Local fibered right adjoints are polynomial

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Abstract. For any locally cartesian closed category $\mathcal{E}$, we prove that a local fibered right adjoint between slices of $\mathcal{E}$ is given by a polynomial. The slices in question are taken in a well known fibered sense.

Introduction

In a locally cartesian closed category $\mathcal{E}$, a diagram of the form

$$I \xleftarrow{\ s\ } E \xrightarrow{\ p\ } B \xrightarrow{\ t\ } J \tag{1}$$

gives rise to a so-called *polynomial functor* $\mathcal{E}/I \to \mathcal{E}/J$, namely the composite functor

$$\mathcal{E}/I \xrightarrow{\ s^*\ } \mathcal{E}/E \xrightarrow{\ p_*\ } \mathcal{E}/B \xrightarrow{\ t_!\ } \mathcal{E}/J. \tag{2}$$

In [2], six different intrinsic characterizations of polynomial functors are listed for the case where $\mathcal{E} = \mathbf{Set}$, one of them being that a functor $P : \mathbf{Set}/I \to \mathbf{Set}/J$ is polynomial if and only if it is a local right adjoint, i.e. the slice of P at the terminal object of $\mathbf{Set}/I$ is a right adjoint. Already for $\mathcal{E}$ a presheaf topos, this characterization fails, as pointed out by Weber [7]. His counter-example is a significant one: the free-category monad on the category of graphs is a local right adjoint, but it is not polynomial.

In this note we adjust the local-right-adjoint characterization so as to be valid in every locally cartesian closed category; for this we pass to the setting of fibered slice categories and fibered functors. The *fibered slice* $\mathcal{E}|I$ is the fibered category over $\mathcal{E}$ whose fiber over an object K is the plain slice $\mathcal{E}/(I \times K)$. A diagram as (1) defines also a fibered polynomial functor

$$\mathcal{E}|I \xrightarrow{s^*} \mathcal{E}|E \xrightarrow{p_*} \mathcal{E}|B \xrightarrow{t_!} \mathcal{E}|J,$$

whose 1-fiber is the plain polynomial functor (2). Hence any polynomial functor has a canonical extension to a fibered functor.

Our main theorem is this: *a fibered functor $P : \mathcal{E}|I \to \mathcal{E}|J$ is polynomial if and only if it is a local fibered right adjoint.*

1 Fibered categories and fibered slices

This section is mainly to fix terminology and notation. We refer to [1], [4], [6] further background on fibered categories.

A category $\mathcal{E}$ is fixed throughout; for the main result, it should be a locally cartesian closed category (lccc) with a terminal object. For some of the considerations, it suffices that it is a category with finite limits, as in [6]. We also assume we have chosen pullbacks once and for all, so that for each arrow $a : J \to I$ we dispose of two functors

$$a_! : \mathcal{E}/J \longrightarrow \mathcal{E}/I \qquad a^* : \mathcal{E}/I \longrightarrow \mathcal{E}/J,$$

and if $\mathcal{E}$ is furthermore a lccc, the a^*s have right adjoints

$$a_* : \mathcal{E}/J \longrightarrow \mathcal{E}/I$$

which we also assume are chosen once and for all; so

$$a_! \dashv a^* \dashv a_*.$$

1.1 Fibered categories. We shall work with categories fibered over $\mathcal{E}$, henceforth just called fibered categories. These form a 2-category $\mathbf{Fib}_{\mathcal{E}}$ whose objects are fibered categories, whose 1-cells are fibered functors (i.e. functors commuting with the structure functor to $\mathcal{E}$ and sending cartesian arrows to cartesian arrows), and whose 2-cells are fibered natural transformations (i.e. natural transformations whose components are vertical arrows). If $\mathcal{F}$ is a fibered category, we denote by $\mathcal{F}^I$ the (strict) fiber over an object $I \in \mathcal{E}$, and if $L : \mathcal{G} \to \mathcal{F}$ is a fibered functor, we write $L^I : \mathcal{G}^I \to \mathcal{F}^I$ for the induced functor between the I-fibers.

1.2 Cleavages. A cleavage of a fibered category $\mathcal{F}$ amounts to giving for each arrow $a : J \to I$ in $\mathcal{E}$ a functor $a^* : \mathcal{F}^I \to \mathcal{F}^J$, which we call *base change*. This is the same as choosing cartesian lifts. We cannot assume that the composite of two base-change functors is again a base-change functor. For the same reason, for a fibered functor $L : \mathcal{G} \to \mathcal{F}$, the functors induced on fibers do not commute strictly with base-change functors: rather there are canonical invertible 2-cells (fibered natural transformations) of the form

$$\begin{array}{ccc} \mathcal{G}^I & \xrightarrow{L^I} & \mathcal{F}^I \\ {\scriptstyle a^*}\downarrow & L^a \nearrow & \downarrow{\scriptstyle a^*} \\ \mathcal{G}^J & \xrightarrow[L^J]{} & \mathcal{F}^J \end{array} \tag{3}$$

whose components are induced by the universal property of cartesian arrows (which is what ensures the required coherence).

1.3 Fibered adjunctions. An adjunction between two fibered functors in opposite directions is given by two fibered natural transformations $\eta : \mathrm{id} \Rightarrow R \circ L$ and $\epsilon : L \circ R \Rightarrow \mathrm{id}$ satisfying the usual triangle identities. In other words, it is an adjunction in the 2-category $\mathbf{Fib}_{\mathcal{E}}$. It is clear that a fibered adjunction induces a fiber-wise adjunction each fiber. Conversely (cf. e.g. [1], 8.4.2), if $R : \mathcal{F} \to \mathcal{G}$ is a fibered functor and for each $I \in \mathcal{E}$ there is a left adjoint $L^I \dashv R^I$, then these L^I assemble into a fibered left adjoint if for every arrow $a : J \to I$ in $\mathcal{E}$, the mate of the canonical invertible 2-cell

$$\begin{array}{ccc} \mathcal{F}^I & \xrightarrow{R^I} & \mathcal{G}^I \\ {\scriptstyle a^*}\downarrow & (R^a)^{-1} \swarrow & \downarrow{\scriptstyle a^*} \\ \mathcal{F}^J & \xrightarrow[R^J]{} & \mathcal{G}^J, \end{array}$$

namely

$$\begin{array}{ccc} \mathcal{F}^I & \xleftarrow{L^I} & \mathcal{G}^I \\ {\scriptstyle a^*}\downarrow & \nwarrow & \downarrow{\scriptstyle a^*} \\ \mathcal{F}^J & \xleftarrow[L^J]{} & \mathcal{G}^J \end{array}$$

is again invertible, and hence turns the family L^I into a fibered functor.

1.4 Bifibrations. A fibered category $\mathcal{F}$ is called *bifibered* if the structure functor $\mathcal{F} \to \mathcal{E}$ is also an opfibration, i.e. has all opcartesian lifts, or equivalently, cobase-change functors, which we denote by lowershriek. In a bifibered category $\mathcal{F}$, cobase change (lowershriek) is left adjoint to base change (upperstar). Indeed, for $a : J \to I$ in $\mathcal{E}$ we have natural bijections

$$\mathcal{F}^J(T, a^*X) \simeq \mathcal{F}^a(T, X) \simeq \mathcal{F}^I(a_! T, X)$$

according to the universal properties of cartesian and opcartesian arrows.

A fibered functor is called *bifibered* if it preserves also opcartesian arrows. In terms of base-change functors, we can say that a fibered functor $L : \mathcal{G} \to \mathcal{F}$ is bifibered if for every arrow $a : J \to I$ in $\mathcal{E}$, the mate of the compatibility-with-base-change square (3):

$$\begin{array}{ccc} \mathcal{G}^I & \xrightarrow{L^I} & \mathcal{F}^I \\ {\scriptstyle a_!}\uparrow & \nwarrow \overline{L}^a & \uparrow{\scriptstyle a_!} \\ \mathcal{G}^J & \xrightarrow[L^J]{} & \mathcal{F}^J \end{array} \tag{4}$$

is invertible.

1.5 $\mathcal{E}$-indexed sums. A fibered category $\mathcal{F}$ is said to have *$\mathcal{E}$-indexed sums* when $\mathcal{F} \to \mathcal{E}$ is bifibered, and the *Beck-Chevalley condition* holds. This condition is easiest to state if we assume chosen cartesian (resp. opcartesian) lifts denoted by upperstar (resp. lowershriek): then the Beck-

Chevalley condition says that for every pullback square in $\mathcal{E}$

$$\begin{array}{ccc} \cdot & \xrightarrow{b} & \cdot \\ {\scriptstyle u}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle v} \\ \cdot & \xrightarrow[a]{} & \cdot \end{array}$$

the fibered natural transformation

$$u_! b^* \Rightarrow a^* v_!$$

is invertible.

1.6 Proposition. *A fibered left adjoint between bifibered categories preserves cobase change (in particular, preserves $\mathcal{E}$-indexed sums if the categories have such).*

Proof. Take left adjoints of all the arrows in the base-change compatibility square (3) for the right adjoint. □

1.7 Fibered slices. The *fibered slice* $\mathcal{E}|I$ is the category whose objects are spans

$$I \xleftarrow{p} M \xrightarrow{q} K,$$

and whose morphisms are diagrams

$$\begin{array}{cccc} & M' & \xrightarrow{q'} & K' \\ {\scriptstyle p'}\swarrow & \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle w} \\ I \xleftarrow{\;p\;} & M & \xrightarrow[q]{} & K. \end{array} \tag{5}$$

The structural functor $\mathcal{E}|I \to \mathcal{E}$ that returns the right-most object (resp. arrow) is a bifibration. The cartesian arrows are the diagrams for which the square is a pullback, while the opcartesian arrows are those for which v is invertible. The vertical arrows are those for which w is an identity arrow.

More conceptually, the fibered slice is obtained from the plain slice as the following pullback, and the structural functor as the left-hand vertical composite:

$$
\begin{array}{ccc}
\mathcal{E}|I & \longrightarrow & \mathcal{E}/I \\
\big\downarrow & \lrcorner & \big\downarrow \text{dom} \\
\mathrm{Ar}(\mathcal{E}) & \xrightarrow{\ d\ } & \mathcal{E} \\
c \big\downarrow & & \\
\mathcal{E}. & &
\end{array}
$$

(Here $\mathrm{Ar}(\mathcal{E})$ is the category of arrows in $\mathcal{E}$, and d and c are the domain and codomain fibrations, respectively.) This is to say that the fibered slice is the so-called family fibration of the fibration $\mathrm{dom} : \mathcal{E}/I \to \mathcal{E}$, cf. [6], 6.2.

For the K-fiber of $\mathcal{E}|I$ we have the canonical identification

$$(\mathcal{E}|I)^K \simeq \mathcal{E}/(I \times K).$$

In particular, the plain slice $\mathcal{E}/I$ sits inside the fibered slice $\mathcal{E}|I$ as the fiber over the terminal object $1 \in \mathcal{E}$:

$$(\mathcal{E}|I)^1 \simeq \mathcal{E}/I.$$

Note also that we have $\mathcal{E}|1 \simeq \mathrm{Ar}(\mathcal{E})$.

In the I-fiber of $\mathcal{E}|I$ we have the canonical object given by the identity span

$$I \xleftarrow{\ \mathrm{id}\ } I \xrightarrow{\ \mathrm{id}\ } I.$$

which, in view of the interpretation in terms of plain slices, we denote by δ (for "diagonal").

We note that $\mathcal{E}|I$ has $\mathcal{E}$-indexed sums: for $a : K' \to K$ in $\mathcal{E}$, the base-change functor $a^* : (\mathcal{E}|I)^K \to (\mathcal{E}|I)^{K'}$ is identified with $(a \times \mathrm{id}_I)^* : \mathcal{E}/(I \times K) \to \mathcal{E}/(I \times K')$ which has left adjoint $(a \times \mathrm{id}_I)_!$, and the Beck-Chevalley condition follows from the case of plain slices.

1.8 Polynomial functors — fibered version. Each arrow $a : J \to I$ in $\mathscr{E}$ induces fibered functors (the third provided $\mathscr{E}$ is a lccc)

$$a_! : \mathscr{E}|J \longrightarrow \mathscr{E}|I \qquad a^* : \mathscr{E}|I \longrightarrow \mathscr{E}|J \qquad a_* : \mathscr{E}|J \longrightarrow \mathscr{E}|I$$

and fibered adjunctions

$$a_! \dashv a^* \dashv a_*.$$

These extend the basic functors on plain slices: for example if $a^* : \mathscr{E}/I \to \mathscr{E}/J$ is the plain pullback, then the K-fiber of the fibered pullback functor a^* is

$$(a \times \mathrm{id}_K)^* : \mathscr{E}/(I \times K) \to \mathscr{E}/(J \times K).$$

A fibered functor of the form

$$\mathscr{E}|I \xrightarrow{\ s^*\ } \mathscr{E}|E \xrightarrow{\ p_*\ } \mathscr{E}|B \xrightarrow{\ t_!\ } \mathscr{E}|J$$

for a diagram in $\mathscr{E}$

$$I \xleftarrow{\ s\ } E \xrightarrow{\ p\ } B \xrightarrow{\ t\ } J$$

is called a (fibered) *polynomial functor*.

2 Fibered left adjoints

In this section, the base category $\mathscr{E}$ is just assumed to have finite limits.

Recall that $\delta \in \mathscr{E}|I$ denotes the identity span $I \xleftarrow{\ \mathrm{id}\ } I \xrightarrow{\ \mathrm{id}\ } I$.

2.1 Main lemma. *Let $\mathscr{F}$ be a fibered category with $\mathscr{E}$-indexed sums. For each $I \in \mathscr{E}$, the functor*

$$\begin{array}{rcl} \mathrm{ev}_\delta : \mathbf{BiFib}_{\mathscr{E}}(\mathscr{E}|I, \mathscr{F}) & \longrightarrow & \mathscr{F}^I \\ L & \longmapsto & L(\delta) \end{array}$$

is an equivalence of categories.

Proof. Relative to chosen base-change functors (upperstar) and cobase-change functors (lowershriek), an explicit pseudo-inverse is given by

$$\begin{array}{rcl} h : \mathcal{F}^I & \longrightarrow & \mathbf{BiFib}_{\mathcal{E}}(\mathcal{E}|I, \mathcal{F}) \\ X & \longmapsto & [\langle p, q\rangle \mapsto q_! p^* X]. \end{array}$$

Here the input span to the functor $h(X)$ is of the form

$$I \xleftarrow{\;p\;} M \xrightarrow{\;q\;} K, \tag{6}$$

so that applying $h(X)$ to it gives an object in $\mathcal{F}^K$. It is routine to check that this assignment is well-defined and functorial, cf. below. (This check depends on the Beck-Chevalley condition, so it would not work by just requiring $\mathcal{F}$ to be a bifibration.) It is clear that for $X \in \mathcal{F}^I$, applying first h and then ev_δ yields X again, since evaluating $\langle p, q\rangle \mapsto q_! p^* X$ at the identity span gives just X again (up to canonical coherent isomorphism). On the other hand, start with $L \in \mathbf{BiFib}_{\mathcal{E}}(\mathcal{E}|I, \mathcal{F})$, evaluate at δ and apply h. The result is the functor $\langle p, q\rangle \mapsto q_! p^* L(\delta)$, and we need to establish an isomorphism of this object with $L(\langle p, q\rangle)$, natural in $\langle p, q\rangle$. With $\langle p, q\rangle$ as in (6), consider the diagram

$$\begin{array}{ccc} (\mathcal{E}|I)^I & \xrightarrow{\;L^I\;} & \mathcal{F}^I \\ {\scriptstyle p^*}\downarrow & L^p & \downarrow{\scriptstyle p^*} \\ (\mathcal{E}|I)^M & \xrightarrow{\;L^M\;} & \mathcal{F}^M \\ {\scriptstyle q_!}\downarrow & \overline{L}^q & \downarrow{\scriptstyle q_!} \\ (\mathcal{E}|I)^K & \xrightarrow[\;L^K\;]{} & \mathcal{F}^K. \end{array}$$

Here L^p is the invertible 2-cell expressing that L is fibered, cf. (3), and $\overline{L}^q$ is the invertible 2-cell expressing that L preserves $\mathcal{E}$-indexed sums, cf. (4). Evaluating at the object $\delta \in (\mathcal{E}|I)^I$ and going clockwise around the diagram yields $q_! p^* L^I(\delta)$, while going counter-clockwise gives $L^K q_! p^*(\delta) = L^K(\langle p, q\rangle)$, as we wanted to prove. □

2.2 Remark. We also have the well known fibered Yoneda lemma

$$\mathbf{Fib}_{\mathcal{E}}(\mathcal{E}/I, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}^I,$$

valid for any fibered category $\mathcal{F}$. When $\mathcal{F}$ has $\mathcal{E}$-indexed sums, we can compose this equivalence with that of the Main Lemma to conclude that $\mathcal{E}|I$ is the $\mathcal{E}$-indexed-sum completion of $\mathcal{E}/I$, which is well-known (cf. e.g. [4], B1.4.16), noting that $\mathcal{E}|I = \mathrm{Fam}(\mathcal{E}/I)$. The Main Lemma could be deduced from these facts, but we have preferred to give the direct proof since we need the explicit construction in it.

2.3 Details regarding the pseudo-inverse. The first thing to check is that for fixed $X \in \mathcal{F}^I$, the assignment $\langle p, q \rangle \mapsto q_! p^* X$ is indeed a functor. We give the value on vertical and cartesian arrows separately. For a vertical arrow in $\mathcal{E}|I$ (i.e. a diagram (5) with w an identity), the required arrow $q'_! p'^* X \to q_! p^* X$ is an instance of a counit for $v_! \dashv v^*$:

$$q'_! p'^* X \simeq q_! v_! v^* p^* X \xrightarrow{\epsilon} q_! p^* X;$$

this is a vertical arrow in $\mathcal{F}$ since we are dealing with a fibered adjunction. For a cartesian arrow in $\mathcal{E}|I$ (i.e. a diagram (5) with cartesian square), the argument is the same, but using also a Beck-Chevalley isomorphism (this is where we use the assumption that $\mathcal{F}$ has $\mathcal{E}$-indexed sums) and a cartesian lift of w (displayed as the last arrow):

$$q'_! p'^* X \simeq q'_! v^* p^* X \overset{\mathrm{BC}}{\simeq} w^* q_! p^* X \longrightarrow q_! p^* X.$$

It is clear that composition and identity arrows are respected, so we do indeed have a functor.

Next we check that the functor sends cartesian arrows to cartesian arrows, and opcartesian to opcartesian. We already described the image of a cartesian arrow: it is a composite of isomorphisms with a cartesian lift of j, hence is cartesian. The image of an opcartesian arrow in $\mathcal{E}|I$ (i.e. a diagram (5) where v is invertible), is this:

$$q'_! p'^* X \longrightarrow w_! q'_! p'^* X \simeq w_! q'_! v^* p^* X \simeq q_! v_! v^* p^* X \simeq q_! p^* X,$$

which is an opcartesian lift of j. So the assignment $\langle p, q \rangle \mapsto q_! p^* X$ is indeed a bifibered functor.

Finally we check that the construction is functorial in X: consider an arrow $\phi : X \to Y$ (in $\mathcal{F}^I$). The image is the fibered natural transformation whose component at $\langle p, q\rangle$ is

$$q_! p^* X \xrightarrow{q_! p^* \phi} q_! p^* Y.$$

2.4 Corollary to the Main Lemma. *If $L : \mathcal{E}|I \to \mathcal{E}|J$ is a fibered functor that preserves $\mathcal{E}$-indexed sums, then it is isomorphic, as a fibered functor, to $q_! \circ p^*$ for the span $I \xleftarrow{p} M \xrightarrow{q} J$ obtained as $L(\delta)$.*

Proof. Take $\mathcal{F} = \mathcal{E}|J$ in the Main Lemma, and let $\langle p, q\rangle := L(\delta)$. To show that $L \simeq q_! \circ p^*$, the lemma implies that it is enough to check at the object $\delta \in \mathcal{E}|I$, for which it is obvious from the definition of p and q. □

2.5 Corollary. *If $L : \mathcal{E}|I \to \mathcal{E}|J$ is a fibered functor that preserves $\mathcal{E}$-indexed sums as well as terminal objects, then it is isomorphic, as a fibered functor, to p^*, for some $p : J \to I$.* □

We record the following special case of Corollary 2.4:

2.6 Theorem. *If $L : \mathcal{E}|I \to \mathcal{E}|J$ is a fibered left adjoint, it is obtained from a span $I \xleftarrow{p} M \xrightarrow{q} J$, as $L \simeq q_! \circ p^*$.* □

3 Local fibered right adjoints

3.1 Local right adjoints. If $\mathcal{F}$ is a category with a terminal object $1_{\mathcal{F}}$, and $R : \mathcal{F} \to \mathcal{G}$ is a functor, there is a well known canonical factorization of R

$$\mathcal{F} \xrightarrow{\overline{R}} \mathcal{G}/R(1_{\mathcal{F}}) \xrightarrow{\text{dom}} \mathcal{G} \tag{7}$$

where $\overline{R}$ takes $A \in \mathcal{F}$ to the value of R on $A \to 1_{\mathcal{F}}$. One says that R is a *local right adjoint* if $\overline{R}$ is a right adjoint. (This implies that all the evident functors $\mathcal{F}/X \to \mathcal{G}/R(X)$ also have right adjoints, but we shall not use this.)

3.2 Local fibered right adjoints. Let $\mathcal{F}$ and $\mathcal{G}$ be fibered over $\mathcal{E}$, assume that $\mathcal{F}$ has a terminal object $1_{\mathcal{F}}$, and let $R : \mathcal{F} \to \mathcal{G}$ be a fibered functor. Then the factorization (7) is actually a factorization of fibered functors, where $\mathcal{G}/R(1_{\mathcal{F}})$ is fibered over $\mathcal{E}$ via the fibration dom. If in this situation $\overline{R}$ is a fibered right adjoint, we say that R is a *local fibered right adjoint*. (In particular, R is then a (plain) local right adjoint.)

In the case where $\mathcal{G}$ is of the form $\mathcal{E}|J$, we shall see that the middle object $(\mathcal{E}|J)/R(1_{\mathcal{F}})$ is itself a fibered slice. For this we need a little preparation:

3.3 Plain slices of fibered slices. In addition to the structural functor $\mathcal{E}|J \to \mathcal{E}$ (which given a span $J \leftarrow M \to K$ returns K), we have also the "apex" functor (also a fibration)

$$\begin{array}{rcl} d : \mathcal{E}|J & \longrightarrow & \mathcal{E} \\ [J \leftarrow M \to K] & \longmapsto & M. \end{array}$$

For a fixed span $(J \xleftarrow{t} M \to K) = Q \in \mathcal{E}|J$, there is induced a forgetful fibered functor

$$(\mathcal{E}|J)/Q \longrightarrow \mathcal{E}|d(Q)$$

which sends an object

$$\begin{array}{ccccc} & & Y & \longrightarrow & X \\ J & \swarrow \nwarrow^{t} & \downarrow & & \downarrow \\ & & M & \longrightarrow & K. \end{array}$$

to the span

$$M \longleftarrow Y \longrightarrow X.$$

This functor is the first leg of a factorization of the domain functor dom:

$$\begin{array}{ccc} (\mathcal{E}|J)/Q & \xrightarrow{\text{dom}} & \mathcal{E}|J \\ & \searrow \qquad \nearrow_{t_!} & \\ & \mathcal{E}|d(Q) & \end{array}$$

3.4 Lemma. *With notation as above, when Q belongs to the 1-fiber for the structural fibration (i.e. is of the form $J \leftarrow M \to 1$), the forgetful functor*

$$(\mathcal{E}|J)/Q \longrightarrow \mathcal{E}|d(Q)$$

is an equivalence of fibered categories over $\mathcal{E}$. $\square$

We now return to the factorization (7) as fibered functors, and take $\mathcal{F} = \mathcal{E}|I$ and $\mathcal{G} = \mathcal{E}|J$. Note that $\mathcal{E}|I$ has a terminal object 1_I, namely the span $I \stackrel{\mathrm{id}_I}{\leftarrow} I \to 1$. It belongs to the 1-fiber, and hence so does $Q := R(1_I)$. Combining the above discussion with the Lemma, we arrive at this:

3.5 Corollary. *Any fibered functor $R : \mathcal{E}|I \to \mathcal{E}|J$ has a canonical factorization by fibered functors*

$$\mathcal{E}|I \xrightarrow{\overline{R}} \mathcal{E}|B \xrightarrow{t_!} \mathcal{E}|J, \tag{8}$$

with $\overline{R}$ a fibered right adjoint. Here $(J \stackrel{t}{\leftarrow} B \to 1) := R^1(I \stackrel{\mathrm{id}_I}{\leftarrow} I \to 1)$. □

We can now prove the Theorem announced in the title. Let $\mathcal{E}$ be a locally cartesian closed category with a terminal object.

3.6 Theorem. *If $R : \mathcal{E}|I \to \mathcal{E}|J$ is a local fibered right adjoint, then it is a polynomial functor (in the sense of 1.8).*

Proof. We need to construct the polynomial

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J \tag{9}$$

representing R, i.e. such that $R \simeq t_! \circ p_* \circ s^*$. By Corollary 3.5 we have $R = t_! \circ \overline{R}$, where $\overline{R} : \mathcal{E}|I \to \mathcal{E}|B$ has a fibered left adjoint L. Explicitly, B and t are determined by

$$(J \stackrel{t}{\leftarrow} B \to 1) \;:=\; R^1(I \stackrel{\mathrm{id}_I}{\leftarrow} I \to 1).$$

By Main Lemma we can write $L \simeq s_! \circ p^*$, hence $\overline{R} \simeq p_* \circ s^*$. The proof of Main Lemma gives the maps s and p explicitly:

$$(I \stackrel{s}{\leftarrow} E \stackrel{p}{\to} B) \;:=\; L^B(B \stackrel{\mathrm{id}_B}{\leftarrow} B \stackrel{\mathrm{id}_B}{\to} B).$$

Altogether we have $R \simeq t_! \circ p_* \circ s^*$ as claimed. □

3.7 Remarks. The converse of the theorem is also true: (fibered) polynomial functors are always local fibered right adjoints. Indeed, if $P : \mathcal{E}|I \to \mathcal{E}|J$ is the fibered functor $P = t_! \circ p_* \circ s^*$, then we have $\overline{P} = p_* \circ s^*$ with fibered left adjoint $s_! \circ p^*$.

It should also be noted that the diagram (9) "representing" a local fibered right adjoint is essentially unique. This follows from Theorem 2.17 in [2]. That theorem establishes a biequivalence between a bicategory whose 1-cells are "polynomials" (1) and a 2-category whose 1-cells are "polynomial functors" (i.e. functors that are isomorphic to one given by a polynomial, and hence have a canonical extension to a fibered functor). Theorem 3.6 gives an intrinsic characterization of this essential image.

3.8 Example: $\mathcal{E}$ = Set. If $\mathcal{E}$ is the category of sets, any category $\mathcal{F}$ can canonically be seen as the 1-fiber of a category fibered over $\mathcal{E}$, namely the category whose I-fiber is the category of I-indexed families of objects in $\mathcal{F}$; and any functor $\mathcal{F} \to \mathcal{G}$ extends canonically to a fibered functor. One often expresses this by saying "any category $\mathcal{F}$ 'is' fibered over Sets, and any functor 'is' fibered". So rather than considering local fibered right adjoints, one can consider just local right adjoints $\mathcal{E}/I \to \mathcal{E}/J$ and prove that they are given by a polynomial (1); see e.g. [2].

If $\mathcal{E}$ is a more general topos, functors $\mathcal{E}/I \to \mathcal{E}/J$ need not "be" fibered, not even for $I = J = 1$, i.e. they may not be the 1-fibers of a fibered functor $\mathcal{E}|I \to \mathcal{E}|J$, as the following examples show.

3.9 Example (Weber). Let $\mathcal{E}$ be the category of (irreflexive) directed graphs, i.e. the presheaf category of $(0 \rightrightarrows 1)$, and let $T : \mathcal{E} \to \mathcal{E}$ be the free-category monad. Weber observes that T is a local right adjoint but not a polynomial functor. The argument (given in detail in [7], Example 2.5) amounts to showing that the left adjoint to $\overline{T}$ does not preserve monos. In contrast, for a polynomial functor $P = t_! p_* s^*$, the left adjoint to $\overline{P} = p_* s^*$ is $s_! p^*$, which does preserve monos (as does every polynomial functor). It follows a posteriori from our Main Theorem that T cannot be the 1-fiber of a local fibered right adjoint $\mathcal{E}|1 \to \mathcal{E}|1$.

3.10 Example. Let $\mathcal{E}$ be the category of G-sets, where G is a non-trivial group. The group homomorphism $p : G \to 1$ induces functors $p_! \dashv p^* \dashv p_*$, where p^* applied to a set S makes it into a G-set, with trivial G-action. (The functors $p_!$ and p_* may be seen as left- and right- Kan extensions, resp.) The endofunctor $R : \mathcal{E} \to \mathcal{E}$ given by $p^* \circ p_*$ converts a G-set X to the subset consisting of the fixpoints for the action (and equipped with trivial action). Now R has a left adjoint, namely $p^* \circ p_!$. So R is a right adjoint $\mathcal{E} \to \mathcal{E}$, (or $\mathcal{E}/1 \to \mathcal{E}/1$), but it is not the 1-fiber of a fibered right adjoint $\mathcal{E}|1 \to \mathcal{E}|1$. For, this would imply, by general theory (cf. below), that R could be equipped with a (monoidal) strength $I \times R(X) \to R(I \times X)$, natural in I and X (objects of $\mathcal{E}$). In particular, take $X = 1$, so we have $I = I \times R(1) \to R(I)$; but if I is a non-empty G-set without stationary points, this is impossible, since then $R(I)$ is empty.

3.11 Example. (Cf. [5] §4.) Let D be an atom in a lccc, meaning that the endofunctor $X \mapsto X^D$ has a right adjoint R. This R cannot be part of a fibered adjunction unless $D = 1$ (although R can actually be extended to a fibered functor, cf. loc. cit., and L clearly extends to a fibered functor since it is polynomial). For, if such a fibered left adjoint L exists, there is a strength $I \times L^1(X) \to L^1(I \times X)$, which is an isomorphism, cf. 3.12. Taking $X = 1$ and using $L^1(X) = 1^D = 1$, we get that the canonical "diagonal" $I \to I^D$ is an isomorphism. By an easy application of the Yoneda Lemma, we get that $D = 1$.

3.12 Fibered functors and strong functors. A plain slice category $\mathcal{E}/I$ is tensored over $\mathcal{E}$: if S is in $\mathcal{E}$ and $\xi : M \to I$ is in $\mathcal{E}/I$, we have $S \otimes \xi = \mathrm{pr}_!(\mathrm{pr}^*(\xi))$, where pr denotes the projection $S \times I \to I$. If L is a fibered functor $\mathcal{E}|I \to \mathcal{E}|J$, we may rewrite $S \otimes L^1(\xi)$ as

$$S \otimes L^1(\xi) = \mathrm{pr}_!(\mathrm{pr}^*(L^1(\xi))) = \mathrm{pr}_!(L^S(\mathrm{pr}^*(\xi))).$$

On the other hand, the 2-cell exhibited in (4) in particular provides a natural transformation $\mathrm{pr}_! \circ L^S \Rightarrow L^1 \circ \mathrm{pr}_!$. Precomposing it with $\mathrm{pr}^* : \mathcal{E}/I \to \mathcal{E}/(S \times I)$ and instantiating at ξ therefore furnishes a map

$$S \otimes L^1(\xi) \to L^1(S \otimes \xi),$$

which is a tensorial strength for the functor L^1. So briefly, "a fibering implies a (tensorial) strength", cf. [3] §3 for the case $I = J = 1$. Note that if L is a fibered left adjoint, it commutes with lowershriek (cobase-change) functors, and therefore the strength is an isomorphism.

There is a partial converse to "fibering implies strength", due to Paré, cf. [3] Proposition 3.3: *a pullback-preserving functor $\mathcal{E} \to \mathcal{E}$ with a strenght extends to a fibered functor $\mathcal{E}|1 \to \mathcal{E}|1$.* Polynomial functors always preserve pullbacks. Therefore, when talking about polynomial functors (endofunctors, at least), (tensorial) strength and fibering are equivalent. In [2], polynomial functors are studied in the setting of functors equipped with a tensorial strength ("strong functors"); in a sense this is more economical, since only plain slices are needed. In the present note our goal is rather to characterize polynomial functors, and to this end, fibering seems necessary.

References

[1] Francis Borceux. *Handbook of categorical algebra. 2: Categories and structures.* Cambridge University Press, Cambridge, 1994.

[2] Nicola Gambino and Joachim Kock. *Polynomial functors and polynomial monads.* Preprint, arXiv:0906.4931.

[3] Peter T. Johnstone. *Cartesian monads on toposes.* J. Pure Appl. Alg. **116** (1997), 199–220.

[4] Peter T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 1*, vol. 43 of Oxford Logic Guides. The Clarendon Press Oxford University Press, New York, 2002.

[5] Anders Kock and Gonzalo E. Reyes. *Aspects of fractional exponent functors.* Theory Appl. Categ. **5** (1999), 251–265 (electronic).

[6] Thomas Streicher. *Fibrations à la Bénabou.* Notes (1999), available from `http://www.mathematik.tu-darmstadt.de/~streicher/`

[7] Mark Weber. *Familial 2-functors and parametric right adjoints.* Theory Appl. Categ. **18** (2007), 665–732 (electronic).

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