Monads and extensive quantities

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Abstract. If T is a commutative monad on a cartesian closed category, then there exists a natural T-bilinear pairing $T(X) \times T(1)^X \to T(1)$ ("integration"), as well as a natural T-bilinear action $T(X) \times T(1)^X \to T(X)$. These data together make the endofunctors T and $T(1)^{(-)}$ (co- and contravariant, respectively) into a system of extensive/intensive quantities, in the sense of Lawvere. A natural monad map from T to a certain monad of distributions (in the sense of functional analysis (Schwartz)) arises from the integration.

Introduction

Another word for "extensive quantitiy", and one which is commonly used outside mathematics, is "distribution" [1]. In this common non-mathematical usage, an extensive quantity (say, of rain on a given day) may be *distributed* over a given space, (say the sidewalk), and its *total* over that space is measured in terms of some absolute quantity: the total *mass* of the rain on the sidewalk, or the total *number* of raindrops on the sidewalk (this number is an integer). So we have different quantity *types* for such totals, say the quantity type "mass", or the quantity type "(integral) number". In this case, both are "positive"; but one has also quantity types like "electric charge" whose quantities admit opposite signs, in the sense that two such quantities can cancel each other. Note that a mass is not a (non-negative real) *number*, but only becomes so after choosing a *unit* of mass. The *ratio* of a given mass distribution with a chosen unit is a "(distributed) dimensionless quantity", and a good approximative mathematical model for many types of totals of such distributed dimensionless quantities is the ring of real numbers – although for mass, say, *non-negative* real numbers would be a more realistic model for such quantity. Or, the dimensionless quantity may be an integer (or a non-negative integer), for the case of "number of raindrops".

A simple approximative mathematical model of these types of dimensionless total quantities is: they form commutative (additively written) monoids, like $\mathbb{R}$, $\mathbb{R}_+$, or $\mathbb{N}$, in fact, are *free* "algebras" on one generator, for a suitable notion of "algebra" (thus $\mathbb{N}$ is the free *commutative monoid* on one generator, and $\mathbb{R}$ is the free *real vector space* on one generator). The notion of "free algebra" may be encoded by the notion of monad T. Thus $\mathbb{R}$ is $T(1)$,

[1] In some Germanic languages, like German or Danish, the commonly non-mathematical word for these kind of distributions is "Verteilung," resp. "fordeling". In mathematics, the word "distribution" has acquired a more specific meaning, namely the distributions in the sense of Laurent Schwartz, where a "distribution" is a continuous linear functional on a space of "test functions".

for the free real vector space monad T. The fact that $T(1)$ is endowed with a T-bilinear multiplication comes out from the strength of the monad.

In the present article, we experiment with the viewpoint that the dimensionless distributions on a space X form themselves a space $T(X)$, where T is a monad (assumed "commutative") on "the" category $\mathscr{E}$ of spaces, assumed to be a cartesian closed category. For instance, T may be the "free commutative monoid" monad, or the "free real vector space monad", – assuming that the field of reals is itself suitably an object of $\mathscr{E}$. In fact, we have in mind the case where $\mathscr{E}$ is for instance the category of convenient vector spaces and the smooth maps in between; or a topos, like the "smooth topos", or a "well adapted model" for synthetic differential geometry; in these cases, the cohesion (say, topology) of $\mathbb{R}$ is retained, by $\mathbb{R}$ being seen as an object of the category $\mathscr{E}$.

The units of the monad, i.e. the maps $\eta_X : X \to T(X)$, assign to $x \in X$ the distribution with total 1, and concentrated in x, in some contexts: the "Dirac distribution at x".

We shall also have a fragment of a theory of how quantities with a physical dimension, like mass, which are not pure quantities, fit into the picture. They are likewise covariant endofunctors M on $\mathscr{E}$, but $M(1)$, unlike $T(1)$, does not carry a natural multiplication or unit; M is, in some sense, a "torsor" over the appropriate dimensionless-quantity monad T. (In our [17], we considered similar torsor structure, but only for "total" quantities, i.e. *not* distributed over an extended space.)

In most of the present article, we consider only dimensionless quantities.

The theory presented here implies an attempt to comment on Schwartz' dictum "*les distribution mathématiques constituent une définition mathématique correcte des distributions rencontrées en physique.*" ([25] p. 84) – but now with "distribution" in the sense given by general commutative monads.

A main thing is that T is a *covariant functor*. An element $P \in T(X)$ is a distribution on X. We have a unique map $X \to 1$; applying $T(X \to 1)$ to P yields an element in $T(1)$, the *total* of the distribution P. This covariant feature of extensive quantities was stressed by Lawvere; in particular, he stressed that distributions (in the sense of functional analysis) are not to be viewed as "generalized functions" (functions behave contravariantly; they are intensive quantities), but rather are extensive quantities, behaving covariantly (at least when restricted to distributions of compact support). We shall return to some of Lawvere's more specific theory of extensive quantities in the last sections.

One main aspect of the theory to be presented here is that there is a canonical comparison transformation τ from the monad T into a Schwartz (double-dualization) type monad S associated to T. To distinguish, we call the elements P of $T(X)$ *concrete* distributions, to distinguish them from distributions in the sense of functional analysis.

The technical underpinning of the present theory is the theory of strong (= $\mathscr{E}$-enriched) monads on a symmetric monoidal closed category $\mathscr{E}$, developed by the author in a series of articles in the early 1970s, [10], [11], [12], [13], and [14]. We begin by recalling and expanding some of the aspects of this theory; however, since we shall be interested in the case of a CCC (= cartesian closed category) $\mathscr{E}$ only, we use notation etc. from this special case throughout (so we write $\times$ rather than $\otimes$), even though the material in Sections 1-3 deal with the general SMC (= symmetric monoidal closed) case.

Following the typographical convention from these papers, we write $X \pitchfork Y$ for the exponential object Y^X. The counit for the adjunction defining the exponential $- \pitchfork Y$ is denoted $ev : X \times (X \pitchfork Y) \to Y$ (for "evaluation").

Because an unspecified endofunctor T is involved throughout, we have in the main preferred to formulate constructions etc. in terms of diagrams, rather than in terms of elements of the objects considered; however, expressions talking about "elements" in "sets" are sometimes more readable than diagrams, so we shall sometimes use such expressions, even though objects of $\mathscr{E}$ may have no (global) elements at all. Alternatively, the elements talked about are *generalized* elements in the sense used in, say, Synthetic Differential Geometry, as expounded in [15]. It is in principle routine to translate equations and constructions, expressed in terms of elements, into commutative diagrams.

Acknowledgements. The dialectics between extensive and intensive quantities, as covariant and contravariant, I learned from Lawvere, and this was a leading guideline in the present research. This was further spurred by reading Cramér's introductory text [4] on "calculus of probabilites," which explicitly stresses the analogy between probability distributions and mass distributions - both important cases of extensive quantities.

– I want to acknowledge several fruitful conversations with Michael Wright on these topics. The diagrams of the article were produced using Paul Taylor's "diagrams" package.

1 Combinators for strong endofunctors and monads

We consider a CCC $\mathscr{E}$; notions of "enrichment" or "strength" refer to this $\mathscr{E}$.

First, we have the evaluation map

$$ev_{X,Y} : X \times (X \pitchfork Y) \to Y$$

and its twin sister $\tilde{ev}_{X,Y} : (X \pitchfork Y) \times X \to Y$; they are the counits for the adjunction $(X \times -) \dashv (X \pitchfork -)$ (resp. $(- \times X) \dashv (X \pitchfork -)$). Often the decorations X,Y on $ev_{X,Y}$ may be omitted, because X and Y are clear from the context.

We consider an endofunctor $T : \mathscr{E} \to \mathscr{E}$, assumed strong (= $\mathscr{E}$-enriched); recall that such enrichment is presented in terms of data

$$st_{X,Y} : X \pitchfork Y \to T(X) \pitchfork T(Y),$$

cf. [7], or [3] II.6.2.3 item (2).

In [10] and [14], we observed that the strength can be encoded as a *tensorial* strength

$$t'_{X,Y} : T(X) \times Y \to T(X \times Y),$$

natural in X and Y. By "conjugating" with the twist map $X \times Y \to Y \times X$, one also gets its "twin sister"

$$t''_{X,Y} : X \times T(Y) \to T(X \times Y),$$

likewise encoding the strength.

Finally, the strength can be encoded as *cotensorial strength*

$$\lambda_{X,Y} : T(X \pitchfork Y) \to X \pitchfork T(Y),$$

cf. [10] and [12].

We give elementwise descriptions of these basic combinators. The elements of $X \pitchfork Y$ are maps $f : X \to Y$. The (Eilenberg-Kelly-) strength $st_{X,Y} : X \pitchfork Y \to T(X) \pitchfork T(Y)$ of T takes such f to $T(f) \in T(X) \pitchfork T(Y)$. The tensorial strength $t'_{X,Y} : T(X) \times Y \to T(X \times Y)$ takes (P,y) to $T(u_y)(x)$, where $u_y : X \to X \times Y$ takes x to (x,y); similarly $t''_{X,Y} : X \times T(Y) \to T(X \times Y)$ takes (x,Q) to $T(\tilde{u}_x)(Q)$ where $\tilde{u}_x : Y \to X \times Y$ takes y to (x,y). Finally, $\lambda_{X,Y} : T(X \pitchfork Y) \to X \pitchfork T(Y)$ takes $S \in T(X \pitchfork Y)$ to the map $x \mapsto T(ev_x)(S)$, where $ev_x : X \pitchfork Y \to Y$ is evaluation at $x \in X$.

The following will be a main actor in the following. Let $B \in \mathscr{E}$ and let $\beta : T(B) \to B$ be a map. (We are ultimately interested in the case where T is a strong monad, and β makes B into a T-algebra). Then for any $X \in \mathscr{E}$, we have the composite

$$T(X) \times (X \pitchfork B) \xrightarrow{t'} T(X \times (X \pitchfork B)) \xrightarrow{T(ev)} T(B) \xrightarrow{\beta} B. \tag{1}$$

Alternatively, by (6) below, this map equals

$$T(X) \times (X \pitchfork B) \xrightarrow{id \times st} T(X) \times (T(X) \pitchfork T(B)) \xrightarrow{ev} T(B) \xrightarrow{\beta} B. \tag{2}$$

In elementwise terms: If $P \in T(X)$ and $\phi \in X \pitchfork B$ (so $\phi : X \to B$ is a map), the value of (2) on (P,ϕ) is $\beta(T(\phi)(P)) \in B$, and is denoted $\int_{X,B} \phi(x)\, dP(x)$ (with 'x' as a dummy variable). Frequently, B and β may be understood from the context (and the most important case is when $B = T(1)$), in which case the definiton of $\int$ reads

$$\int_X \phi(x)\, dP(x) := \beta(T(\phi)(P)), \tag{3}$$

or, with increasing pedantry (rarely needed)

$$\int_X \phi(x)\, dP(x) = \int_{X,B} \phi(x)\, dP(x) = \int_{X,(B,\beta)} \phi(x)\, dP(x).$$

We are interested in the case where β provides B with structure of T-algebra. (In this case, when $\mathscr{E}$ is the category of sets, this "integration" relationship between monad/algebra theory, and algebraic theories, dates back to the early days of monad theory with Linton, Wraith and others, in the mid 1960s: they knew that the elements of $T(X)$ can be interpreted as X-ary operations $X \pitchfork B \to B$ on T-algebras B.) If in particular X is B itself, and we put $\phi = id_B$, we end up with

$$\int_{B,B} x\, dP(x) = \beta(P). \tag{4}$$

We are ultimately to read this as the *expectation* of P, see Section 8 below.

We collect some further definitions and basic relations concerning the combinators related to a strong endofunctor $T : \mathscr{E} \to \mathscr{E}$ on a symmetric monoidal closed category $\mathscr{E}$. We continue to use notation as if $\mathscr{E}$ were actually cartesian closed, i.e. we write $\times$ rather than $\otimes$.

We already considered the counit ev for the adjunction $(X \times -) \dashv (X \pitchfork -)$. The unit for this adjunction is not used so often, it is denoted u, with suitable decorations. Similarly, $\tilde{u}$ is the unit corresponding to the counit $\tilde{ev}$. Thus

$$u_{X,Y} : Y \to X \pitchfork (X \times Y) \quad \tilde{u}_{X,Y} : Y \to X \pitchfork (Y \times X).$$

The decorations are usually omitted from notation; even the tilde may often safely be omitted; one case where it is useful to retain the tilde is in the characterizing diagram for δ (canonical map to double dual; "δ" for "Dirac"); we have that the following diagram commutes

$$\begin{tikzcd} X \times (X \pitchfork B) \arrow[r, "\delta \times id"] \arrow[d, "ev"'] & ((X \pitchfork B) \pitchfork B) \times (X \pitchfork B) \arrow[d, "\tilde{ev}"] \\ B \arrow[r, "="'] & B. \end{tikzcd} \tag{5}$$

Next, a diagram relating the tensorial strength t' with the Eilenberg-Kelly strength (= $\mathscr{E}$-enrichment) st of T:

$$\begin{tikzcd} TA \times (A \pitchfork B) \arrow[r, "t'_{A,A\pitchfork B}"] \arrow[d, "TA \times st_{A,B}"'] & T(A \times (A \pitchfork B)) \arrow[d, "T(ev)"] \\ TA \times (TA \pitchfork TB) \arrow[r, "ev"'] & TB. \end{tikzcd} \tag{6}$$

The proof of this comes about, via manipulation by exponential adjointness, of the definition of $t'_{X,Y}$ (as given in [10] p.2 for t'') in terms of st, namely as the exponential adjoint of the composite

$$Y \xrightarrow{u_{X,Y}} X \pitchfork (X \times Y) \xrightarrow{st} TX \pitchfork T(X \times Y).$$

(put $Y := A \pitchfork B$ and $X := A$).

Next, a diagram relating the tensorial strength t' with the cotensorial strength λ:

$$\begin{tikzcd} T(Y \pitchfork Z) \times Y \arrow[r, "\lambda_{Y,Z} \times Y"] \arrow[d, "t'_{Y\pitchfork Z,Y}"'] & (Y \pitchfork TZ) \times Y \arrow[d, "\tilde{ev}"] \\ T((Y \pitchfork Z) \times Y) \arrow[r, "T(\tilde{ev})"'] & TZ. \end{tikzcd} \tag{7}$$

This also follows by exponential adjointness manipulations; a proof is given in [12] Lemma 1.2.

We shall also have occasion to use commutativity of the outer diagram in

$$\begin{tikzcd}
TX \times (X \pitchfork B) \arrow[r, "T\delta \times id"] \arrow[dd, "t'"'] & T((X \pitchfork B) \pitchfork B) \times (X \pitchfork B) \arrow[r, "\lambda \times id"] \arrow[d, "t'"] & ((X \pitchfork B) \pitchfork TB) \times (X \pitchfork B) \arrow[dd, "\tilde{ev}"] \\
& T((X \pitchfork B) \pitchfork B) \times (X \pitchfork B)) \arrow[dr, "T(\tilde{ev})"] & \\
T(X \times (X \pitchfork B)) \arrow[ur, "T(\delta \times id)"'] \arrow[rr, "T(ev)"'] & & TB;
\end{tikzcd} \tag{8}$$

here, the left hand square commutes by naturality of t', the right hand square by (7), and the triangle by applying T to (5).

We will take the tensorial strength t' (or equivalently t'') as the primary encoding. If $\mathscr{E}$ is the category of sets, $t'_{X,Y} : T(X) \times Y \to T(X \times Y)$ is the map which for $P \in T(X)$ and $y \in Y$ returns the value $T(u_y)(P) \in T(X \times Y)$, where $u_y : X \to X \times Y$ is the map $x \mapsto (x, y)$.

The combinator t' satisfies a unit law and an associative law. The unit law says that $t'_{X,1} : T(X) \times 1 \to T(X \times 1)$ is the composite of the two canonical isomorphisms $T(X) \times 1 \cong T(X) \cong T(X \times 1)$. The associative law says that the composite

$$T(X) \times Y \times Z \xrightarrow{t'_{X,Y} \times Z} T(X \times Y) \times Z \xrightarrow{t'_{X \times Y, Z}} T(X \times Y \times Z)$$

equals $t'_{X, Y \times Z}$. (For simplicity, we write as if $\times$ were strictly associative.) There are similar unit- and associative laws for t''. All these laws follow from the standard laws for the $\mathscr{E}$-enrichment, cf. [10].

We shall have occasion to use a "derived" combinator,

$$t_{X,Y,Z} : X \times T(Y) \times Z \to T(X \times Y \times Z); \tag{9}$$

it can be defined in several ways which are equivalent in view of the associative law for t' and the construction of t'' in terms of t'. One way to define it is to consider

$$t''_{X \times Z, Y} : X \times Z \times T(Y) \to T(X \times Z \times Y) \tag{10}$$

and conjugate it by interchange of Z and $T(Y)$, respectively, Z and Y. It can, by the associative law, also be defined as the composite

$$X \times T(Y) \times Z \xrightarrow{X \times t'_{Y,Z}} X \times T(Y \times Z) \xrightarrow{t''_{X, Y \times Z}} T(X \times Y \times Z). \tag{11}$$

A natural transformation $\tau : T \Rightarrow S$ between two strong functors is *strong* if all squares of the form

$$\begin{tikzcd} T(X)\times Y \arrow[r, "t'_{X,Y}"] \arrow[d, "\tau_X \times Y"'] & T(X\times Y) \arrow[d, "\tau_{X\times Y}"] \\ S(X)\times Y \arrow[r, "s'_{X,Y}"'] & S(X\times Y) \end{tikzcd} \tag{12}$$

commute, where t' and s' are the tensorial strengths of T and S, respectively[2].

Let (T,η,μ,t') be a strong monad, (so T is equipped with a strength t', and η and μ are strong natural transformations; recall that t' induces a natural strength on $T\circ T$; "strengths compose"; an explicit expression for the composite strength (in t''-terms) appears in the center line of (31) below).

It is easy to deduce from the strength of the natural transformation $\eta : id \Rightarrow T$ that the following diagram commutes:

$$\begin{tikzcd} X\times(X\pitchfork B) \arrow[rr, "ev"] \arrow[d, "\eta_X\times id"'] & & B \arrow[d, "\eta_B"] \\ T(X)\times(X\pitchfork B) \arrow[r, "t'"'] & T(X\times(X\pitchfork B)) \arrow[r, "T(ev)"'] & T(B). \end{tikzcd} \tag{13}$$

Together with a monad T on $\mathscr{E}$ comes the notion of (Eilenberg-Moore-) *T-algebra* (B,β) where $\beta : T(B)\to B$ satisfies a unit- and associative law. In particular $\beta\circ\eta_B = id_B$. So from the above, we deduce that if $B=(B,\beta)$ is a T-algebra, then precomposing $\int_X : T(X)\times(X\pitchfork B)\to B$ with $\eta_X\times id$ just yields the evaluation map (use the description (1)); let us record this:

$$\big[X\times(X\pitchfork B)\xrightarrow{\eta_X\times id} T(X)\times(X\pitchfork B)\xrightarrow{\int_X} B\big] = \big[X\times(X\pitchfork B)\xrightarrow{ev} B\big]. \tag{14}$$

The T-algebras form a category $\mathscr{E}^T$, whose maps are called *T-homomorphisms*; we shall also use the term *T-linear* maps, because this will allow us to talk about *T-bilinear* maps, a notion introduced in the strong-monad context in [22] and [13]. We shall recall and expand some of the theory from loc.cit. As long as the monad T is fixed, we may say "linear" instead of T-linear, and similarly for "bilinear".

The following result (Theorem 2.1 in [10]) is important for our present aims:

the functor part T of a strong monad carries two canonical structures as a monoidal functor; with respect to each of these, η is a monoidal transformation.

[2] There is another use of the word "strong" for a natural transformation, namely "all the naturality squares are pull-backs". This is not how we use the word here.

These two monoidal structures are in loc.cit. denoted ψ and $\tilde{\psi}$, respectively; one is just the "twisted" version of the other; $\psi_{X,Y}$ is the composite

$$T(X) \times T(Y) \xrightarrow{t'_{X,T(Y)}} T(X \times TY) \xrightarrow{T(t''_{X,Y})} T^2(X \times Y) \xrightarrow{\mu_{X \times Y}} T(X \times Y), \tag{15}$$

and $\tilde{\psi}_{X,Y}$ similarly is

$$T(X) \times T(Y) \xrightarrow{t''_{T(X),Y}} T(TX \times Y) \xrightarrow{T(t'_{X,Y})} T^2(X \times Y) \xrightarrow{\mu_{X \times Y}} T(X \times Y). \tag{16}$$

The "nullary" part of both the monoidal structures is $\eta_1 : 1 \to T(1)$, where 1 is the unit object of $\mathscr{E}$ (i.e. the terminal object, in the CCC case).

In distribution theory, if P an Q are distributions of compact support on spaces X and Y respectively, the distribution $\psi_{X,Y}(P,Q)$ on $X \times Y$ is called the *tensor product* of P and Q, cf. [25] III.1.

Recall from [10] that the strong monad T is called *commutative* if $\psi = \tilde{\psi}$. In Theorem 3.2 in [10] it is proved that if T is commutative in this sense, then μ is a monoidal transformation (and hence T a monoidal monad, since η is in any case a monoidal transformation).

(There is a converse result contained in Theorem 2.3 in [14]; it contains the assertion that the strength t' of T can be reconstructed from the structure of monoidal monad; however, in the present article, we prefer to have strength as a principle underlying everything – almost a part of the logic. For $\mathscr{E}$ = Sets, strength is canonically present in all functors and transformations.

We proceed to describe some of the relationships we need between the various combinators associated to strong monads.

Proposition 1 *Precomposing $\psi_{X,Y}$ with $\eta_X \times T(Y)$ yields $t''_{X,Y}$. Similarly, precomposing $\tilde{\psi}_{X,Y}$ with $T(X) \times \eta_Y$ yields $t'_{X,Y}$.*

Diagrammatically, the first assertion says that the outer diagram in the following diagram commutes; the inner square commutes by naturality, the left hand triangle commutes since η is strong, and the right hand triangle commutes by a monad law. So the total diagram is likewise commutative, and this proves the first assertion of the Proposition.

$$\begin{tikzcd}
 & X \times T(Y) \arrow[r, "t''_{X,Y}"] \arrow[dl, "\eta_X \times TY"'] \arrow[d, "\eta_{X \times TY}"] & T(X \times Y) \arrow[d, "\eta_{T(X \times Y)}"'] \arrow[dr, "id"] & \\
TX \times TY \arrow[r, "t'_{X,TY}"'] & T(X \times TY) \arrow[r, "T(t''_{X,Y})"'] & T^2(X \times Y) \arrow[r, "\mu_{X \times Y}"'] & T(X \times Y).
\end{tikzcd}$$

The proof and diagram for the second assertion are similar.

Let $A = (A,\alpha)$ and $C = (C,\gamma)$ be T-algebras. A map $f : A \times X \to C$ is called *1-linear* (or *linear* (T-linear) *in the first variable*), cf. [13], if the following pentagon commutes

$$\begin{tikzcd} T(A)\times X \arrow[r, "t'_{A,X}"] \arrow[d, "\alpha\times X"'] & T(A\times X) \arrow[r, "T(f)"] & T(C) \arrow[d, "\gamma"] \\ A\times X \arrow[rr, "f"'] & & C. \end{tikzcd} \tag{17}$$

Similarly, a map $X \times A \to C$ is called *2-linear* (or *linear in the second variable*) if a similar diagram, now using $t''_{X,A} : X \times T(A) \to T(X \times A)$ commutes. Finally, if further $B = (B,\beta)$ is a T-algebra, a map $A \times B \to C$ is called *bilinear* if it is both 1-linear and 2-linear.

(One may define the notion of n-linear map $A_1 \times \ldots A_n \to C$ (where the A_is and C are (underlying objects of) algebras), and in this way, one should get a multicategory; however, to substantiate this, there are some coherence conditions that need to be worked out.)

Recall that an object of the form $T(Z)$ carries a canonical algebra structure, namely with structure map $\mu_Z : T^2(Z) \to T(Z)$ (this is the *free* T-algebra on Z). The algebras in the following Proposition are free.

Proposition 2 *For any X and Y in $\mathscr{E}$, the map $t'_{X,Y} : T(X) \times Y \to T(X \times Y)$ is 1-linear. Similarly $t''_{X,Y} : X \times T(Y) \to T(X \times Y)$ is 2-linear.*

Proof. The pentagon (17) above, with $A = T(X)$ and $X = Y$ has as top line the map $T(t'_{X,Y}) \circ t'_{T(X),Y}$, and this is an instance of the tensorial strength for the composite functor $T \circ T$; and then the commutativity of the pentagon is seen to be an instance of the assumption that μ is a strong natural transformation. The proof of the second assertion follows by suitable conjugation by twist maps.

A consequence (cf. [13]) is that $\psi_{X,Y} : T(X) \times T(Y) \to T(X \times Y)$ is 1-linear, and that $\tilde{\psi}_{X,Y}$ is 2-linear. If all instances of ψ are 2-linear (or equivalently, bilinear), then the monad is commutative, and vice versa, cf. loc.cit. Proposition 1.5.

Recall that if $C = (C,\gamma)$ is a T-algebra, any map $X \to C$ extends uniquely over $\eta_X : X \to T(X)$ to a linear map $T(X) \to C$; this is the "free" property of $T(X)$. We have a closely related "universal" property of $T(X) \times Y$:

Proposition 3 *Any map $f : X \times Y \to C$ extends uniquely over $\eta_X \times Y : X \times Y \to T(X) \times Y$ to a 1-linear map : $T(X) \times Y \to C$.*

For, there are natural bijective correspondences

$$\hom(X \times Y, C) \cong \hom(X, Y \pitchfork C) \cong \hom_T(T(X), Y \pitchfork C)$$

(where the second occurrence of $Y \pitchfork C$ is the cotensor $Y \pitchfork C$ in $\mathscr{E}^T$, recalled in (20) below, and where the second bijection is induced by precomposition by η_X); finally, the set

$\hom_T(T(X), Y \pitchfork C)$ is in bijective correspondence with the set of 1-linear maps $T(X) \times Y \to C$, by [13] Proposition 1.3 (i).

It is useful to be have an explicit formula for the 1-linear extension of $f : X \times Y \to C$; it is the composite

$$T(X) \times Y \xrightarrow{t'_{X,Y}} T(X \times Y) \xrightarrow{T(f)} T(C) \xrightarrow{\quad\gamma\quad} C. \tag{18}$$

For, $t'_{X,Y}$ is 1-linear, by Proposition 2, and the two other maps in (18) are linear, so the composite is 1-linear. Also, it is easy to see that the restriction of (18) along $\eta_X \times Y$ gives f back (use the unit law $t'_{X,Y} \circ (\eta_X \times Y) = \eta_{X \times Y}$, and also $\gamma \circ \eta_C = id_C$). So (18) satisfies the two conditions in Proposition 3.

Proposition 4 *The map $\psi_{X,Y} : T(X) \times T(Y) \to T(X \times Y)$ is characterized by the following two properties: it is 1-linear, and its precomposition with $\eta_X \times TY$ is $t''_{X,Y}$. Similarly $\tilde{\psi}$: $T(X) \times T(Y) \to T(X \times Y)$ is characterized by the two properties: it is 2-linear, and its precomposition with $TX \times \eta_Y$ is $t'_{X,Y}$.*

Proof. We prove the first assertion. We already observed that $\psi_{X,Y}$ satisfies these two conditions, cf. Propositions 1 and the quotation from [13] after Proposition 2. The converse follows from Proposition 3.

Assume now that $B = (B, \beta)$ is an algebra, and consider a map $f : X \times B \to C$ which is 2-linear. It extends, by the above, to a 1-linear map $\overline{f} : T(X) \times B \to C$, and we may ask whether this $\overline{f}$ inherits from f the property of being 2-linear (and is thus bilinear). A sufficient condition is commutativity of T:

Proposition 5 *Let T be commutative. Let $B = (B, \beta)$ and $C = (C, \gamma)$ be T-algebras, and assume that $f : X \times B \to C$ is 2-linear. Then its 1-linear extension $\overline{f} : T(X) \times B \to C$ is bilinear.*

Proof. We use the formula (18) with $Y = B$ for the extension. It remains to prove 2-linearity of this map, i.e. to prove commutativity of the following diagram (where the bottom line is $\overline{f}$, according to (18))

$$\begin{tikzcd}
TX \times TB \arrow[r, "t''"] \arrow[d, "TX \times \beta"'] & T(TX \times B) \arrow[r, "T(t')"] & T^2(X \times B) \arrow[r, "T^2 f"] & T^2C \arrow[r, "T\gamma"] & TC \arrow[d, "\gamma"] \\
TX \times B \arrow[rr, "t'"'] & & T(X \times B) \arrow[r, "Tf"'] & TC \arrow[r, "\gamma"'] & C.
\end{tikzcd}$$

Consider the composite $\gamma \circ T\gamma \circ T^2 f$ of the last three arrows in the clockwise composite. By pure monad- and algebra theory, we have

$$\gamma \circ T\gamma \circ T^2 f = \gamma \circ Tf \circ \mu, \tag{19}$$

and having performed this replacement, the definition of $\tilde{\psi}$ appears at the beginning of the clockwise composite. Since T was assumed commutative, we may replace $\tilde{\psi}$ by ψ, and after this replacement, the clockwise composite comes out as the composite

$$TX \times TB \xrightarrow{t'} T(X \times TB) \xrightarrow{T(t'')} T^2(X \times B) \xrightarrow{\mu} T(X \times B) \xrightarrow{Tf} TC \xrightarrow{\gamma} C.$$

Now we can use (19) once more, in the opposite direction, and we end up with the composite

$$TX \times TB \xrightarrow{t'} T(X \times TB) \xrightarrow{T(t'')} T^2(X \times B) \xrightarrow{T^2 f} T^2 C \xrightarrow{T\gamma} TC \xrightarrow{\gamma} C.$$

After these manipulations with the clockwise composite, the diagram to be proved commutative has the following shape

$$\begin{tikzcd}
TX \times TB \arrow[r, "t'"] \arrow[d, "TX \times \beta"'] & T(X \times TB) \arrow[r, "T(t'')"] \arrow[d, "T(X \times \beta)"] & T^2(X \times B) \arrow[r, "T^2 f"] \arrow[d, phantom, "*"] & T^2 C \arrow[d, "T\gamma"] & \\
TX \times B \arrow[r, "t'"'] & T(X \times B) \arrow[rr, "Tf"'] & & TC \arrow[r, "\gamma"'] & C
\end{tikzcd}$$

Here the pentagon * commutes: it comes about by applying the functor T to the diagram expressing the assumption that $f : X \times B \to C$ is 2-linear. This proves the desired 2-linearity of $\overline{f}$.

For a commutative T, we get as an immediate corollary that a map $f : X \times Y \to C$ (where $C = (C, \gamma)$ is a T-algebra) extends uniquely to a bilinear $T(X) \times T(Y) \to C$. Since also f extends uniquely to a linear $T(X \times Y) \to C$, we may conclude that $T(X \times Y)$ may serve as $T(X) \otimes T(Y)$ in $\mathscr{E}^T$, with $\psi_{X,Y}$ as the universal bilinear map; but we shall not prove or need existence of such tensor products for general T-algebras.

The following is hardly surprising, and the routine proof is omitted:

Proposition 6 *If τ is a strong natural transformation from one strong monad T to another one, S, compatible with the monad structures, then τ will also be compatible with the monoidal strucures ψ^T, ψ^S, i.e. it will be a monoidal transformation. Similarly, τ will be compatible with the monoidal structures $\tilde{\psi}^T$, $\tilde{\psi}^S$.*

2 Monads and double dualization

Given a commutative monad T on $\mathscr{E}$. Some of the formal properties of the construction $\int_X \phi(x)\, dP(x)$ is best stated in terms of a transformation τ from T to a certain "double dualization" monad associated to T. In essence, τ_X will be exponential adjoint of $\int_X$: $T(X) \times (X \pitchfork B) \to B$ (where B is a T-algebra).

We assume that $\mathscr{E}$ has equalizers (or sufficiently many – only a few are needed; we study this question in more detail in Section 9). In this case, the category $\mathscr{E}^T$ of algebras for a strong monad $T = (T, \eta, \mu, t')$ becomes enriched over $\mathscr{E}$: if (A, α) and (B, β) are algebras, the $\mathscr{E}$-valued hom-object $[(A, \alpha), (B; \beta)]_T$ is carved out of $A \pitchfork B$ by an evident equalizer diagram involving α and β, expressing the T-homomorphism condition diagrammatically. (This goes back to [2].) We write $(A, \alpha) \pitchfork_T (B, \beta)$ for this hom object, and often omit α and β from notation; they are then to be understood from the context. Note that $(A, \alpha) \pitchfork_T (B, \beta)$ is a subobject of $A \pitchfork B$. In short notation, $A \pitchfork_T B \subseteq A \pitchfork B$.

Also, $\mathscr{E}^T$ is cotensored over $\mathscr{E}$: if $X \in \mathscr{E}$ and $(B, \beta) \in \mathscr{E}^T$, the cotensor $X \pitchfork (B, \beta)$ is the object $X \pitchfork B$ in $\mathscr{E}$, equipped with the T-structure

$$T(X \pitchfork B) \xrightarrow{\lambda_{X,B}} X \pitchfork T(B) \xrightarrow{X \pitchfork \beta} X \pitchfork B, \tag{20}$$

using the cotensorial strength λ of T.

In [12], we proved that if T is a commutative monad, then $(A, \alpha) \pitchfork_T (B, \beta)$, as a subobject of $A \pitchfork B$, is actually a sub-T*algebra* (with the algebra structure of $A \pitchfork B$ given by the recipe above, with $X = A$). This in fact makes $\mathscr{E}^T$ into a closed category in its own right, cf. Theorem 2.2 in [12]. (It is even a *symmetric* closed category, in the sense suggested in loc.cit. ; this was substantiated in [5], [6].)

The notion of cotensor and $\mathscr{E}$-valued hom are related by an ($\mathscr{E}$-strong) adjointness, as is well known, cf. [9] (3.42). This implies that for $B = (B, \beta)$ in $\mathscr{E}^T$, we have contravariant functors

$$- \pitchfork (B, \beta) : \mathscr{E} \to \mathscr{E}^T$$

and

$$- \pitchfork_T (B, \beta) : \mathscr{E}^T \to \mathscr{E}$$

which are strongly adjoint to each other on the right, so that we get a strong monad on $\mathscr{E}$, with functor part $X \mapsto (X \pitchfork (B, \beta)) \pitchfork_T (B, \beta)$, or with slight abuse of notation

$$X \mapsto (X \pitchfork B) \pitchfork_T B, \tag{21}$$

a "restricted double dualization" functor (terminology from [16]). These double dualization monads are rarely commutative (even for commutative T); and their categories of algebras are often hard to analyze.

In case where $\mathscr{E}$ is the category of sets, and $T(X)$ is the monad whose algebras are boolean algebras, $(X \pitchfork 2) \pitchfork_T 2$ is the set of ultrafilters on X, and the category of algebras for $(- \pitchfork 2) \pitchfork_T 2$ is the category of compact Hausdorff spaces (Manes' Theorem, cf. [8] III.2.4).

If T is the identity functor, and $D \in \mathscr{E}$ is any object, we have the "plain" double-dualization monad $(- \pitchfork D) \pitchfork D$, studied in detail in [11]. It is the "full algebraic theory of D", if we identify monads on the category of sets with infinitary Lawvere theories (as is done in [23], or [26]).

It is easy to see that B itself is an algebra for the (unrestricted) double dualization monad $(- \pitchfork B) \pitchfork B$; the structure is the map $ev_{id} : (B \pitchfork B) \pitchfork B \to B$ which is "evaluation at the identity map $id_B \in B \pitchfork B$". In particular

$$ev_{id} \circ \delta_B = id_B. \tag{22}$$

Another significant example: if R is a commutative ring object in $\mathscr{E}$, there is (under suitable completeness conditions on $\mathscr{E}$) a monad T whose category of algebras are the R-module objects. So R itself is a T-algebra (in fact, $R = T(1)$), and we have the restricted double dualization monad $(- \pitchfork R) \pitchfork_T R$. In some examples $(X \pitchfork R) \pitchfork_T R$ can be analyzed as an internal version of *distributions with compact support* on X (distributions in the sense of Schwartz); see [24] Prop. II.3.6 for some toposes of C^∞ spaces. An algebraic analysis is given in [16] for the case where R is the generic commutative ring.

We return to the general case of a restricted double dualization monad $X \mapsto (X \pitchfork B) \pitchfork_T B$, where T is a strong monad on $\mathscr{E}$ and $B = (B, \beta)$ a T-algebra. The unit for this monad is denoted δ, so

$$\delta_X : X \to (X \pitchfork B) \pitchfork_T B.$$

Post-composing with the inclusion $(X \pitchfork B) \pitchfork_T B \subseteq (X \pitchfork B) \pitchfork B$ gives the combinator δ_X considered in Section 1. If $\mathscr{E}$ is the category of sets, it is the map which takes $x \in X$ to the T-algebra map $\delta_X(x) : X \pitchfork B \to B$, "evaluating at $x \in X$". This "evaluation at x" is a T-homomorphism, thus an element in $(X \pitchfork B) \pitchfork_T B$. In distribution terms, it is the Dirac distribution on X at x, whence the notation δ. The $\overline{\mu}$ of the monad $(- \pitchfork B) \pitchfork_T B$ can also ultimately be described in terms of δ. We describe it when, and to the extent we need it, in the proof of Theorem 1 below.

These double-dualization monads depend on the choice of the object (T-algebra) B. The most important case for us is where B is $(T(1), \mu_1)$ (later on, we shall denote this particular T-algebra by the letter R; it plays the role of a number line). Recall that for any $X \in \mathscr{E}$, the algebra $(T(X), \mu_X)$ is the *free* T-algebra on X. In particular, $T(1)$ is the free algebra in one generator.

The construction of the restricted double dualization monad (21) does not depend on commutativity of the given monad T, however, the following does. So let T be a *commutative* monad on $\mathscr{E}$, and let $B = (B, \beta)$ be a T-algebra. Then by [12], $A \pitchfork_T B$ carries structure of a T-algebra whenever $A = (A, \alpha)$, $B = (B, \beta)$ are T-algebras. Therefore, the map $\delta : X \to (X \pitchfork B) \pitchfork_T B$ extends uniquely to a T-homomorphism on the free T-algebra $T(X)$, so that we have a canonical T-homomorphism

$$\tau_X : T(X) \longrightarrow (X \pitchfork B) \pitchfork_T B.$$

Its relationship to $\int_X$ is made explicit in Proposition 7 below.

Theorem 1 *Let T be a commutative monad. Then the maps τ_X form a strongly natural morphism $\tau : T \Rightarrow (- \pitchfork B) \pitchfork_T B$; it is a morphism of monads, and it is compatible with the canonical monoidal structures on the functors in question.*

(This holds, whether one takes ψ or $\tilde{\psi}$ as the monoidal structure on the double dualization monad; and for T, there is anyway only one canonical monoidal struture, since T is assumed commutative.)

Proof. By construction, $\tau \circ \delta = \eta$, so τ is compatible with the units of the two monads in question. Let us prove compatibility with the μ , $\overline{\mu}$ (the latter being the multiplication of the double dualization monad in question). The unit and counit of the adjunction that gave rise to the monad $(- \pitchfork B) \pitchfork_T B$ are $\delta_X : X \to (X \pitchfork B) \pitchfork_T B$ in $\mathscr{E}$ (already considered), and $\varepsilon_A : A \to (A \pitchfork_T B) \pitchfork B$ in $\mathscr{E}^T$, given by essentially the same recipe which gave δ (the counit goes the "wrong way" because of the contravariant nature of the two adjoint functors in question). Since the multiplication of a monad arising from an adjoint pair is an instance of the counit of it, we conclude that $\overline{\mu}$ does indeed live in $\mathscr{E}^T$. Therefore, the two maps to be compared to prove that τ is compatible with $\mu, \overline{\mu}$ are two maps $T^2(X) \to (X \pitchfork B) \pitchfork_T B$ both of which are T-linear. By Proposition 3, it therefore it suffices to see that they agree when precomposed with η_{TX}. Here is the relevant diagram:

$$\begin{tikzcd}
TX \arrow[r, "\eta_{TX}"] & T(TX) \arrow[r, "T(\tau)"] \arrow[d, "\mu_X"'] & T((X \pitchfork B) \pitchfork_T B) \arrow[r, "\tau"] & ((((X \pitchfork B) \pitchfork_T B) \pitchfork B) \pitchfork_T B) \arrow[d, "\overline{\mu}_X"] \\
& TX \arrow[rr, "\tau"'] & & (X \pitchfork B) \pitchfork_T B
\end{tikzcd}.$$

The counterclockwise composite gives τ_X, by the unit law for the monad T. The top composite may be rewritten, using naturality of η, into the composite

$$TX \xrightarrow{\tau_X} (X \pitchfork B) \pitchfork_T B \xrightarrow{\eta} T((X \pitchfork B) \pitchfork_T B) \xrightarrow{\tau} ((((X \pitchfork B) \pitchfork_T B) \pitchfork B) \pitchfork_T B);$$

but $\tau \circ \eta$ is δ, by construction of τ, and δ composed with $\overline{\mu}$ is an identity map (one of the monad laws for the double dualization monad here). So the clockwise composite likewise ends up as τ_X. So τ is indeed a morphism of monads. Since everything is compatible with the strengths, we conclude from Proposition 6 that τ also preserves the monoidal structure.

Remark. This theorem is analogous to Theorem 3.2 in [11]; there, however, one considers the *full* double dualization monad $(- \pitchfork B) \pitchfork B$ for an *unstructured* object B.

The transformation τ in the Theorem is in fact an exponential adjoint version of the "integral" studied in Section 1:

Proposition 7 *The map $\tau_X : T(X) \to (X \pitchfork B) \pitchfork B$ has for its exponential adjoint the map $\int_{X,B} : T(X) \times (X \pitchfork B) \to B$.*

Proof. Since τ is the T-linear extension of $\delta : X \to (X \pitchfork B) \pitchfork B$, τ may be described explicitly as the composite

$$TX \xrightarrow{T(\delta)} T((X \pitchfork B) \pitchfork B) \xrightarrow{\lambda} (X \pitchfork B) \pitchfork TB \xrightarrow{id \pitchfork \beta} (X \pitchfork B) \pitchfork B. \tag{23}$$

Thus, the exponential adjoint of τ appears as the clockwise composite in (8) composed with β. On the other hand, if one follows the counterclockwise composite in (8) by β, we see from (1) that we have the map $\int_{X,B}$-

Let (B,β) be a T-algebra. By the Theorem, we have for each $X \in \mathscr{E}$ a map $\tau_X : T(X) \to (X \pitchfork B) \pitchfork_T B$, defined in terms of β, and with good properties, in particular, it is T-linear. For $X = B$, we have in particular $\tau_B : T(B) \to (B \pitchfork B) \pitchfork_T B$. We have also a map $ev_{id} : (B \pitchfork B) \pitchfork_T B \to B$ "evaluation at id_B", and thus get by composition a map $T(B) \to B$.

Proposition 8 *The composite*

$$T(B) \xrightarrow{\tau_B} (B \pitchfork B) \pitchfork_T B \xrightarrow{ev_{id}} B$$

equals $\beta : T(B) \to B$.

Proof. Both maps to be compared are T-linear, so it suffices to see that they agree when precomposed with η_B. We have $\beta \circ \eta_B = id_B$, by the unit law for T-algebras. On the other hand,

$$ev_{id} \circ \tau_B \circ \eta_B = ev_{id} \circ \delta_B$$

by construction of τ, and $ev_{id} \circ \delta_B = id_B$, as we observed above (22) for unrestricted double dualization into B; it holds then, by restriction, also for the restricted double dualization monad.

Theorem 1 allows us to describe $\tau_{X\times Y}(\psi(P,Q))$ in terms of $\tau_X(P)$ and $\tau_Y(Q)$, and similarly for $\tilde{\psi}$; note the formal similarity with Fubini's Theorem.

Theorem 2 *Let $P \in T(X)$ and $Q \in T(Y)$, and let $\phi \in (X \times Y) \pitchfork B$. Then $\tau_{X\times Y}(\psi(P,Q))(\phi)$ and $\tau_{X\times Y}(\tilde{\psi}(P,Q))(\phi)$ appear as the left and right hand side, respectively, of the following equation (which therefore* holds *for all P,Q,ϕ when T is assumed to be a commutative monad)*

$$\int_X \left(\int_Y \phi(x,y)\ dQ(y) \right) dP(x) = \int_Y \left(\int_X \phi(x,y)\ dP(x) \right) dQ(y). \tag{24}$$

Proof. We first argue that $\tau_{X\times Y}(\psi_{X,Y}(P,Q))(\phi) \in B$ is given by the expression on the left hand side. We denote the combinators for the strong monad $S = (- \pitchfork B) \pitchfork_T B$ by $\overline{t}'$, $\overline{\psi}$, etc. Then by Theorem 1

$$\tau_{X\times Y}(\psi(P,Q)) = \overline{\psi}_{X,Y}(\tau_X(P), \tau_Y(Q)).$$

Therefore, it is a matter of analyzing $\overline{\psi}_{X,Y}$ for the monad S, and this is pure λ-calculus; in fact, S can easily be proved to be a submonad of the full double dualization monad $D = (- \pitchfork B) \pitchfork B$. We claim that the monoidal structure $\overline{\psi}$ for this monad is given, in elementwise terms, as follows, for $\overline{P} \in (X \pitchfork B) \pitchfork B$, $\overline{Q} \in (Y \pitchfork B) \pitchfork B)$ and $\phi \in (X \times Y) \pitchfork B$:

$$\overline{\psi}(\overline{P},\overline{Q})(\phi) = \overline{P}[x \mapsto \overline{Q}[y \mapsto \phi(x,y)]].$$

This is an elementwise reformulation of the following (writing ψ for $\overline{\psi}$ etc.:

Proposition 9 *The monoidal structure $\psi_{X,Y} : D(X) \times D(Y) \to D(X \times Y)$ on the functor $D = (- \pitchfork B) \pitchfork B$ may be described as follows: for $P \in D(X)$ and $Q \in D(Y)$, the value of $\psi_{X,Y}(P,Q)$ on $\phi \in (X \times Y) \pitchfork B$ is given as the composite*

$$(X \times Y) \pitchfork B \xrightarrow{\cong} X \pitchfork (Y \pitchfork B) \xrightarrow{X \pitchfork Q} X \pitchfork B \xrightarrow{P} B. \tag{25}$$

Similarly, the value of $\tilde{\psi}(P,Q)$ on ϕ is given as the composite

$$(X\times Y)\pitchfork B \xrightarrow{\cong} Y\pitchfork(X\pitchfork B) \xrightarrow{X\pitchfork P} Y\pitchfork B \xrightarrow{Q} B.$$

Proof. One may prove this by brute force, by λ-calculus, but since $D(D(X\times Y))$ is involved, this means that a four times dualization into B is involved, and this is not easy to handle; some ML type program on a computer would be useful here! However, we can use the fact that ψ is characterized by being linear in the first variable, and to restrict along the unit (here: δ) to t'', cf. Proposition 4. So we shall prove that (25), as a function of P,Q, satisfies these two criteria. We shall be content with arguing elementwise (synthetically). So consider $P\in D(X)$ (so $P: X\pitchfork B\to B$) and $Q\in D(Y)$ (so $Q: Y\pitchfork B\to B$). Then (25) returns with P, Q as input the composite described. We must argue that it, for fixed Q, depends linearly on P; recall that "linear" presently means "D-linear", i.e. "homomorphisms of D-algebras". The function of P given by (25) is the map

$$(X\pitchfork B)\pitchfork B \xrightarrow{s\pitchfork B} ((X\times Y)\pitchfork B)\pitchfork B,$$

where s is the map

$$(X\times Y)\pitchfork B \xrightarrow{\cong} X\pitchfork(Y\pitchfork B) \xrightarrow{X\pitchfork Q} X\pitchfork B.$$

Now any object of form $U\pitchfork B$ is canonically a D-algebra, and any morphism $V\pitchfork B\to U\pitchfork B$ of the form $s\pitchfork B$ (for $s: U\to V$) is a D-algebra homomorphism, since $-\pitchfork B:\mathscr{E}\to(\mathscr{E}^D)^{op}$ is the left adjoint of the two adjoint functors that together produced the monad D.

To prove the other condition, "precomposing with η", consider what happens if one puts $P=\delta_X(x)=$ evaluation at x, where $x\in X$ (recall that η now is Dirac delta formation). Then P gets replaced by ev_x, so the value of (25) is

$$(X\times Y)\pitchfork B \xrightarrow{\cong} X\pitchfork(Y\pitchfork B) \xrightarrow{X\pitchfork Q} X\pitchfork B \xrightarrow{ev_x} B.$$

But $ev_x\circ(X\pitchfork Q)=Q\circ ev_x$, and precomposing ev_x with the isomorphism $(X\times Y)\pitchfork B\cong X\pitchfork(Y\pitchfork B)$ yields $Q\circ(\tilde{u}_x\pitchfork B)$, and thus we arrive at (the value of $t''_{X,Y}$ at (x,Q)), as given at the beginning of Section 1 (replacing T by $(-\pitchfork B)\pitchfork B$).

3 Monads and actions by monoids

Exploiting the fact that the functor T carries two monoidal structures, we get in particular that $T(1)$ carries two natural monoid structures, namely, first,

$$T(1)\times T(1) \xrightarrow{\psi_{1,1}} T(1\times 1)\cong T(1) \tag{26}$$

and, secondly, the one obtained by replacing ψ with $\tilde{\psi}$. They of course agree when T is commutative. The monoid multiplication m (26) may be described equivalently as the composite

$$T(1)\times T(1) \xrightarrow{t'_{1,T(1)}} T(1\times T(1))\cong T^2(1) \xrightarrow{\mu_1} T(1). \tag{27}$$

This follows by recalling the construction of ψ in terms of t', t'' and μ, and noting that $t''_{1,1}$ may be eliminated, since it equals the composite of two "trivial" isomorphisms $1 \times T(1) \cong T(1) \cong T(1 \times 1)$, cf. [10] Lemma 1.8 (in fact, in the cartesian closed case, one has more generally commutativity of

$$\begin{tikzcd} X \times TY \arrow[r, "t''_{X,Y}"] \arrow[d, "pr"'] & T(X \times Y) \arrow[d, "T(pr)"] \\ TY \arrow[r, "="'] & TY \end{tikzcd}$$

where pr denotes the projection). From either description (26) or (27) follows that the multiplication of $T(1)$ is 1-linear. (It is not necessarily 2-linear, even when $T(1)$ happens to be commutative. However, if T is commutative, the multiplication is bilinear.)

The unit e of the monoid $T(1)$ is $\eta_1 : 1 \to T(1)$, also sometimes denoted 1.

Any object of the form $T(X)$ carries a left action by $T(1)$, and also a right action by $T(1)$, the latter (which will be our main concern) given by

$$T(X) \times T(1) \xrightarrow{\psi_{X,1}} T(X \times 1) \cong T(X).$$

This action is unitary and associative, using the monoid structure on $T(1)$ given by ψ; if one prefers, one can replace simultaneously $\psi_{X,1}$ and $\psi_{1,1}$ by the corresponding $\tilde{\psi}$s. (For the left action by $T(1)$, one uses either ψ for both the action and the monoid structure, or uses $\tilde{\psi}$ for both the action and the monoid structure.) We stick to right action, defined by ψ, as in the displayed formula. It is immediate to see that if $f : X \to Y$ is any map, then the map $T(f) : T(X) \to T(Y)$ is equivariant for the action.

The action of the monoid $T(1)$ on $T(X)$ may be discussed (for some of its aspects) in more generality as follows: Let $T = (T, \eta, \mu, t')$ be a strong monad on $\mathscr{E}$, and let $R = (R, e, m)$ be a monoid in $\mathscr{E}$ (with $e : 1 \to R$ the unit and $m : R \times R \to R$ the multiplication). There is an evident notion of a T-linear right action of R on T, namely a family of unitary and associative actions (natural in $X \in \mathscr{E}$) $\vdash_X : T(X) \times R \to T(X)$, with $\vdash_X$ 1-linear.

A 1-linear action by a monoid R on the monad T is by Proposition 3 determined by its restriction (for each X) along $\eta_X \times R$, i.e. by maps $\rho_X : X \times R \to T(X)$, natural in X. So the unit and associativity constraints for the action can be encoded in terms of ρ. We have

Proposition 10 *The 1-linear extension of a map $\rho : X \times R \to T(X)$ satsfies the unit constraint iff*

$$\left(X \cong X \times 1 \xrightarrow{X \times e} X \times R \xrightarrow{\rho} T(X) \right) = \eta_X,$$

and it satisfies the associativity constraint iff the following diagram commutes:

$$\begin{tikzcd} X \times R \times R \arrow[r, "\rho \times R"] \arrow[d, "X \times m"'] & T(X) \times R \arrow[r, "t'_{X,R}"] & T(X \times R) \arrow[r, "T(\rho)"] & T^2(X) \arrow[d, "\mu_X"] \\ X \times R \arrow[rrr, "\rho"'] & & & T(X). \end{tikzcd} \tag{28}$$

Proof. We leave the proof of the first assertion to the reader. Assume now that (28) commutes. To prove that the action is associative means proving equality of two maps $T(X) \times R \times R \to T(X)$, both of which are 1-linear. So it suffices to prove that these two maps agree when precomposed with $\eta_X \times R \times R$. The resulting diagram is then seen to be (28); note that the three last arrows in the clockwise composite in (28) is just the action, by the explicit formula (18) for how a map $X \times R \to T(X)$ extends to a 1-linear map $T(X) \times R \to T(X)$. – We leave to the reader the proof that associativity of the action implies commutativity of (28) (and we shall not need this implication).

We return to the special case of the right action by the monoid $T(1)$ on $T(X)$. We denote this action simply by a dot, $P \cdot \lambda$, for $P \in T(X)$ and $\lambda \in T(1)$. We think of $T(1)$ as "scalars".

We ask the question whether not only *free* T-algebras, but general T-algebras carry an action by $T(1)$. For this, we need commutativity of T; we have the following (which is not used in the sequel).

Proposition 11 *Let T be a commutative monad, and let (A, α) be a T-algebra. Then A carries a unique action by the monoid $T(1)$, in such a way that $\alpha : T(A) \to A$ is equivariant. The action is unitary and associative, and any homomorphism of T-algebras is equivariant.*

Proof. By general monad theory, we have that $\alpha : T(A) \to A$ sits in a canonical absolute coequalizer diagram in $\mathscr{E}$

$$T^2(A) \rightrightarrows T(A) \to A,$$

where the two parallel maps are $T(\alpha)$ and μ_A respectively. The map $T(\alpha)$ is equivariant for the action, without any assumptions on T. We shall use commutativity of T to prove equivariance of μ_A. When this is established, it is clear that the action descends along α from $T(A)$ to A, and the rest is then easy. Equivariance of μ_A means that the right hand region in the following diagram commutes:

$$\begin{tikzcd}
TA \times 1 \arrow[r, "\eta_{TA} \times \eta_1"] \arrow[d, "pr"', "\cong"] \arrow[dr, "\eta_{TA\times 1}"'] & T^2A \times T1 \arrow[r, "\mu_A \times T1"] \arrow[d, "\psi_{TA,1}"] & TA \times T1 \arrow[d, "\psi_{A,1}"] \\
TA \arrow[dr, "\eta_{TA}"'] & T(TA \times 1) \arrow[d, "\cong"', "T(pr)"] & T(A \times 1) \arrow[d, "\cong"', "T(pr)"] \\
 & T^2A \arrow[r, "\mu_A"'] & TA
\end{tikzcd}$$

Here *pr* denotes the relevant projections. The triangle commutes because η is a monoidal transformation. The (slanted) square commutes by naturality. So the counterclockwise composite equals $pr : TA \times 1 \to TA$. The top line composite is just $TA \times \eta_1$. The clockwise composite of the total diagram is $T(pr) \circ \psi_{A,1} \circ (TA \times \eta_1)$; this, however, is again just $pr : TA \times 1 \to TA$, by a general law for the relationship between η, ψ and the unit isomorphisms (here the *pr*), cf. [10], diagram (2.3). So we conclude that the total diagram commutes. Now, the two composites in the right hand region are both T-bilinear, because the ψs are T-bilinear by commutativity of T. So to prove commutativity of the right hand region, it suffices to prove that it commutes after precomposotion with $\eta \times \eta$, which is what the commutativity of the total diagram expresses. This proves the Proposition.

It is easy to see that if the monad T is $M \times -$ for a non-commutative monoid M in the category of sets, then μ_A will not be equivariant; so for Proposition 11, one cannot dispense with the commutativity assumption for the monad T.

Even though the projection $pr : A \times 1 \xrightarrow{\cong} A$ appears in the above construction and argument, all the constructions and arguments work in general symmetric monoidal closed categories, using the unit object I instead of 1, and using the unit isomorphisms (part of the data of a monoidal closed category) $A \otimes I \cong A$ instead of *pr*. The construction in the following Sections, however, depend in a crucial way of the assumption that our category is *cartesian* closed.

4 Action of functions on distributions

We consider a strong (not necessarily commutative) monad T on $\mathscr{E}$. But from now, we assume not only that $\mathscr{E}$ is symmetric monoidal closed, but that it is *cartesian* closed (as the notation in the previous sections anticipated). Then the object 1 is terminal, and we have the notion of the total; for $P \in T(X)$, its *total* is $T(!)(P) \in T(1)$, where $! : X \to 1$ is the unique such map.

Recall that $T(1)$ carries a canonical monoid structure, m,e, with m defined in terms of $\psi_{1,1}$, cf. (26).

Proposition 12 *Let $P \in T(X)$ and $Q \in T(Y)$. Then the total of $\psi_{X,Y}(P,Q)$ is the product by m of the totals of P and Q.*

This is an immediate consequence of the definition of m together with naturality of ψ with respect to the maps $! : X \to 1$ and $! : Y \to 1$.

The space $X \pitchfork T(1)$ inherits a monoid structure m_X, e_X in a standard ("pointwise") way. We shall equip any free T-algebra $T(X)$ with a (right) T-linear action $\vdash$ by the monoid $X \pitchfork T(1)$. The construction does not depend on commutativity of the monad T. We shall construct a map

$$T(X) \times (X \pitchfork T(1)) \xrightarrow{\vdash} T(X) \tag{29}$$

It is defined as the unique 1-linear extension over $\eta_X \times T(1)$ of the following composite map:

$$\rho := \qquad X \times (X \pitchfork T(1)) \xrightarrow{\langle pr, ev\rangle} X \times T(1) \xrightarrow{t''_{X,1}} T(X \times 1) \cong T(X). \tag{30}$$

Here, pr denotes the projection $X \times (X \pitchfork T(1)) \to X$ to the first factor, and ev denotes the evaluation map $X \times (X \pitchfork T(1)) \to T(1)$. The composite map displayed is actually 2-linear, so if the monad T happens to be commutative, the 1-linear extension of it to $\vdash$ will be bilinear, by Proposition 5.

By Proposition 1, it is clear that an alternative description of ρ is:

$$X \times (X \pitchfork T1) \xrightarrow{\langle pr, ev\rangle} X \times T1 \xrightarrow{\eta_X \times id} TX \times T1 \xrightarrow{\psi_{X,1}} T(X \times 1) \cong TX.$$

The action $\dashv$ of $X \pitchfork T(1)$ on $T(X)$ presented here ("action by *functions* on distributions") restricts to the action of $T(1)$ on $T(X)$ ("by *scalars* on distributions") considered in Section 3, via the monoid map $! \pitchfork T(1) : 1 \pitchfork T(1) \to X \pitchfork T(1)$ induced by $! : X \to 1$; expressed synthetically, if $\phi : X \to T(1)$ has constant value $\lambda \in T(1)$, then $P \vdash \phi = P \cdot \lambda$, where $\vdash$ denotes the action of $X \pitchfork T(1)$, and the dot denotes the action of $T(1)$ on $T(X)$.

Theorem 3 *The action $\vdash: T(X) \times (X \pitchfork T(1)) \to T(X)$ is associative and unitary.*

Proof. Our proof is not quite straightforward; there ought to be a better one. To prove the associativity assertion, we should compare two map $T(X) \times (X \pitchfork T(1) \times (X \pitchfork T(1) \to T(X)$ which both are 1-linear, so it suffices to prove that their precomposite with $\eta_X \times id$ are equal. This is achieved by a contemplation of the following diagram. (For the arrow denoted "$\langle pr, ev\rangle$", the middle factor $T1$ does not participate in the $\langle pr, ev\rangle$-formation (so elementwise, the map takes (x, λ, ϕ) to $(x, \lambda, \phi(x))$); also, isomorphisms $X \times \cong X$ are omitted from notation.)

$$
\begin{tikzcd}
X \times (X \pitchfork T1) \times (X \pitchfork T1) \arrow[r, "\langle pr, ev\rangle \times id"] \arrow[d, "X \times m_X"'] & X \times T1 \times (X \pitchfork T1) \arrow[r, "t_{X,1,X\pitchfork T1}"] \arrow[d, "``\langle pr, ev\rangle\text{''}"'] & T(X \times (X \pitchfork T1)) \arrow[d, "T\langle pr, ev\rangle"] \\
X \times (X \pitchfork T1) \arrow[dr, "\langle pr, ev\rangle"'] & X \times T1 \times T1 \arrow[r, "t_{X,1,T1}"'] \arrow[d, "X \times m"] & T(X \times T1) \arrow[d, "T(t''_{X,1})"] \\
 & X \times T1 \arrow[dr, "t''_{X,1}"'] & T^2 X \arrow[d, "\mu_X"] \\
 & & TX;
\end{tikzcd}
$$

the left hand region commutes, by definition of m_X in terms of m, and the upper right hand square is essentially just a twisted version of the naturality square for t'' w.r.to the map $\langle pr, ev\rangle : X \times (X \pitchfork T(1)) \to X \times T(1)$, recalling that th combinator t in (9) came about by a twisting of $t''_{X\times Z,Y}$ (here with $Y = 1$ and $Z = T(1)$). The lower right hand region deserves a more detailed argument. Let us prove its commutativity, without using identifications like $X \times 1 \cong X$. Consider namely

$$
\begin{tikzcd}
X \times T1 \times T1 \arrow[rr, "t_{X,1,T1}"] \arrow[d, "X \times t'_{1,T1}"'] & & T(X \times 1 \times T1) \arrow[dd, "T(t''_{X\times 1,1})"] \\
X \times T(1 \times T1) \arrow[d, "X \times T(t''_{1,1})"'] & * & \\
X \times T^2(1 \times 1) \arrow[r, "t''_{X,T(1\times 1)}"] \arrow[d, "X \times \mu_{1\times 1}"'] & T(X \times T(1 \times 1)) \arrow[r, "T(t''_{X,1\times 1})"] & T^2(X \times 1 \times 1) \arrow[d, "\mu_{X\times 1\times 1}"] \\
X \times T(1 \times 1) \arrow[rr, "t''_{X,1\times 1}"'] & & T(X \times 1 \times 1).
\end{tikzcd}
\tag{31}
$$

After the identification of 1×1 with 1, the left hand column is (X times) the defining construction of m; and after the identification of $X \times 1 \times 1$, the lower right hand object is $T(X)$. The lower region commutes because μ is a strong natural transformation, thus compatible with the tensorial strengths of T^2 and T; and the upper region * is an instance of the generalized associativity of the tensorial strengths t', t''. In more detail, writing Y and Z for 1, to keep them apart, consider

$$\begin{tikzcd}
X \times TY \times TZ \arrow[r, "X \times t'_{Y,TZ}"] & X \times T(Y \times TZ) \arrow[r, "t''_{X,Y \times TZ}"] \arrow[d, "X \times T(t''_{Y,Z})"'] & T(X \times Y \times TZ) \arrow[d, "T(X \times t''_{Y,Z})"] \\
 & X \times T^2(Y \times Z) \arrow[r, "t''_{X,T(Y \times Z)}"'] & T(X \times T(Y \times Z)) \arrow[d, "T(t''_{X,Y \times Z})"] \\
 & & T^2(X \times Y \times Z).
\end{tikzcd}$$

The top composite is $t_{X,Y,Z}$, by (11). The right hand vertical composite is $T(t''_{X \times Y,Z})$, by the associative law for t''. So the clockwise composite in this diagram equals the clockwise composite of * in (31) (when we put $Y = Z = 1$) ; and the counterclockwise similarly equals the counterclockwise in *. This proves the associativity.

To prove the unitary law, we must prove that $id \times e_X : T(X) \times 1 \to T(X) \times (X \pitchfork T1)$ followed by $\vdash$ is the identity map of $T(X)$ (modulo the identification $T(X) \times 1 \cong T(X)$). The two maps $T(X) \times 1 \to T(X)$ to be compared are 1-linear, so it suffices to prove that they agree when precomposed with η_X. Consider the diagram

$$\begin{tikzcd}
X \times 1 \arrow[r, "X \times e_X"] \arrow[dr, "X \times \eta_1"'] & X \times (X \pitchfork T1) \arrow[r, "\eta_X \times id"] \arrow[d, "\langle pr, ev \rangle"] & TX \times (X \ T1) \arrow[d, "\vdash"] \\
 & X \times T1 \arrow[r, "t''"'] & T(X \times 1)
\end{tikzcd}$$

The square commutes by the construction of $\vdash$, and the triangle commutes by the pointwise nature of e_X in terms of $e = \eta_1$. Finally, the lower composite is $\eta_{X \times 1}$. After the identification of $X \times 1$ with X, we thus get η_X, and this proves the unitary law.

We next address naturality questions for the action $\vdash$, both with respect to X, and with respect to the monad T. It does not immediately make sense to ask for plain naturality of $\vdash$ w.r.to X, since the domain $T(X) \times (X \pitchfork T(1))$ depends both covariantly and contravariantly on X, but we do have

Proposition 13 (Frobenius reciprocity) *If $f : X \to Y$ is any map, the map $T(f) : T(X) \to$*

$T(Y)$ is $Y \pitchfork T(1)$-equivariant, where $Y \pitchfork T(1)$ acts on $T(X)$ via the monoid homomorphism $f^* : Y \pitchfork T(1) \to X \pitchfork T(1)$.

Here, f^* is short for $f \pitchfork T(1) : Y \pitchfork T(1) \to X \pitchfork T(1)$. The statement can be expressed diagrammatically as commutativity of the right hand region in the diagram

$$\begin{tikzcd}
X \times (Y \pitchfork T1) \arrow[r, "\eta_X \times id"] \arrow[d, "f \times id"'] & TX \times (Y \pitchfork T1) \arrow[r, "TX \times f^*"] \arrow[d, "T(f) \times id"'] & TX \times (X \pitchfork T1) \arrow[r, "\vdash"] & TX \arrow[d, "T(f)"] \\
Y \times (Y \pitchfork T1) \arrow[r, "\eta_Y \times id"'] & TY \times (Y \pitchfork T1) \arrow[rr, "\vdash"'] & & TY
\end{tikzcd} .$$

In this region, both composites are 1-linear, so as in the proof of Proposition 11, it suffices to prove commutativity of the diagram when precomposed with $\eta_X \times id$. Then the ηs may trivially be pushed to the right, using naturality of η and bifunctorality of $\times$. When the ηs come next to the $\vdash$s, we can use the defining equations (30) to eliminate $\vdash$, so that the total diagram above is rewritten as

$$\begin{tikzcd}
X \times (Y \pitchfork T1) \arrow[r, "id \times f^*"] \arrow[d, "f \times id"'] & X \times (X \pitchfork T1) \arrow[r, "\langle pr, ev \rangle"] & X \times T1 \arrow[r, "t''_{X,1}"] \arrow[d, "f \times id"'] & T(X \times 1) \arrow[r, phantom, "\cong"] \arrow[d, "T(f \times id)"] & T(X) \arrow[d, "T(f)"] \\
Y \times (Y \pitchfork T1) \arrow[rr, "\langle pr, ev \rangle"'] & & Y \times T1 \arrow[r, "t''_{Y,1}"'] & T(Y \times 1) \arrow[r, phantom, "\cong"] & T(Y)
\end{tikzcd} .$$

Here, the left hand region commutes for pure "λ-calculus" reasons, and the rest commutes by naturality. This proves the Proposition.

We next consider the naturality w.r.to morphisms of monads $\tau : T \to S$. This is simpler:

Proposition 14 *Let $\tau : T \Rightarrow S$ be a morphism of strong monads. Then $\tau_X : T(X) \to S(X)$ is $X \pitchfork T(1)$-equivariant, where $S(X)$ is equipped with action by $X \pitchfork T(1)$ via the monoid homomorphism $X \pitchfork \tau_1 : X \pitchfork T(1) \to X \pitchfork S(1)$.*

In diagrammatic terms, this says that the following diagram commutes:

$$\begin{tikzcd}
TX \times (X \pitchfork T1) \arrow[r, "\vdash"] \arrow[d, "\tau_X \times (X \pitchfork \tau_1)"'] & TX \arrow[d, "\tau_X"] \\
SX \times (X \pitchfork S1) \arrow[r, "\vdash"'] & SX
\end{tikzcd} .$$

For, it is standard monad theory that a monad morphism $\tau : T \Rightarrow S$ induces a "forgetful" functor $\mathscr{E}^S \to \mathscr{E}^T$, compatible with the "underlying" functors; and then $\tau_X : T(X) \to S(X)$ is a T-homomorphism; similarly for "T-homomorphisms in the first variable", like the left hand vertical map in the displayed diagram. Since both composites thus are T-linear in the first variable, it suffices by Proposition 3 to see that we get a commutative diagram when we precompose the displayed diagram by $\eta_X \times id$, and this is straightforward.

We address the question of the relation between the action $\vdash : T(X) \times (X \pitchfork T(1)) \to T(X)$ of functions on distributions, and $T(1)$-valued integration $\int_X : T(X) \times (X \pitchfork T(1)) \to T(1)$. We express this elementwise, and leave the diagrammatic description to the reader. We first note

Proposition 15 *Let $P \in T(X)$ and let $\phi \in X \pitchfork T(1)$. Then $\int_X \phi(x)\, dP(x)$ equals the total of $P \vdash \phi$.*

Proof. We are comparing the value at P, ϕ of two maps $T(X) \times (X \pitchfork T(1)) \to T(1)$. Both are 1-linear, so it suffices to see that they agree when precomposed with $\eta_X \times id : X \times (X \pitchfork T(1)) \to T(X) \times (X \pitchfork T(1))$. Precomposing $\int_X$ yields by (14) the evaluation map $X \times (X \pitchfork T(1)) \to T(1)$. For the other composite, we recall the description of $\vdash$ as 1-linear extension of the map $\rho : X \times (X \pitchfork T(1)) \to T(X)$ in (30), here appearing as the top composite in

$$\begin{tikzcd}
X \times (X \pitchfork T1) \arrow[r, "\langle pr, ev \rangle"] \arrow[d, "ev"'] & X \times T1 \arrow[r, "t''"] \arrow[d, "! \times id"'] & T(X \times 1) \arrow[r, phantom, "\cong"] \arrow[d, "T(! \times id)"'] & T(X) \arrow[d, "T(!)"] \\
T1 \arrow[r, "\cong"'] & 1 \times T1 \arrow[r, "t''"'] & T(1 \times 1) \arrow[r, phantom, "\cong"] & T(1).
\end{tikzcd}$$

The clockwise composite is the total in question, the counterclockwise is again the evaluation map. This proves the Proposition.

Combining with Theorem 3, we therefore have the following integration theoretic significance of the action $\vdash$; again, we express it in elementwise terms. The monad T is assumed commutative.

Theorem 4 *For $P \in T(X)$ and ϕ_1 and ϕ_2 in $X \pitchfork T(1)$, we have*

$$\int_X \phi_1(x)\, d(P \vdash \phi_2)(x) = \int_X (\phi_1 \cdot \phi_2)(x)\, dP(x).$$

Proof. The left hand side is, by the Proposition, the total of the distribution $(P \vdash \phi_2) \vdash \phi_1$, and the right hand side is by the Proposition the total of the distribution $P \vdash (\phi_1 \cdot \phi_2)$. The result now follows from the associative law (Theorem 3) for the action of the (commutative) monoid $X \pitchfork T(1)$ on $T(X)$.

This Theorem can also be obtained by using the naturality of the various combinators with respect to transformation of monads, as expressed in Proposition 14; namely, one uses the transformation $\tau : T \Rightarrow (- \pitchfork T(1)) \pitchfork_T T(1)$ considered in Theorem 1.

5 Tensor product and convolution

If $P \in T(X)$ and $Q \in T(Y)$, one has $\psi(P,Q) \in T(X \times Y)$. This is, for classical distributions, the "tensor product" of the distributions P and Q. One has also $\tilde{\psi}(P,Q)$, which agrees with $\psi(P,Q)$ if the monad is commutative. We henceforth stick to the commutative case.

If now $m : X \times Y \to Z$ is a map, we may form the *convolution* of P and Q along m; this is $T(m)(\psi(P,Q)) \in T(Z)$. Thus in element-free terms, convolution formation along m is the composite

$$T(X) \times T(Y) \xrightarrow{\psi} T(X \times Y) \xrightarrow{T(m)} T(Z).$$

It is T-bilinear.

We have encountered special cases already, namely the (right) action of $T(1)$ on $T(X)$, which is convolution along the isomorphism $X \times 1 \to X$. The multiplication making $T(1)$ into a monoid is the special case where $X = 1$, so this multiplication is likewise a convolution.

The convolutions that are most important in functional analysis are the convolutions along the addition map $+ : V \times V \to V$ for an abelian monoid V; this will be a map $* : T(V) \times T(V) \to T(V)$ making $T(V)$ in to an abelian semigroup. Assuming that the monad T is of the kind studied in Section 7 below, all objects $T(X)$ carry a natural addition structure $+$, and $*$ and $+$ together will make $T(V)$ into a commutative rig. Distributivity of $*$ over $+$ follows from R-bilinearity of $*$.

6 Physical quantities as torsors

To motivate the following, consider the 1-dimensional vector space k over a field k. Then a k-linear isomorphism $k \to k$ is multiplication by an invertible scalar $r \in k$, and r in fact defines a natural isomorphism $\rho : T \Rightarrow T$, where T is the free-vector space monad, namely $\rho_X : T(X) \to T(X)$ is the homothety "multiplication by r". This transformation is compatible with the μ of the monad, since each instance of ρ is a linear map; but it is not compatible with η, since $\rho_1(1) = r \in k$ is not necessarily $1 \in k$.

Proposition 16 *Let $\rho : T \Rightarrow S$ be a strong natural transformation between endofunctors on $\mathscr{E}$. Then for any pair of objects X, Y, the following diagram commutes:*

$$\begin{tikzcd}
TX \times (X \pitchfork TY) \arrow[r] \arrow[d, "\rho_X \times (X \pitchfork \rho_Y)"'] & T^2Y \arrow[d, "\rho^2_Y"] \\
SX \times (X \pitchfork SY) \arrow[r] & S^2Y
\end{tikzcd}$$

where the horizontal maps are "strength in the righthand factor, followed by evaluation", and where natural transformation ρ^2 denotes the natural transformation $T^2 \Rightarrow S^2$ derived from ρ.

Thus the top arrow is

$$TX \times (X \pitchfork TY) \xrightarrow{TX \times st^T} TX \times (TX \pitchfork T^2Y) \xrightarrow{ev} T^2Y,$$

where st^T is the strength (enrichment) of T; similarly the bottom arrow one is obtained from the strength st^S of S. The natural transformation ρ^2 is more explicitly given by $\rho^2_Y = S(\rho_Y) \circ \rho_{TY}$. The proof of this Proposition is in principle elementary; it uses that ρ is a *strong* natural transformation, which means in particular (cf. [3] II.6.2.8) that diagrams of the form

$$\begin{tikzcd} X \pitchfork Y \arrow[r, "st_T"] \arrow[d, "st_S"'] & TX \pitchfork TY \arrow[d, "TX \pitchfork \rho_Y"] \\ SX \pitchfork SY \arrow[r, "\rho_X \pitchfork Y"'] & TX \pitchfork SY \end{tikzcd}$$

commute (the equivalence of this notion of strong natural transformation with the one of (12) is proved in [10] Lemma 1.1).

Let $B = (B, \beta)$ be a T-algebra. Recall from Section 1 that we have a map

$$\int_X : T(X) \times (X \pitchfork B) \to B.$$

Similarly for S. Inspecting the explicit construction (2) (with $(T(1); \mu_1)$ for (B, β)), we note that the construction depends on μ, but it does not depend on η. Therefore, the following is not surprising:

Proposition 17 *Let T and S be strong monads on $\mathscr{E}$, and let $\rho : T \Rightarrow S$ be a strong natural transformation, compatible with the μs, but not necessarily with the ηs. Then the $\int_X$-formation for the monads T and S is compatible with ρ, in the sense that the following diagram commutes:*

$$\begin{tikzcd} T(X) \times (X \pitchfork T(1)) \arrow[r, "\int_X"] \arrow[d, "\rho_X \times (X \pitchfork \rho_1)"'] & T(1) \arrow[d, "\rho_1"] \\ S(X) \times (X \pitchfork S(1)) \arrow[r, "\int_X"'] & S(1). \end{tikzcd}$$

Proof. Use the explicit form (2) for the $\int_X$ in question; then the desired diagram comes about from the diagram in Proposition 16 by putting $Y = 1$, and concatenating it with the commutative square expressing compatibility of ρ with the μs of the monads:

$$\rho_1 \circ \mu_1^T = \mu_1^S \circ \rho_1^2,$$

where μ^T and μ^S denote the multiplication of the monads T and S, respectively.

Let T be a commutative monad on $\mathscr{E}$. Consider another strong endofunctor M on $\mathscr{E}$, equipped with an action ν by T,

$$\nu : T(M(X)) \to M(X)$$

natural in X, and with ν satisfying a unitary and associative law. Then every $M(X)$ is a T-algebra by virtue of $\nu_X : T(M(X)) \to M(X)$, and morphisms of the form $M(f)$ are T-linear. Let M and M' be strong endofunctors equipped with such T-actions. There is an evident notion of when a strong natural transformation $\lambda : M \Rightarrow M'$ is compatible with the T-actions, so we have a category of T-actions. The endofunctor T itself is an object in this category, by virtue of μ. We say that M is a T-*torsor* if it is isomorphic to T in the category of T-actions. Note that no particular such isomorphism is chosen; this is just like a 1-dimensional vector space over k: it is isomorphic to k, but no particular isomorphism is chosen.

Our contention is that the category of T-torsors is a mathematical model of (not necessarily pure) quantities of type T (which is the corresponding pure quantity). Thus if T is the free $\mathbb{R}$-vector space monad, the functor M which to a space $X \in \mathscr{E}$ associates the space of distributions of electric charges over X, is a T-torsor.

The following Proposition expresses that isomorphisms of actions $\lambda : T \cong M$ are determined by $\lambda_1 : T(1) \to M(1)$; in the example, the latter data means: choosing a *unit* of electric charge.

Proposition 18 *If g and $h : T \Rightarrow M$ are isomorphisms of T-actions, and if $g_1 = h_1 : T(1) \to M(1)$, then $g = h$.*

Proof. By replacing h by its inverse $M \to T$, it is clear that it suffices to prove that if $\rho : T \to T$ is an isomorphism of T-actions, and $\rho_1 = id_{T(1)}$, then ρ is the identity transformation. As a morphism of T-actions, ρ is in particular a *strong* natural transformation, which implies that right hand square in the following diagram commutes for any $X \in \mathscr{E}$; the left hand square commutes by assumption on ρ_1:

$$\begin{tikzcd}
X \times 1 \arrow[r, "X \times \eta_1"] \arrow[d, "="'] & X \times T(1) \arrow[r, "t''"] \arrow[d, "X \times \rho_1"'] & T(X \times 1) \arrow[d, "\rho_{X \times 1}"] \\
X \times 1 \arrow[r, "X \times \eta_1"'] & X \times T(1) \arrow[r, "t''"'] & T(X \times 1)
\end{tikzcd}$$

Now both the horizontal composites are $\eta_{X \times 1}$, by general theory of tensorial strengths. Also $\rho_{X \times 1}$ is T-linear. Then uniqueness of T-linear extensions over $\eta_{X \times 1}$ implies that the right hand vertical map is the identity map. Using the natural identification of $X \times 1$ with X, we then also get that ρ_X is the identity map of $T(X)$.

7 Monads and biproducts

Let T be a commutative monad. We summarize some of the relations between the covariant functor $T : \mathscr{E} \to \mathscr{E}$, and the contravariant $- \pitchfork T(1) : \mathscr{E} \to \mathscr{E}$. The latter is actually valued in the category of commutative monoids in $\mathscr{E}$.

- There is a T-bilinear pairing $T(X) \times (X \pitchfork T(1)) \to T(1)$, namely the exponential adjoint $\int_X$ of the map $\tau_X : T(X) \to (X \pitchfork T(1)) \pitchfork_T T(1)$.
- There is an associative and unitary T-bilinear action $\vdash$ of $X \pitchfork T(1)$ on $T(X)$; it satisfies a "Frobenius reciprocity" naturality condition.

These are two of the axioms laid down by Lawvere in his description of relations between extensive quantities and intensive quantities, except for the T-bilinearity, cf. e.g. [19], Lecture IV; in Lawvere's axiomatics, one deals rather with bilinearity in the sense of an additive structure.

We shall in the present Section describe a simple categorical property of the monad T, which will guarantee that "T-linearity implies additivity", even "R-linearity" in the sense of a rig $R \in \mathscr{E}$ ("rig"= commutative semiring), namely $R = T(1)$. This condition will in fact imply that $\mathscr{E}^T$ is a "linear category".

We begin with some standard general category theory, namely a monad $T = (T, \eta, \mu)$ on a category which has finite products and finite coproducts. (No distributivity is assumed.) So $\mathscr{E}$ has an initial object $\emptyset$. If $T(\emptyset) \in \mathscr{E}$ is a terminal object, then the object $(T(\emptyset), \mu_\emptyset)$ is a zero object in $\mathscr{E}^T$, i.e. it is both initial and terminal. It is initial because T, as a functor $\mathscr{E} \to \mathscr{E}^T$, is a left adjoint, hence preserves initials; and since $T(\emptyset) = 1$, it is also terminal (the terminal object in $\mathscr{E}^T$ being $1 \in \mathscr{E}$, equipped with the unique map $T(1) \to 1$ as structure). This zero object in $\mathscr{E}^T$ we denote 0. Existence of a zero object in a category implies that the category has distinguished zero maps $0_{A,B} : A \to B$ between any two objects A and B, namely the unique map $A \to B$ which factors through 0. For $\mathscr{E}^T$, we can even talk about the zero map $0_{X,B} : X \to B$, where $X \in \mathscr{E}$ and $B = (B, \beta) \in \mathscr{E}^T$, namely η_X followed by the zero map $0_{T(X),B} : T(X) \to B$. We have a canonical map $X + Y \to T(X) \times T(Y)$: the composite $X \to X + Y \to T(X) \times T(Y)$ is $(\eta_X, 0_{X,T(Y)})$ (here, the first map is the coproduct inclusion map). Similarly, we have a canonical map $Y \to T(X) \times T(Y)$. Using the universal property of coproducts, we thus get a canonical map $\phi_{X,Y} : X + Y \to T(X) \times T(Y)$. It extends uniquely over $\eta_{X+Y} : X + Y \to T(X + Y)$ to a T-linear map

$$\Phi_{X,Y} : T(X + Y) \to T(X) \times T(Y),$$

and Φ is natural in X and in Y. We say that $T : \mathscr{E} \to \mathscr{E}^T$ *takes binary coproducts to products* if $\Phi_{X,Y}$ is an isomorphism (in $\mathscr{E}$ or equivalently in $\mathscr{E}^T$) for all X, Y in $\mathscr{E}$. Note that the definition presupposed that $T(\emptyset) = 1$; it is the zero object in $\mathscr{E}^T$, so that if T takes binary coproducts to products, it in fact takes finite coproducts to products, in a similar sense. So we can also use the phrase "T *takes finite coproducts to products*" for this property of T.

We define an "addition" map in $\mathscr{E}^T$; it is a map $+ : T(X) \times T(X)$ to $T(X)$, namely the composite

$$T(X) \times T(X) \xrightarrow{\Phi^{-1}_{X,Y}} T(X + X) \xrightarrow{T(\nabla)} T(X)$$

where $\nabla : X + X \to X$ is the codiagonal. So in particular, if in_i denotes the ith inclusion ($i = 1,2$) of X into $X + X$, we have

$$id_{TX} = \quad TX \xrightarrow{T(in_i)} T(X+X) \xrightarrow{\Phi_{X,X}} TX \times TX \xrightarrow{+} TX. \tag{32}$$

Under the identification $T(X) \cong T(X+\emptyset) \cong T(X) \times 1$, the equation (32) can also be read: $T(!) : T(\emptyset) \to T(X)$ is right unit for $+$, and similarly one gets that it is a left unit.

We leave to the reader the easy proof of associativity and commutativity of the map $+ : T(X) \times T(X) \to T(X)$. It follows that $T(X)$ acquires structure of an abelian monoid in $\mathscr{E}^T$ (and also in $\mathscr{E}$).

Proposition 19 *Every T-linear map $T(X) \to T(Y)$ is compatible with the abelian monoid structure.*

Proof. This means that we should prove commutativity of the square * in the following diagram

$$\begin{tikzcd} T(X+X) \arrow[r, "\Phi"] & T(X) \times T(X) \arrow[r, "f \times f"] \arrow[d, "+"'] \arrow[dr, phantom, "*"] & T(Y) \times T(Y) \arrow[d, "+"] \\ & T(X) \arrow[r, "f"'] & T(Y) \end{tikzcd}$$

for f any T-linear map; so f is not necessarily of the form $T(g)$, but it has the property that it preserves 0. To prove commutativity of the diagram *, it suffices to precompose with the linear isomorphism Φ. Now the two maps to be compared are both T-linear, and $T(X+X)$ is a coproduct in $\mathscr{E}^T$, so it suffices to see that their composite with the inclusion $T(in_i) : T(X) \to T(X+X)$ (where $i = 1$ or $= 2$) are equal. Now $(f \times f) \circ \Phi \circ T(in_i)$ is seen to be f, using that 0 is neutral for the addition.

Recall that we have the T-bilinear action $T(X) \times T(1) \to T(X)$. It follows from the Proposition that it is additive in each variable separately.

We have in particular the T-bilinear commutative multiplication $m : T(1) \times T(1) \to T(1)$, likewise bi-additive, $m(x+y,z) = m(x,z) + m(y,z)$, or in the notation one also wants to use,

$$(x+y) \cdot z = x \cdot z + y \cdot z,$$

so that $T(1)$ carries structure of a rig (= commutative semiring). This rig we also denote R. The category of modules over a rig R is the category of abelian monoids equipped with a bi-additive action by R, and maps which preserve the addition and the action. We may summarize:

Proposition 20 *Each $T(X)$ is a module over the rig $R = T(1)$; each T-linear map $T(X) \to T(Y)$ is an R-module morphism.*

It is more generally true that T-linear maps $A \to B$ (for A and $B \in \mathscr{E}^T$) are R-module maps. We shall not use this fact; it is proved in analogy with the proof of Proposition 11.

Let us finally note

Proposition 21 *If T takes finite coproducts to products, then so does the associated Schwartz monad $S = (- \pitchfork T(1)) \pitchfork_T T(1)$.*

Proof (sketch). We have

$$(\emptyset \pitchfork T(1)) \pitchfork_T T(1) \cong 1 \pitchfork_T T(1) \cong 1$$

the last isomorphism because $1 = 0$ is an initial T-algebra. Similarly,

$$\begin{aligned} S(X) \times S(Y) &= [(X \pitchfork T(1)) \pitchfork_T T(1)] \times [(Y \pitchfork T(1)) \pitchfork_T T(1)] \\ &= [(X \pitchfork T(1)) \oplus (Y \pitchfork T(1))] \pitchfork_T T(1) \end{aligned}$$

because $\oplus$ is coproduct in $\mathscr{E}^T$,

$$= [(X+Y) \pitchfork T(1)] \pitchfork_T T(1)$$

because $\oplus$ is product in $\mathscr{E}$

$$= S(X+Y).$$

8 Expectation and other moments

We consider now a commutative monad $T = (T, \eta, \mu, t')$ on $\mathscr{E}$ (a CCC with coproducts and finite inverse limits), such that (T, η, μ) takes finite coproducts to products. Thus $\mathscr{E}^T$ is a semi-additive category with biproducts $\oplus$; all its objects are modules over the rig $R = T(1)$, and all morphisms are R-linear (as well as T-linear, of course). We call R the rig of *scalars*.

Talking synthetically, we call the elements of $T(X)$ *concrete* distributions on X. We also have the object $S(X) = (X \pitchfork R) \pitchfork_T R$ of Schwartz distributions on X, i.e. T-linear functionals $X \pitchfork R \to R$; and we have the map $\tau_X : T(X) \to S(X)$ taking concrete distributions on X to such functionals. We have by Proposition 7 that $\int_X \phi(x)\, dP(x)$ is the value of the functional $\tau_X(P) : X \pitchfork R \to R$ on $\phi : X \to R$ (ϕ a "test function" on X, in Schwartz terminology). The *total* of P is the $T(X \to 1)(P) \in T(1) = R$, and may be written as $\int_X 1_X\, dP(x)$ where $1_X : X \to R$ is the constant function with the multiplicative unit $1 = e = \eta_1$ of R.

For concrete distributions P on the space R itself, (so $P \in T(R)$) there are other characteristic scalars, called "moments", namely some of the values of the functional $\tau_R(P) : R \pitchfork R \to R$ on some particular functions $R \to R$. In view of the universal role which the identity map has in e.g. Yoneda's Lemma, it is no surprise that the value of this functional on id_R plays a particular role. It is the *expectation* of P, denoted $E(P) \in R$,

$$E(P) := \int_R x\, dP(x).$$

We have

Proposition 22 *Let $P \in T(R)$. Then any value of the functional $\tau_R(P) : (R \pitchfork R) \to R$ is the expectation of some $P' \in T(R)$: for any $\phi \in R \pitchfork R$ and $P \in T(R)$,*

$$\tau_R(P)(\phi) = E(T(\phi)(P)).$$

Proof. By naturality of τ with respect to $\phi : R \to R$,

$$\tau_R \circ T(\phi) = [(\phi \pitchfork R) \pitchfork_T R] \circ \tau_R.$$

When postcomposed with $ev_{id} : (R \pitchfork R) \pitchfork_T T \to R$, the left hand side gives $E(T(\phi)(P))$, the right hand side gives $\tau_R(P)(\phi)$, because $ev_{id} \circ ((\phi \pitchfork R) \pitchfork_T R) = ev_\phi$.

Note that for any T-algebra $B = (B, \beta)$, and $P \in T(B)$, we have $E(P) = \int_B x \, dP(x) = \beta(P)$; this is just a reformulation of (4).

Since R is a rig, we have for each natural number n a map $R \to R$, elementwise described by $x \mapsto x^n$. The *nth moment* $\alpha_n(P)$ of $P \in T(R)$ is defined as $\int_R x^n \, dP(x)$, thus $\alpha_0(P)$ is the total of P, and $\alpha_1(P)$ the expectation of P. Note that $\alpha_1(P) = E(P) = \mu_1(P)$, where $\mu_1 : T^2(1) \to T(1) = R$ comes from the monad-multiplication $\mu : T^2 \Rightarrow T(1) = R$.

In [4] (5.5.6), one finds the formula $E\{X+Y\} = E\{X\} + E\{Y\}$ where X, Y is a joint distribution of two simultaneous random variables, valued in R. The formula looks deceptively just like it were a consequence of linearity of $E : T(R) \to R$ ($= \mu_1 : T^2(1) \to T(1)$); but recall that X, Y is not a pair of distributions; rather, it is meant to denote a *simultaneous* distribution, i.e. an element $P \in T(R \times R)$, and $X + Y$ refers to the distribution $\in T(R)$ obtained by applying $T(+) : T(R \times R) \to T(R)$ to P. So the formula is not a simple linearity. It is rather a formulation of the following:

Proposition 23 *The following diagram * commutes:*

$$\begin{tikzcd}
T^2(2) \arrow[r, "T(\Phi)"] \arrow[d, "\mu_2"'] & T(R \times R) \arrow[r, "T(+)"] \arrow[d, "\beta"'] \arrow[dr, phantom, "*"] & T(R) = T^2(1) \arrow[d, "\mu_1"] \\
T(2) \arrow[r, "\Phi"'] & R \times R \arrow[r, "+"'] & R = T(1)
\end{tikzcd}$$

where β is the coordinatewise T-algebra structure on $R \times R$.

Proof. Write $T(1)$ for R, and write $1+1$ for 2, and let Φ be the comparison isomorphism, expressing that T takes finite coproducts to products. Then the left hand square commutes, since Φ is T-linear, and the outer diagram commutes by naturality of μ with respect to the map $\nabla : 2 \to 1$. (Here, of course, ∇ is the unique map $2 \to 1$, but we write it for systematic reasons; in fact, the Proposition and the proof immediately generalizes when R is replaced by R^n, in which case $\nabla : 2n \to n$ is not so trivial.)

For comparison with the quoted formula from [4], if X,Y denotes P, the clockwise composite takes P to $E\{X+Y\}$, and the counterclockwise takes it into $E\{X\}+E\{Y\}$.

If $P \in T(R)$ has total 1, the physical significance of $E(P) \in R$ is "center of gravity" of P (thinking of P as a mass distribution). However, physically it is clear that the center of gravity of a mass distribution on the line R does not the depend on the location of the origin $0 \in R$, but only of the *affine* structure of R, in other words, it is invariant under affine maps $R \to R$. Here, we may take "affine map $R \to R$" to mean maps of the form $x \mapsto a \cdot x + b$ where a and b are scalars $\in R$.

Proposition 24 *Let $P \in T(R)$ have total 1. Then for any affine $\phi : R \to R$, $\phi(E(P)) = E(T(\phi)(P))$.*

Proof. We may write $\phi \in R \pitchfork R$ as a linear combination of the identity map $id : R \to R$, and $1 : R \to R$ (the map with constant value $1 \in R$), $\phi(x) = a \cdot x + b$. By Proposition 22, we have

$$E(T(\phi)(P)) = \tau_R(P)(\phi) = \tau_R(P)(a \cdot id + b \cdot 1).$$

Then since $\tau_R(P)$ is T-linear, it is R-linear (Proposition 20), so we may continue the equation

$$= a \cdot \tau_R(P)(id) + b \cdot \tau_R(P)(1),$$

which is $a \cdot E(P) + b \cdot 1$, the last term since P has total 1.

The notion of moments make sense not only for distributions on $R = T(1)$, but for instance also for distributions on $R^2 = T(2)$. Thus if $P \in T(2)$, we have for any $\phi : R^2 \to R$ the scalar $\int_{R^2} \phi(z)\, dP(z)$. Since the dummy variable z here ranges over R^2, it is more natural to write it $z = (x,y)$, where x and y range over R, and thus the scalar in question is written $\int_{R^2} \phi(x,y)\, dP(x,y)$. The *mixed second order momemt* of P is the scalar obtained by taking ϕ to be the multiplication map $R \times R \to R$, so is $\int_{R^2} x \cdot y\, dP(x,y)$. It is in terms of this that one can define the *correlation* coefficient of P.

9 Examples.

The simplest example is where $\mathscr{E}$ is the category of sets (strength is automatic here), and T is the free-commutative-monoid monad. This is related to the notion of "multiset", since $T(X)$ also may be seen as the set of multi-subsets of X; an element of $T(X)$ consists in an assignement P of multiplicities $\{n(x) \in \mathbb{N} \mid x \in X\}$, with $n_x = 0$ for all but a finite number of xs, "P is if compact support". Then $T(1) = \mathbb{N}$, and $X \pitchfork T(1)$ is the set of assignements ϕ of multiplicities $\{n(x) \in \mathbb{N} \mid x \in X\}$, but without the requirement of compact support. Consider $X = T(1) = \mathbb{N}$. One can easily see that $T(\mathbb{N})$ may be identified with the set of polynomials in one variable with coefficients from $\mathbb{N}$, and then convolution along the addition map $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ becomes identified with multiplication of polynomials. (Similarly for finite products $\mathbb{N}^k$.)

An example where the conceptual machinery (strength) has to be brought in explicitly is the following, which was one of the motivations for the present research: Consider the category $\mathscr{E}$ of convenient vector spaces and the smooth (i.e. C^∞ but not necessarily linear) maps in between. It is a cartesian closed category, cf. [18] and [1], and there exists free vector spaces ($\mathbb{R}$-modules) in it, hence a commutative monad T. The category $\mathscr{E}$ does (probably) not have equalizers, at least it is clear that the zero set of a nonlinear map, say $V \to \mathbb{R}$, does not have a natural vector space structure. On the other hand, the equalizer of two parallel *linear* maps in $\mathscr{E}$ does exist. The following piece of general theory shows that therefore $\mathscr{E}$ has *enough* equalizers to form the subobject $A \pitchfork_T B \subseteq A \pitchfork B$, which was crucial in the construction of restricted double dualization monads (as in Section 2), and in [12].

We recall from [2], or [12] the two parallel maps whose equalizer, if it exists, gives $A \pitchfork_T B \subseteq A \pitchfork B$, (where $A = (A, \alpha)$ and $B = (B, \beta)$ are two T-algebras). The two maps $A \pitchfork B \to T(A) \pitchfork B$ are $\alpha \pitchfork B$, on the one hand, and the composite

$$A \pitchfork B \xrightarrow{st} T(A) \pitchfork T(B) \xrightarrow{T(A) \pitchfork \beta} T(A) \pitchfork B. \tag{33}$$

The map $\alpha \pitchfork B$ is clearly T-linear. For the map (33), this is not immediately clear; in fact, it depends on commutativity of the monad T:

Proposition 25 *Let T be a commutative monad, and $A = (A, \alpha)$ and $B = (B, \beta)$ two T-algebras. Then the composite (33) is T-linear.*

Proof. In the diagram

$$\begin{tikzcd}
T(A \pitchfork B) \arrow[r, "T(st)"] \arrow[d, "\lambda"'] & T(TA \pitchfork TB) \arrow[r, "T(id \pitchfork \beta)"] \arrow[d, "\lambda"] & T(TA \pitchfork B) \arrow[d, "\lambda"] \\
A \pitchfork TB \arrow[r, "st"'] \arrow[d, "id \pitchfork \beta"'] & TA \pitchfork T^2B \arrow[r, "id \pitchfork T\beta"'] \arrow[d, shift right, "id \pitchfork \mu"'] \arrow[d, shift left, "id \pitchfork T\beta"] & TA \pitchfork TB \arrow[d, "id \pitchfork \beta"] \\
A \pitchfork B \arrow[r, "st"'] & TA \pitchfork TB \arrow[r, "id \pitchfork \beta"'] & TA \pitchfork B,
\end{tikzcd}$$

the vertical outer edges are the T-algebra structures on $A \pitchfork B$ and $TA \pitchfork B$, respectively, expressed in terms of the cotensorial strength λ. We are thus required to prove that the outer square commutes.Three of the inner squares commute for obvious reasons (ignore for the moment the arrow $id \pitchfork \mu$), but the upper left square does not. Now the associative law for the structure β allows us to replace the "doubled" arrow $id \pitchfork T\beta$ with $id \pitchfork \mu$. But for a commutative monad T, the upper left hand square postcomposed with $id \pitchfork \mu$ commutes; this condition is in fact equivalent to commutativity of T, as stated in [12] Definition 2.1 (in loc.cit., it is presented as an alternative equivalent definition of commutativity of T in terms of the cotensorial strength λ). From this follows that the outer diagram above commutes, and this proves the Proposition.

A concrete description of the monad T for vector spaces over $\mathbb{R}$ in this category was given in [1]. The authors in fact prove that it is"carved it out" by topological means from a Schwartz type double dualization monad, described also in [18]. They provide a categorical study of this monad along different lines than ours, namely in terms of an "exponential modality" (essentially the comonad ! considered in linear logic).

10 Probability theory

To justify some measure- and probability- theoretic terminology, one may think of an element $X \pitchfork T(1)$ not just as a "test function", in the sense of Schwartz distribution theory, but as a generalized "measurable subset" of X, or as a generalized "event" in the "outcome space" X. The connection is that a subset $X' \subset X$ (for suitable $\mathscr{E}$ and suitable T) gives rise to a function $X \to T(1)$, namely the characteristic function (whose value is 1 for $x \in X'$, and 0 else). Like for $X \pitchfork T(1)$, the set of subsets of X depends contravariantly on X, via inverse image formation. Instead of the T-linearity requirement for Schwartz distributions $X \pitchfork T(1) \to T(1)$, there are other well known algebraic requirements for measures, viewed as functions from the boolean algebra of subsets of X to the rig $T(1)$. This shall not concern us in detail here; the observation is just that test functions on X may be viewed as generalized measurable subsets/events in X, and thereby it gives us access to terminology and notions borrowed from measure theory or probability theory. We already anticipated this import of terminology when we, for $P \in T(R)$, used the word "expectation of P" for $\int_R x\, dP(x)$.

A strong monad T on a CCC $\mathscr{E}$ is called *affine* if $T(1) = 1$. For algebraic theories, this was introduced in [26]. For strong monads, it was proved in [13] that this is equivalent to the assertion that for all X,Y, the map $\psi_{X,Y} : T(X) \times T(Y) \to T(X \times Y)$ is split monic with $(T(pr_1), T(pr_2)) : T(X \times Y) \to T(X) \times T(Y)$ a retraction. In [21], it was proved that if $\mathscr{E}$ has finite limits, any commutative monad T has a maximal affine submonad T_0, the "affine part of T". It is likewise a commutative monad. Speaking in elementwise terms, $T_0(X)$ consists of those concrete distributions whose total is $1 \in T(1)$. We consider in the following a commutative monad T and its affine part T_0.

Probability distributions have by definition total $1 \in R$, (recall that R denotes the rig $T(1)$) and take values in the interval from 0 to 1. We do not in the present article consider any order relation on R, so there is no "interval from 0 to 1"; so we are stretching terminology a bit when we use the word "probability distribution on X" for the elements of $T_0(X)$, but we shall do so. So a "probability distribution" is here just a concrete distribution $P \in T(X)$ with total 1, or in the notation from Section 1,

$$\int_X 1_X \, dP(x) = 1$$

where $1_X : X \to R$ is the function with constant value $1 \in R$. Since the object 1 is terminal, it is clear that for any $f : X \to Y$, if $P \in T(X)$ is a probability distribution, then so is $T(f)(P) \in T(Y)$. (Alternatively: T_0 is a subfunctor of T.)

If $P \in T_0(X)$ and $Q \in T_0(Y)$, then $\psi(P,Q) \in T_0(X \times Y)$, cf. Proposition 12; this also follows since the inclusion of strong monads $T_0 \subseteq T$ is compatible with the monoidal structure

ψ. From this in turn follows that e.g. probability distributions are stable under convolution.

The assertion that $\psi_{X,Y}$ for the monad T_0 is split monic, quoted above, may in terminology from probability theory be rendered: "the distribution for independent random variables may be reconstructed from marginal distributions"; recall that if $Q \in T(X \times Y)$, then its marginal distributions are $T(pr_i)(Q)$ $(i = 1,2)$. If Q is a probability distribution, then so are its marginal distributions.

The subobject $T_0(X) \subseteq T(X)$ is clearly not stable under the multiplication by scalars $\lambda \in R$; in fact, formation of totals is the map $T(!) : T(X) \to T(1) = R$, hence is T-linear, and therefore commutes with multiplication by scalars. In particular, $T_0(X) \subseteq T(X)$ is not stable under multiplication $\vdash$ by functions $\phi \in X \pitchfork R$. However, this multiplication still plays a role in the formulation of probability theory presented here. Let $P \in T_0(X)$, and let $\phi \in X \pitchfork R$ be such that $\lambda := \int_X \phi(x)\, dP(x)$ is invertible in the multiplicative monoid of R. Then we have $P \vdash \phi \in T(X)$. We may form the element in $T(X)$

$$P_\phi := (P \vdash \phi) \cdot \lambda^{-1};$$

this is a probability distribution. For by Theorem 4, its total is calculated as λ^{-1} multiplied on

$$\int_X 1_X \; d(P \vdash \phi)(x) = \int_X 1_X \cdot \phi(x)\; dP(x) = \int_X \phi(x)\; dP(x) = \lambda.$$

So we get 1.

Let us think of ϕ in the above consideration as a (generalized) "event" A, writing A for ϕ; also, let us write $P(B)$ for $\int_X B(x)\, dP(x)$, for general $B \in X \pitchfork R$. Then we have $\lambda = P(A)$, and the value of $P \vdash A$ on the "event" B is $P(A \cdot B)$. Now $A \cdot B$ is the event $A \cap B$ (for the case of characteristic functions of subsets of X). So P_A is $P(A \cap B)/P(A)$, the classical "conditional probability of B given A".

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