots,f(\varepsilon_n))$, we similarly get the map $v$ making the triangles commute. The fact that $u$ and $v$ are mutually inverse follows from the uniqueness assertion contained in the definition of formal étaleness, by considering the squares $\beta \circ f = \beta \circ f$ and $\tau \circ \bar{f}[\varepsilon_1,\dots,\varepsilon_n] = \tau \circ \bar{f}[\varepsilon_1,\dots,\varepsilon_n]$.

## REFERENCES

[1] Artin, M., Grothendieck, A., Verdier, J.L.: Theorie des topos et cohomologie étale des schémas (SGA 4), Springer Lecture Notes, Vol. 269, 270 (1972), 305 (1973).

[2] Barr, M., Exact Categories, in Barr, Grillet, van Osdol: Exact categories, Springer Lecture Notes Vol.236 (1971).

[3] Hakim, M., Topos annelés et schémas relatifs, Ergebnisse der Math. und ihrer Grenzgebiete Vol.64, Springer 1972.

[4] Iversen, B., Generic local structure in commutative algebra, Springer Lecture Notes Vol. 310 (1973).

[5] Kock, A., Linear algebra and projective geometry in the Zariski topos, Aarhus Preprint Series 1974/75 No.4. Revised version: Universal projective geometry via topos theory, Journ. Pure Appl.Alg.9 (1976), 1-24.

[6] Kock, A., A simple axiomatics for differentiation, Aarhus Preprint Series 1975/76 No. 12 (to appear in Math. Scand. 40 (1977).)

[7] Kock, A., Taylor Series calculus for ring objects of line type, Aarhus Preprint Series 1976/77 No. 4.

[8] Lawvere, F.W., Categorical dynamics, lecture Chicago 1967 (unpublished).

[9] MacLane, S., Categories for the working mathematician, Graduate Texts in Mathematics Vol.5, Springer 1971.

[10] Reyes, G.E., From sheaves to logic, in: Studies in algebraic logic, MAA studies vol. 9 (1975) (ed. Daigneault), 143-204.

[11] Reyes, G.E., Wraith G., A note on tangent bundles in a category with a ring object, to appear in Math. Scand.

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