TWO THEOREMS OF LIE ON INFINITESIMAL SYMMETRIES OF DIFFERENTIAL EQUATIONS

Dedicated to the memory of Bill Lawvere with thanks

ANDERS KOCK

Abstract

We give an account, in terms of synthetic differential geometry, of some of Sophus Lie's geometric theory of first order differential equations. This theory is, in modern terms, formulated in terms of vector fields on manifolds.

I owe credit to Gonzalo Reyes for long collaboration on the particular subject in the present note; some of this is documented in our joint preprint [10], as well as in our article [11].

1. Some algebra of nilpotent elements

We consider a commutative ring R. Recall that $a \in R$ is *nilpotent* if $a^k = 0$ for some natural number $k = 1, 2, 3, \ldots$. We are mainly concerned with the case $k = 2$, and define

$$D := \{d \in R \mid d^2 = 0\}.$$

The letter 'd' will hence forward be reserved to elements of D.

Note that commutativity of R implies that $d \in D \Rightarrow r \cdot d \in D$, for any $r \in R$, in particular $-d \in D$. So the subset D is closed under multiplication. However, it is not closed under addition; indeed $(d_1 + d_2)^2 = d_1^2 + d_2^2 + 2d_1 \cdot d_2 = 0 + 0 + 2d_1 \cdot d_2$, which is not necessarily 0.

The 2-dimensional analogue of $D \subseteq R$ is the subset $D(2) \subseteq D \times D$ given by

$$D(2) := \{(d_1, d_2) \in R \times R \mid d_1^2 = 0,\ d_2^2 = 0,\ d_1 \cdot d_2 = 0\}.$$

For x and y in R, we write $x \sim y$ if $(x - y)^2 = 0$. If R has the property that $x + x = 0$ implies $x = 0$ (which we henceforth assume), we therefore have

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1.1. PROPOSITION. *For d_1 and d_2 in D,*

$$d_1 + d_2 \in D \text{ iff } d_1 \cdot d_2 = 0 \text{ iff } d_1 - d_2 \in D \text{ iff } (d_1, d_2) \in D(2) \text{ iff } d_1 \sim d_2.$$

The n-dimensional analogue of $D(2)$ is $D(n) \subseteq D^n$, described by

$$D(n) := \{(d_1 \ldots, d_n) \in R^n \mid d_i \cdot d_j = 0 \text{ for all } i, j = 1, \ldots, n\}.$$

By binomial expansion, one sees that $(d_1, \ldots, d_n) \in D(n)$ implies $\sum d_i \in D$.

We shall not in the present note have occasion to consider nilpotent elements of higher degree, such as $D_2 \subseteq R$ given by $\{x \in R \mid x^3 = 0\}$. Note that we have $D \subseteq D_2$. Also note that, for $(d_1, d_2) \in D \times D$, we have $d_1 + d_2 \in D_2$; this follows by binomial expansion.

2. Synthetic differential geometry

We present a short review of the version of synthetic differential geometry (SDG) which will be used here; see e.g. [5] or [12]. It is an axiomatic theory dealing with a Cartesian closed category $\mathscr{E}$ equipped with a commutative ring object R. We think of R as the number line, or as the ring of scalars. We use language as if $\mathscr{E}$ were the category of sets. We sometimes call the objects of $\mathscr{E}$ *spaces*.

The use of nilpotent scalars provides a tool that allows definition of notions of "infinitesimals" of various order; thus the intuition behind $d \in D$ is that d is an infinitesimal of order 1 (and elements in D_2 are infinitesimals of order 2, etc.)

2.1. THE SDG AXIOMATICS. We shall need only one axiom scheme, Axioms 2.7 for $n = 1, 2, \ldots$. The $n = 1$ case is particularly important, and we state it separately: [1]

2.2. AXIOM. *For every $f : D \to R$, there exists unique a and b in R such that f is of the form $f(d) = a + d \cdot b$.*

In diagrammatic terms, this basic Axiom 2.2 can be expressed by saying that a certain map

$$\gamma : R \times R \to R^D$$

is an isomorphism. In set theoretic terms, this γ has the description,

$$(a, b) \mapsto [d \mapsto a + d \cdot b].$$

The assumption that $\mathscr{E}$ is Cartesian closed implies that this description does indeed make sense, with the displayed square bracket expression defining an element in the function space object R^D in $\mathscr{E}$.

Putting $d = 0$, one gets immediately that $a = f(0)$; the unique b may suggestively be denoted $f'(0)$. From uniqueness of b, one immediately gets that

$$(\forall d \in D : d \cdot b = 0) \Rightarrow b = 0 \tag{1}$$

[1] There is an axiom scheme, comprising all the various axioms that have been used in the development of SDG, one axiom for each "Weil algebra", see e.g. [5], [12] or [1].

("principle of cancelling universally quantified ds"), and hence, by iteration, one also gets

$$(\forall (d_1, d_2) \in D \times D : d_1 \cdot d_2 \cdot b = 0) \Rightarrow b = 0.$$

From the Axiom 2.2, one gets

2.3. PROPOSITION. *For any $f : D \times D \to R$,*

$$f(d_1, d_2) = f(0,0) + d_1 \cdot b_1 + d_2 \cdot b_2 + c \cdot d_1 \cdot d_2$$

for unique b_1, b_2 and $c \in R$.

A special case is:

2.4. PROPOSITION. *Any $f : D \times D \to R$ which for all $d \in D$ satisfies $f(d,0) = f(0,d) = f(0,0)$ is of the form $f(d_1, d_2) = g(d_1 \cdot d_2)$ for a unique function $g : D \to R$.*

PROOF. The assumptions give $b_1 = b_2 = 0$; and then the unique function claimed is given by $(d_1, d_2) \mapsto f(0,0) + c \cdot d$ (where c is as in Proposition 2.3). ■

We note that Proposition 2.4 implies the following parametrized version:

2.5. PROPOSITION. *A function $f : M \times D \times D \to R$ which for all $d \in D$ satisfies $f(m,d,0) = f(m,0,d) = f(m,0,0)$ is of the form*

$$f(m, d_1, d_2) = f(m,0,0) + g(m) \cdot d_1 \cdot d_2$$

for a unique function $g : M \to R$.

2.6. DEFINITION. *An object $M \in \mathscr{E}$ has the* property W *if any $f : D \times D \to M$ which satisfies $f(d,0) = f(0,d) = f(0,0)$, is of the form $f(d_1, d_2) = g(d_1 \cdot d_2)$ for a unique $g : D \to M$.*

Note that Proposition 2.4 says that the object R has property W.

We leave to the reader to prove that if M and N are objects which have the property W, then so does $M \times N$. In particular R^n has the property W. More generally, if M has property W, then so does M^X for any object X.

The basic Axiom 2.2 is the special case (for $n = 1$) of the following Axiom Scheme, whose nth case is:

2.7. AXIOM. *For every $f : D(n) \to R$, there exists unique $b_1, \ldots, b_n$ in R, such that f is of the form*

$$f(d_1, \ldots, d_n) = f(0, \ldots, 0) + d_1 \cdot b_1 + \ldots + d_n \cdot b_n.$$

From this axiom, one immediately gets (taking $n = 2$)

2.8. PROPOSITION. *Given maps $f_1 : D \to R$ and $f_2 : D \to R$ with $f_1(0) = f_2(0)$, there exists a unique $g : D(2) \to R$ with $g(-,0) = f_1$, and with $g(0,-) = f_2$.*

2.9. DEFINITION. *An object* $M \in \mathscr{E}$ *has the* property IL2 *if for* $\tau_1 : D \to M$ *and* $\tau_2 : D \to M$ *with* $\tau_1(0) = \tau_2(0)$*, there exists a unique* $g : D(2) \to M$ *with* $g(-,0) = \tau_1$ *and with* $g(0,-) = \tau_2$.

Proposition 2.8 says that R has property IL2. And just as for property W, we have that if M and N are objects which have the property IL2, then so does $M \times N$. In particular R^n has the property IL2. Also, if M has property IL2, then so does M^X for any object X.

There are completely analogous properties ILn for $n = 3, 4, \ldots$.

2.10. DEFINITION. *A* tangent vector τ *to a space* M *is a map* $D \to M$*; the point* $\tau(0) = m$ *is called the* base point *of* τ.

This is essentially a paraphrasing of Ehresmann's notion that a tangent vector at $m \in M$ is the 1-jet at 0 of a map $R \to M$.

2.11. TANGENT VECTORS AND TANGENT BUNDLES. The space of tangent vectors τ to M is the space M^D. It comes equipped with a base-point map $M^D \to M$ (or natural projection), with description $\tau \mapsto \tau(0)$. Writing $T(M)$ for M^D, we have the base point map $T(M) \to M$, making $T(M)$ the tangent *bundle* of M. The fibre over $m \in M$ in $T(M)$, i.e. the space of tangent vectors with base point m, is denoted $T_m(M)$.

There is an *action* of the multiplicative monoid of R on any $T_m(M)$, given by $(t \cdot \tau)(d) := \tau(t \cdot d)$ for $t \in R$. In fact, if M has properties IL2 and IL3, $T_m(M)$ is canonically an R-module, see e.g. [12] 3.1.1.

The functor T given by $T(M) := M^D$ is an endofunctor on $\mathscr{E}$. (Many differential geometric notions and arguments can be expresse entirely in terms of such endofunctor T, see Rosicky [19], and the Canadian school, see e.g. [2] and the references therein).

The property IL2 for an object M can be used to define addition of tangent vectors on M with same base point: define $\tau_1 + \tau_2$ by putting $(\tau_1 + \tau_2)(d) := g(d,d)$, where g is as in the Definition 2.9. If property IL3 holds for M, one can prove associativity of this addition; more completely (cf. e.g. [5] Proposition I.7.2):

2.12. PROPOSITION. *If M has the properties IL2 and IL3, $T_m(M)$ carries a canonical structure of R-module.*

2.13. DEFINITION. *An R-module V is called a* Euclidean *R-module (or a* KL vector space*) if it satisfies the following generalization of Axiom 2.2: For every* $f : D \to V$*, there is a unique* $b \in V$ *in R so that f is of the form* $f(d) = f(0) + d \cdot b$.

This unique b deserves the name $f'(0)$, or *the principal part* of f.

2.14. REMARK. The reason we consider the properties W and ILn is that they are coordinate free; they are basic for the early developments of SDG. They will be sufficient for the arguments we are to give for the Theorems of Lie referred to in the Introduction, as (partly) paraphrased in Theorems 3.10 and 5.5 below. The letter 'W' refers to Wraith, who made this property explicit (1972), as subsumed in [18]; the letters ILn stand for "Infinitesimal Linear", a notion originally made explicit by Bergeron (1980). The properties W and ILn are instances of a "property scheme", subsumed under the name "microlinearity"; see Appendix D in the 2006 edition of [5]. Common to all the microlinearity properties is that the axioms in the basic axiom scheme for SDG (as parametrized by the category of Weil algebras) imply that R satisfy all of these microlinearity properties. For a succinct account of Weil algebras and microlinearity, see Chapter 2 in [12].

3. "Vector fields = Infinitesimal transformations"

The notion of *transformation* is central in Lie's work. In modern terms, a transformation on a (smooth) manifold M is a diffeomorphism $M \to M$. They form a group $Aut(M)$, and although not finite dimensional, $Aut(M)$ allows for a rich differential geometry, relevant for instance to the theory of differential equations, cf. e.g. Lie's book [16]. In this book, he considers in particular *infinitesimal* transformations, reasoning synthetically and geometrically with infinitesimals; but (in a related paper) he admits that he found it "difficult" to "give a clear exposition on synthetic investigations" except by expressing them in analytic terms.[2] The present note is hopefully a contribution to give a synthetic exposition of some of these investigations, by utilizing concepts and methods that have developed since the time of Lie, notably category theoretic ones.

Lawvere's seminal 1967 conception (in [13]) was that *the category $\mathscr{E}$ of spaces should be Cartesian closed, and that the tangent bundle formation should be representable by an object D as the functor $(-)^D$.*

So the tangent bundle $T(M) = M^D$ is the space of maps $D \to M$. In the setup presented in Section 2 with the R and D as described there, there is a "base point" map: $M^D \to M$ (evaluation at $0 \in D$). This gives the projection map of the tangent bundle. So a vector field on M is a section $M \to M^D$ of the projection map. This is the formulation in (3) below.

But, as Lawvere pointed out, exponential adjointness then gives two further equivalent formulations of the notion of vector field on M:

$$M \times D \xrightarrow{\quad X \quad} M \qquad \text{with } X(m,0) = m; \tag{2}$$

$$M \xrightarrow{\quad \hat{X} \quad} M^D \qquad \text{with } \hat{X}(m)(0) = m; \tag{3}$$

$$D \xrightarrow{\quad \overline{X} \quad} M^M \qquad \text{with } \overline{X}(0) = \text{ id: } M \to M. \tag{4}$$

We shall also use the notation $\overline{X}(d) = X_d$, or equivalently $X_d(m) = X(m,d)$.

We shall mainly use the formulation (2), an *action* of D on M. The formulation (3) is the classical "section of the tangent bunde"; and (4) is the formulation which is considered by Lie, and he uses the name "infinitesimal transformation". In our formulation, this is justified by the fact that each individual X_d (for $d \in D$) is a map $M \to M$ bringing each m to an "infinitesimal

[2] A complete quotation of these statements of Lie may be found (in translation) in the preface to [5], and (in the original language) in [15], 1876. The latter is paraphrased in SDG terms in [9].

neighbour" of m, (written $X_d(m) \sim m$, see the Subsection 3.3). So in our exposition, it is the individual X_ds that are the infinitesimal transformations, where with Lie, it is the *collection* of all the X_ds, i.e. the map $\overline{X}$, which is an infinitesimal transformation.

3.1. PROPOSITION. *Let X be a vector field on M, and assume that M has the property IL2. Then for $(d_1, d_2) \in D(2)$, we have*

$$X(X(m,d_1),d_2) = X(m,d_1+d_2). \tag{5}$$

PROOF. Note that the right hand side makes sense, since $d_1 + d_2 \in D$ by Proposition 1.1 and the assumption $(d_1, d_2) \in D(2)$. Now use the uniqueness assertion in the property IL2 (Definition 2.9). ∎

The following is an immediate consequence:

3.2. PROPOSITION. *For all $d \in D$, we have $X(X(m,d),-d) = m$. In particular, the map $X_d : M \to M$ is invertible with inverse X_{-d}.*

3.3. A NEIGHBOUR RELATION $\sim$. A genuine differential-geometric theory of spaces M should preferably allow the use of some notion of (first order) *neighbour* relation between points of M. It should be a reflexive and symmetric relation. There are several ways for how this can be done, depending on the category $\mathscr{E}$ and the object in question. In algebraic geometry, one has the notion of *first neighbourhood of the diagonal* of an affine scheme; it is derived from the idea "$A \otimes A/I^2$," where A is the coordinate ring of the scheme, and $I \subseteq A \otimes A$ is the kernel of the multiplication map $A \otimes A \to A$.

In the setting of SDG, the relevant first order $\sim$ derives ultimately from D. Thus, on R itself, there is the canonical $\sim$, namely with $x \sim y$ iff $(x-y)^2 = 0$. Therefore, for d_1 and d_2 in D , we have $d_1 \sim d_2$ iff $d_1 - d_2 \in D$ iff $d_1 + d_2 \in D$ iff $d_1 \cdot d_2 = 0$, cf. Proposition 1.1.

For the purpose of the present article: for m_1 and m_2 points in a space M, a *sufficient* condition for $m_1 \sim m_2$ is that there exists a $\tau : D \to M$ and a $d \in D$ such that *either* $\tau(0) = m_1$ and $\tau(d) = m_2$, *or* $\tau(0) = m_2$ and $\tau(d) = m_1$. In this case, we say that $m_1 \sim m_2$ is *witnessed by* τ *and* d. Note that $\sim$ thus defined is a reflexive relation. The "*either... or ...*" in the definition is to ensure that $\sim$ is a symmetric relation. For $M = R$, $m \sim n$ is equivalent to $(m-n)^2 = 0$.

If $m_1 \sim m_2$ in M, witnessed by τ and $d \in D$, then for any map $f : M \to N$, we have $f(m_1) \sim f(m_2)$, witnessed by $f \circ \tau$ and d. For future reference, we record this fact:

3.4. PROPOSITION. *Any map $f : M \to N$ preserves the relation $\sim$. Hence any invertible map preserves and reflects $\sim$.*

A subspace M of a space V is called *formally open* if it is stable under the $\sim$-relation, i.e. if $m \sim n$ and $m \in M$ imply $n \in M$. Assume that V is a Euclidean R-module, and $M \subseteq V$ a formally open subspace. A tangent vector field X on such a $M \subseteq V$ is of the form $X(m,d) = m + d \cdot f(m)$, for $m \in M$ and with $f(m) \in V$, so $f(m)$ is the principal part of the tangent vector $X(m,-)$. So we have a map $f : M \to V$, called the *principal part function* of X. We call X a *proper* vector field if all values $f(m)$ of f are "cancellable" in the sense $\forall t \in R, t \cdot f(m) = 0$ implies that $f(m) = 0$. For instance, in R^n, any element (n-tuple) with at least one invertible entry, is proper.

More generally, for any space M, a tangent vector $\tau : D \to M$ is *proper* if it preserves and reflects $\sim$, and a vector field X on M is proper if for all $m \in M$, X_m is a proper tangent vector.

3.5. FLOW AND STREAMLINES OF A VECTOR FIELD. Given a vector field $X : M \times D \to M$. A map $F : M \times R \to M$, with $F(m,d) = X(m,d)$ for all $m \in M$ and $d \in D$, is called a *complete flow* of X, if it satisfies, for all t_1 and t_2 in R,

$$F(F(m,t_1),t_2) = F(m,t_1+t_2), \tag{6}$$

in particular, for $d_2 \in D$

$$X(F(m,t_1),d_2) = F(m,t_1+d_2) \tag{7}$$

and likewise $F(X(m,d_1),t_2) = F(m,d_1+t_2)$, for all $m \in M$ and for all t_1 and t_2 in R, and d_1 and d_2 in D, cf. [11]. Note that validity of the equation (5) is a necessary condition for the existence of such an extension F of X.

3.6. REMARK. Existence and uniqueness of a flow of a vector field in this sense is an integration question, and is therefore a question of, say, real analysis, or it may be posed axiomatically.

Example: the vector field on $M = R$ given by $X(m,d) = m + d \cdot m$ has the complete flow $F(m,t) = m \cdot e^t$ (provided that the category $\mathscr{E}$ contains the exponential function $e^t : R \to R$). Not all vector fields admit a complete flow $M \times R \to M$. For example, the vector field X on R, given by $X(m,d) = m + d \cdot m^2$, is an example of a vector field which does not admit a complete flow. So there are less ambitious notions of flow. An example of this is a map $F : M \times D_\infty \to R$ satisfying (6), where $D_\infty \subset R$ is the space of all nilpotent scalars, i.e. $t \in R$ with $t^k = 0$ for some natural number k. The equation makes sense, since D_∞ is a subgroup of the additive group of R. Call such a $F : M \times D_\infty \to R$ a *formal* flow. For M microlinear (in a suitably strong sense), formal flows always exist, and uniquely so, see [10] Theorem 2. This is in essence a solution in terms of a formal power series.

For simplicity of exposition, we shall only consider vector fields X on M which have a complete flow F. Such F is in fact unique, using an induction principle, essentially: if f' is constant 0, then f is constant. Then for fixed $m \in M$, the map $F(m,-) : R \to M$ is a parametrized curve, so it is of kinematic nature. Its image, as a subset of M, is called a *streamline* or *orbit* of X (or of F). By being unparametrized, it is of a geometric/static nature, and pictures can be drawn (cf. [16]). The family of streamlines of such X is the subject of Section 5.4 below.

The flow of a vector field X is in Lie's terminology called the "1-parameter group" ("eingliedriche Gruppe") generated by X.

3.7. DIRECTIONAL DERIVATIVES. We consider a Euclidean R-module V (cf. Definition 2.13). A tangent vector τ with base point $m \in V$ is of the form $d \mapsto m + d \cdot v$ for some unique $v \in V$. The vector v is called the *principal part* of τ. The basic Axiom 2.2 in SDG says that R itself is a Euclidean R-module.

A vector field X on a Euclidean R-module V (or on a formally open subset U of V) is therefore of the form

$$X(u,d) = u + d \cdot g(u)$$

for some unique $g : U \to V$, called the *principal part function* of the vector field.

If X is a vector field on M and $f : M \to V$ is a function to a Euclidean R-module V, one may construct a new function $\partial_X(f) : M \to V$, called the *directional derivative of f in the direction X*, namely: $\partial_X(f)$ maps $m \in M$ to the principal part of the tangent vector $f \circ X(m,-)$. So $\partial_X(f)(m)$ is characertized by the validity, of

$$\forall d \in D : f(X(m,d)) = f(m) + d \cdot \partial_X(f)(m). \tag{8}$$

Another commonly used notation for $\partial_X(f)$ is $X(f)$.

3.8. VECTOR FIELDS AS DIFFERENTIAL OPERATORS. For $V = R$, the differential operator $f \mapsto \partial_X(f)$ is a derivation in the algebraic sense. Lie calls this differential operator the *symbol* of X. Classically (for smooth manifolds), X can be reconstructed from its symbol, and one may for suitable spaces M *define* a vector field to be such a differential operator. This depends, however on existence of enough functions $M \to R$; and it is not a geometric viewpoint allowing pictures to be drawn. The description of the Lie bracket of two vector fields given in Sections 4 and 5 below does not use the "vector fields as differential operators" viewpoint, but is purely geometric.

3.9. LIE "THEOREM 7". Recall that a map $f : M \to U$ is a *regular epimorphism* if it is a coequalizer. Then it is also a coequalizer of its kernel pair $Ker(f)$, as displayed in the top of the following diagram; and f is said to *admit* an invertible $\xi : M \to M$ if ξ preserves the kernel pair of f (i.e. if the left hand square is "pairwise" commutative):

$$\begin{array}{ccccc} M \times_U M & \overset{Ker(f)}{\rightrightarrows} & M & \xrightarrow{\ f\ } & U \\ {\scriptstyle \xi \times_U \xi}\downarrow & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \widehat{\xi}} \\ M \times_U M & \underset{Ker(f)}{\rightrightarrows} & M & \xrightarrow[\ f\ ]{} & U \end{array}$$

Verbally in the category of sets: ξ maps each fibre (level set) of f to some fibre of f. In this case, ξ descends to a map $\widehat{\xi} : U \to U$ with $\widehat{\xi} \circ f = f \circ \xi$ as displayed in the right hand square, using the universal property of f as a coequalizer.

If $X : M \times D \to M$ is a vector field on M, we say that f *admits* X if f admits the infinitesimal transformations $X_d : M \to M$, for all $d \in D$, or that X is an *infinitesimal symmetry* of f. So for each $d \in D$, we have a similar diagram, with ξ replaced by X_d and $\widehat{\xi}$ replaced by $\widehat{X_d} : U \to U$:

$$\begin{array}{ccccc} M \times_U M & \overset{Ker(f)}{\rightrightarrows} & M & \xrightarrow{\ f\ } & U \\ {\scriptstyle X_d \times_U X_d}\downarrow & & \downarrow{\scriptstyle X_d} & & \downarrow{\scriptstyle \widehat{X_d}} \\ M \times_U M & \underset{Ker(f)}{\rightrightarrows} & M & \xrightarrow[\ f\ ]{} & U \end{array} \tag{9}$$

This family of maps $\widehat{X_d}$, as d ranges over D, may be subsumed in a map $\widehat{X} : U \times D \to U$,

$$\widehat{X}(v,d) := \widehat{X_d}(v).$$

Now X_0 is the identity map on M, so $\widehat{X_0}$ is the identity map of U. So $\widehat{X}$ is a vector field on U. Since U is a formally open subset $U \subseteq V$ of a Euclidean R-module V, $\widehat{X}$ is of the form

$$\widehat{X}(v,d) = v + d \cdot g(v)$$

for a unique $g : U \to V$, i.e. g is the principal part function of the vector field $\widehat{X}$ on U.

The following is (a contemporary formulation of) "Theorem 7" in Lie's [16], p. 91:

3.10. THEOREM. *Let $f : M \to U$ be a regular epimorphism, with U a formally open subspace of a Euclidean R-module V. If the kernel pair of f admits a vector field X, then there exists a function $g : U \to V$ (necessarily unique) so that $\partial_X(f) = g \circ f$.*

PROOF. The description of a vector field $\widehat{X}$ on U, and its principal part function $g : U \to V$, has been given above. We shall prove that this g satisfies $\partial_X(f)(m) = g(f(m))$ for each $m \in M$. For this, it suffices (by the principle of cancelling universally quantified ds) to see that for all $d \in D$, we have $d \cdot \partial_X(f)(m) = d \cdot g(f(m))$, or equivalently, by adding $f(m)$ to both sides, to prove

$$f(m) + d \cdot \partial_X(f)(m) = f(m) + d \cdot g(f(m)).$$

The left hand side is $f(X(m,d))$ by definition (8), and the right hand side is similarly $\widehat{X}(f(m),d)$; they agree by commutativity of the right hand square in (9). This proves the Theorem. ∎

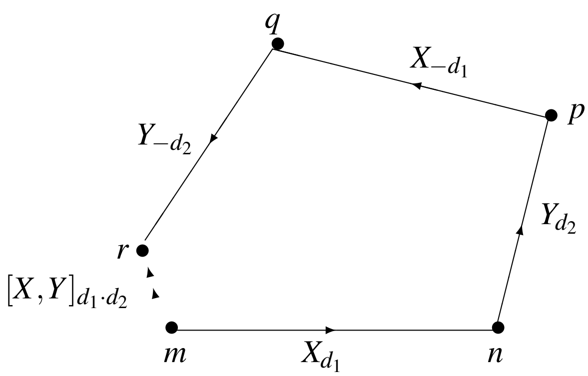
4. The Lie bracket of two vector fields

The property W for an object M (cf. Definition 2.6) is at the heart of the construction of the Lie bracket of two vector fields on M.

4.1. GROUP THEORETIC COMMUTATORS GIVE LIE BRACKETS. We consider two vector fields X and Y on a space M (where M is assumed to have the properties IL2 and W). Recall that X_d denotes the map $m \mapsto X(m,d)$. Thus X_d is a map $M \to M$ (for $d \in D$). It is invertible with inverse $(X_d)^{-1} = X_{-d}$, by Proposition 3.2. Following the paraphrasing of Lie provided by Reyes and Wraith in [18], we may therefore, for d_1 and d_2 in D, consider the group theoretic commutator of X_{d_1} and Y_{d_2}: composing from right to left, this is the map $M \to M$ given as

$$\{X_{d_1}, Y_{d_2}\} := Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1},$$

and its value on a given $m \in M$ is therefore the r, displayed in the following geometric figure. This "pentagonal" picture is (for fixed m) a member of a family of figures (parametized by $(d_1,d_2) \in D \times D$).



From [18], we quote

4.2. THEOREM. *Given two vector fields X and Y on M, there exists a unique vector field* $[X,Y]$ *on M such that, for any* $(d_1,d_2) \in D \times D$,

$$[X,Y]_{d_1 \cdot d_2} = \{X_{d_1}, Y_{d_2}\}. \tag{10}$$

So $[X,Y]_{d_1 \cdot d_2}(m) = r$, as indicated by the dotted arrow. Note that the bracket $[X,Y]$ is constructed without reference to vector fields as differential operators.

In this figure, each straight line connect neighbour points (in the sense of $\sim$), by construction, and the dotted arrow also connect two neighbour points, witnessed by $[X,Y]$ and $d_1 \cdot d_2$. But we in fact also have

4.3. PROPOSITION. *The points n and r are neighbours:* $n \sim r$.

PROOF. We have $p \sim q$. By Proposition 3.4, Y_{-d_2} preserves the neigbour relation $\sim$, so we get $n \sim r$. ■

5. Proper vector fields

5.1. PROPER TANGENT VECTORS. We say that $\tau : D \to M$ is a *proper* tangent vector if τ is a monic map and it preserves and reflects the relation $\sim$.

For a tangent vector τ on the Euclidean R-module R^k, it is clear that if the principal part of τ is a proper vector, then the tangent vector τ itself is proper. (A vector $(v_1, \ldots, v_k) \in R^k$ is proper if at least one of the v_is is invertible; this is a positive way of formulating that it is not the null vector.) A *proper vector field* on a space M is a vector field X where all the individual $X(m,-) : D \to M$ are proper.

Let X be a proper vector field on a space M. Let m and n be points on M. Then we say that n is an X*-neigbour* of m, if n is of the form $n = X(m,d)$ for some $d \in D$ (which is unique, by properness of X). In this case, we write $n \approx_X m$. This is a reflexive relation: $m \approx_X m$ since $X(m,0) = m$; if M has property IL2, it is also symmetric, because X_d has X_{-d} as inverse, by Proposition 3.2, so that $X(m,d) = n$ iff $X(n,-d) = m$. So we have

5.2. PROPOSITION. *Assume M has the property IL2. Then the relation $\approx_X$ is reflexive and symmetric. And $m \approx_X n$ implies $m \sim n$.*

The relation $\approx_X$ is not transitive in general. However, $\approx_X$ is "transitive relative to $\sim$"[3] in the sense of the following key Lemma

5.3. LEMMA. *Assume that X is a proper vector field on M, and that M has the property IL2. Then given m, n and r in M, we have*

$$(m \approx_X n \approx_X r \text{ and } m \sim r) \text{ implies } m \approx_X r.$$

PROOF. We have by assumption that $X(n,d_1) = m$ and $X(n,d_3) = r$ for some d_1 and $d_3 \in D$; (note that the d_1 here corresponds to $-d_1$ in the above pentagon picture); they are unique with this property, by properness of X. Trivially, we have $d_1 = d_3 + (d_1 - d_3)$. Now $X(n,-) : D \to M$ reflects the neighbour relation $\sim$, by properness of $X(n,-)$, so the assumption $m \sim r$ implies $d_1 \sim d_3$ in D, hence $d_1 - d_3 \in D$, by Proposition 1.1 Therefore, Proposition 3.1 implies the *-marked equality sign in

$$m = X(n,d_1) = X(n,d_3 + (d_1 - d_3)) \stackrel{*}{=} X(X(n,d_3), d_1 - d_3) = X(r, d_1 - d_3)$$

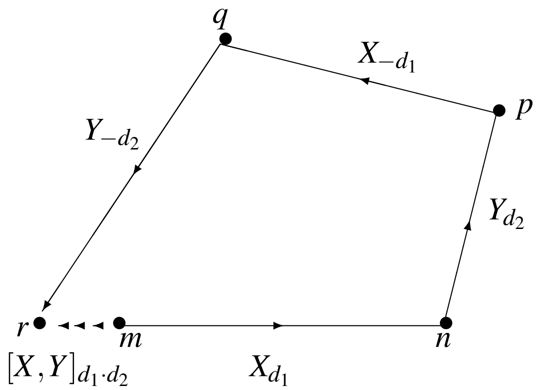
proving $m \approx_X r$, as witnessed by $d_2 := d_1 - d_3$. ∎

5.4. PERMUTING THE STREAMLINES. Let M be a space which has the properties IL2 and W. We consider two vector fields X and Y on M, with X proper. We shall study the question: when does X admit all the infinitesimal transformations Y_d belonging to Y, i.e. when do the Y_ds permute the streamlines (viewed as unparametrized subsets) of X. In [16] Theorem 9 (p. 105), Lie provides an answer. The geometric clue in the Theorem is the following infinitesimal version of it (seeing $\{n \in M \mid n \approx_X m\}$ as an infinitesimal part of a streamline of X).

5.5. THEOREM. *If X is a proper vector field, and Y is any vector field (both vector fields on M), then the following conditions are equivalent*
(i) each Y_d preserves the relation $\approx_X$,
(ii) $[X,Y] = g \cdot X$ for some $g : M \to R$ (which is unique, since X is proper).

PROOF. Assume (i). Consider the pentagon which is displayed below, so r is the value at m of the commutator $\{X_{d_1}, Y_{d_2}\}$.

[3] Such "relative transitivity" of a reflexive symmetric refinement $\approx$ of $\sim$ was studied in [6] to formulate a combinatorial/geometrical version of Frobenius integrability; the phrase "$\approx$ is involutive" was in loc.cit. used for such relative transitivity.



The five points depend on m, d_1 and d_2, thus in particular $r = r(m, d_1, d_2)$. We have $m \approx_X n$ and $q \approx_X p$, both assertions witnessed by $\pm d_1$. Since by assumption, Y_{-d_2} preserves $\approx_X$, and $q \approx_X p$, we have $r \approx_X n$. Finally, $m \sim r$, as witnessed by $[X,Y]$ and $d_1 \cdot d_2$. So Lemma 5.3 yields that $m \approx_X r$, so $r(m,d_1,d_2) = X(m, \delta(m,d_1,d_2))$ for some $\delta : M \times D \times D \to D$ (unique since X is proper). If $d_1 = 0$, δ returns 0, and similarly if $d_2 = 0$. So by Proposition 2.5, $\delta : M \times D \times D \to D \subseteq R$ is of the form $\delta(m,d_1,d_2) = g(m) \cdot d_1 \cdot d_2$ for some unique $g : M \to R$. We therefore have

$$[X,Y](m, d_1 \cdot d_2) = r(m,d_1,d_2) = X(m, g(m) \cdot d_1 \cdot d_2),$$

and cancelling the universally quantified d_1 and d_2, we get $[X,Y](m,-) = g(m) \cdot X(m,-)$. Since this holds for all $m \in M$, we have $[X,Y] = g \cdot X$.

Conversely, assume (ii). Let $q \approx_X p$ by virtue of d_1. Apply the transformation Y_{d_2}, to get points r and n. We shall prove $r \approx_X n$. Define m to be $X(n, -d_1)$, so $n = X(m, d_1)$. So we have $r = [X,Y](m, d_1 \cdot d_2)$, as in the figure. By assumption, this equals $X(m, g(m) \cdot d_1 \cdot d_2)$, hence $r \approx_X m$. Since $m \approx_X n$ by construction of m, and $r \sim n$ by Proposition 4.3, Lemma 5.3 gives $r \approx_X n$, as desired. ∎

5.6. STREAMLINES OF A COMPLETE VECTOR FIELD. Let M be a space having the property IL2. Let X be a complete and proper vector field on M. So it has a flow $F : M \times R \to M$. For any $m \in M$, we have the map $F(m,-) : R \to M$, the flow of m, generated by X. It is a "kinematic" entity, describing a motion of m. We want to describe its image $C(m) \subseteq M$, called the *streamline* of m. It is no longer kinematic, but a purely geometric entity. Likewise, the "infinitesimal" version of the flow of m, namely the subspace $\{n \in M \mid n \approx_X m\}$ is a geometric entity (even though it admits a parametrization by $X(m,-)$).

For this, it is convenient to introduce the notion of an *étale* map. Let R and M be two spaces, with R equipped with a symmetric reflexive relation $\sim$, and let M be equipped with a symmetric reflexive relation $\approx$.

5.7. DEFINITION. *A map $p : R \to M$ is* étale *if*

1) it has the preservation property that $s \sim t$ in R implies $p(s) \approx p(t)$ in M,

and

2) $p(s) \approx m \Rightarrow \exists! t \sim s$ with $p(t) = m$ (for s and t in R and m in M).

5.8. PROPOSITION. *Let X be a proper vector field on M, with a complete flow $F : R \to M$. Then for any $m \in M$, the image $C(m)$ of $F(m,-)$, as a subspace of M, admits a surjective $R \to M$ which is étale w.r.to the relations $\sim$ on R and $\approx_X$ on M.*

PROOF. The map $F(m,-)$ itself maps R onto its image C. Also, if $s \sim t$ in R, we have $t = s+d$ for some $d \in D$, by definition of $\sim$ on R; so $F(m,t) = F(m,s+d) = X(F(m,s),d)$, by the flow equation. Therefore $F(m,s) \approx_X F(m,t)$, witnessed by $X(F(m,s),-)$ and d. This is the required preservation property 1). To see 2), we have to that prove if $F(m,t) \approx_X n$ in C, then there exists a unique $d \in D$ such that $F(m,t+d) = n$. Let the assumed $F(m,t) \approx_X n$ be witnessed by $X(F(m,t),d) = n$. This d is unique since $X(F(m,t),-)$ is a proper tangent vector. But $F(m,t+d) = X(F(m,t),d)$ by the flow equation, and $X(F(m,t),d) = F(F(m,t),d)$, So there is a unique $d \in D$ with $F(F(m,t),d) = n$. ∎

5.9. REMARK. Note that the étaleness condition in some sense says that $F(m,-)$ is "locally bijective". However, one will not in general expect that it is globally bijective; it is well known that there are complete proper vector fields, where the streamlines of the flow are closed curves, e.g. on $M =$ a punctured plane, or on $M =$ a circle.

I conjecture that the property in Proposition 5.8 characterizes the streamlines of m under the flow F of X. And these conditions are expressed entirely in terms of $\approx_X$. So (under the two equivalent conditions stated in Proposition) the transformations Y_d preserve $\approx_X$, hence preserve the property of being a streamline. So assuming the conjecture holds, we can augment the Theorem 5.5 by a third equivalent condition:

(iii) The family of streamlines X *admits all transfornations* Y_d. This is Lie's "Theorem 9", [16] p. 105.

The proof of (iii) given in [4], Proposition 5, depends on existence of "integrals" of the vector field X, meaning functions $\phi : M \to R$ which are constant on the streamlines.

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Aarhus University, Dept. of Mathematics
Email: `kock@math.au.dk`