MATH. SCAND. 20 (1967), 131–150

COCHAIN FUNCTORS FOR GENERAL COHOMOLOGY THEORIES I

ANDERS KOCK, LEIF KRISTENSEN, and IB MADSEN

1. Introduction.

The purpose of this and a forthcoming paper [6] is to study general cohomology theories (on the category of CSS-complexes) by means of cochain functors. The emphasis is on cohomology theories of finite type (i.e. H^n (point) $=0$ except for finitely many n). These cohomology theories are closely related to higher order cohomology operations. We hope that the setup presented here will enable us to generalize to higher order operations the results proved in [10].

Part I and Part II are different in the sense that in I we only treat the usual cochain functor, whereas in II we treat cochain functors for general theories. The setup, however, in Part I is done in such a way that it generalizes to the situation considered in Part II.

The only new result contained in Part I is Theorem 3.6, which we shall state here in a slightly weaker form than in Section 3. Let $\mathcal{O}_{(m)}$ denote the set of natural transformations in m variables from the cochain functor $C=C(\cdot,Z_2)$ into itself,

$$\theta:\ C\times C\times\ldots\times C\to C$$

satisfying

$$\deg\theta(x_1,\ldots,x_m) \;=\; q+\sum\deg x_i$$

for some fixed integer q ($q=\deg\theta$). We also assume that $\theta(x_1,\ldots,x_m)=0$ if $x_i=0$ for some i, $1\leqq i\leqq m$ (see Definitions 3.1–3.3). The set $\mathcal{O}_{(m)}$ has the structure of a graded vectorspace over Z_2. It also has a differential ∇ of degree $+1$. Before stating our main result on the cohomology of $(\mathcal{O}_{(m)},\nabla)$, we shall mention that Steenrod's cup-i-product is an element of $\mathcal{O}_{(2)}$ of degree $-i$.

THEOREM 3.6. *Let $\mathcal{A}$ denote the mod 2 Steenrod algebra; then*

$$H(\mathcal{O}_{(m)},\nabla)\;\cong\;\mathcal{A}\otimes\mathcal{A}\otimes\ldots\otimes\mathcal{A}\qquad(m\text{ copies})\,.$$

Received September 28, 1966.

This theorem (or rather as it is stated in Section 3) generalizes the exact sequences given in [8, Theorem 3.3] and [9, Theorem 2.2].

Theorem 3.6 has several applications. We shall here take the opportunity to mention some of these:

It is possible to give cochain expressions for higher order cohomology operations. Such expressions have made possible evaluations of secondary and tertiary operations in low dimensions. This in turn has enabled us to determine Whitehead products of the form $[\alpha_n, \iota_n]$, where $\alpha_n \in \pi_{n+i}(S^n)$ is detected by a secondary operation and $\iota_n \in \pi_n(S^n)$ is the generator. Elements of this type are for instance $\bar{\nu}_n \in \pi_{n+6}(S^n)$, $\omega_n \in \pi_{n+16}(S^n)$ or $\xi_n \in \pi_{n+18}(S^n)$ (notation as in Toda [13]).

In case α_n is detected by a primary operation, such a determination was carried out by Mahowald [11].

Unstable operations are sometimes not additive. The deviation from additivity can be determined in the case of secondary and tertiary operations, [5, Corollary 3.6 and Theorem 4.3] and [8, Section 2 (12) and Section 3 (11)], respectively. A special case of this was used by E. Brown [1] to give a definition of the Kervaire–Arf invariant in terms of cohomology operations. This made it possible for E. Brown and F. Peterson [2] to solve the Arf invariant problem in dimensions $8k+2$.

Let us also mention that it is possible to deduce various relations between higher order operations, a Cartan formula for secondary operations (not quite satisfactory), various Peterson–Stein formulas, and higher order product structures.

In Part II we consider general cochain functors. The aim here is, among other things, to develop a theory which in a "natural" way can yield information about higher order operations. We believe, however, that the theory in itself has some interest. The treatment of tertiary operations given in [10] is rather involved and not quite satisfactory. We hope it be possible to improve this by use of general cochain functors.

It is well known that general cohomology functors $h=\{h^n\}$ are representable, that is, h^n is naturally equivalent to $[\cdot, k(n)]$, for some space $k(n)$. The representing objects $k(n)$ form an Ω-spectrum, thus the $k(n)$'s have stable k-invariants. The Moore–Postnikov decomposition of $k(n)$ tells us in which way h is built up from usual cohomology functors. The idea of Part II is, correspondingly, to build up general cochain functors. Let us consider the special case

$$h^n(\text{point}) = \begin{cases} \pi_0 & \text{for } n=0\,, \\ \pi_1 & \text{for } n=-m\,, \\ 0 & \text{elsewhere}\,. \end{cases}$$

Then $k(n)$ is a two stage space with a k-invariant $\hat{\lambda}$ of degree m. Choose (Theorem 3.6) a cochain representative for $\hat{\lambda}$,

$$\lambda : \; C^i(\cdot\,;\pi_0) \to C^{i+m}(\cdot\,;\pi_1)\,.$$

We construct a new "cochain" functor $C[\lambda]$ which is essentially the cone-construction of homological algebra,

$$C^i[\lambda](X) \;=\; C^i(X;\pi_0)\times C^{i+m}(X;\pi_1)$$

with differential

$$\delta(x,w) \;=\; (\delta x,\, \delta w+\lambda(x))\,,$$

and addition

$$(x,w)+(y,v) \;=\; (x+y, w+v+d(\lambda;\, x,y))\,.$$

The term $d(\lambda;x,y)$ is a cochain operation measuring the deviation of λ from additivity. Although this addition turns $C[\lambda]$ only into a loop-valued functor, we are able to define an associated cohomology functor $H[\hat{\lambda}]$. The functor $H[\hat{\lambda}]$ is representable and the representing object is a two stage space with k-invariant $\hat{\lambda}$. Thus we have $H[\lambda]=h$. In agreement with Theorem 3.6 of Part I, we let $\mathcal{O}(C[\lambda],C)$ denote all natural transformations (preserving zero) from $C[\lambda]$ to a usual cochain functor C. It is equipped with a differential ∇ of degree $+1$. We have

$$H(\mathcal{O}(C[\lambda],C),\nabla) \;=\; \mathfrak{a}(H[\hat{\lambda}],H)\,, \tag{1}$$

where $\mathfrak{a}(H[\hat{\lambda}],H)$ is the set of stable operations from $H[\hat{\lambda}]$ to H. Note that elements of $\mathfrak{a}(H[\hat{\lambda}],H)$ may be considered secondary cohomology operations associated with relations $0=\hat{a}\hat{\lambda}$. The formula (1) enables us to repeat the construction. In this way we obtain "cochain" functors at least for all cohomology functors h of finite type (see Theorem 5.1 in Part II). Especially one gets cochain formulas for all higher order stable cohomology operations (see Proposition 4.4 and Definition 5.3 in Part II).

2. The primary cochain functor.

We shall deal with the following categories:

CSS-complexes	CSS
CSS-complexes with base points	CSS_*
CSS abelian groups	CSS Ab
non-negatively graded sets	Grad Ens
non-negatively graded sets with base points	Grad Ens_*
non-negatively graded abelian groups	Grad Ab
Z-graded sets	Grad Z Ens
Z-graded sets with base points	Grad Z Ens_*
Z-graded abelian groups	Grad Z Ab.

For definition of CSS, see Kan [4]. The category CSS_* consists of CSS-complexes with a base point $*$ in each dimension; $d_i * = *$, $s_i * = *$ for all the face and degeneracy-operators d_i and s_i. Similarly, Grad Ens_* and Grad Z Ens_* have base points in each dimension.

There is of course a system of obvious forgetful functors between the listed categories. We shall give name to only one of them, namely

$$\square\colon\ \mathrm{CSS} \to \mathrm{Grad\ Ens}\ . \tag{1}$$

We shall make use of the following important fact:

THEOREM 2.1. *The functor $\square$ has a left adjoint Δ and a right adjoint L,*

$$\Delta, L\colon\ \mathrm{Grad\ Ens} \to \mathrm{CSS}\ .$$

The proof is an exercise in category theory, and is a special case of the general construction of adjoints for induced functors between diagram categories. In case where $[n] \in$ Grad Ens is the graded set with one element in dimension n and no element in the other dimensions, $\Delta[n]$ is the standard n-simplex, a CSS-complex whose geometric realization is a closed n-simplex.

In case where $(\pi, n) \in$ Grad Ens is the graded set having (the underlying set of) the abelian group π in dimension n and the 0 group in the other dimensions, $L(\pi, n)$ is the usual (acyclic) Eilenberg–MacLane complex.

The adjoint pair $\square, L$ gives a functor-transformation ("end-adjunction")

$$e\colon\ \square L \to \mathrm{Id}_{\mathrm{Grad\ Ens}}\ ,$$

i.e. for $U \in$ Grad Ens a mapping of graded sets

$$e_U\colon\ \square LU \to U\ , \tag{2}$$

which is called the *fundamental cochain on LU*. Cochains in general are defined as follows:

DEFINITION 2.2. The 0-cochain functor $C^0(\cdot\,; U)$ for $U \in$ Grad Ens is defined for $X \in$ CSS as

$$C^0(X;\, U) = \mathrm{Hom}_{\mathrm{Grad\ Ens}}(\square X, U)\ .$$

For $U \in$ Grad Z Ens the n-cochain functor $C^n(\cdot\,; U)$, $-\infty < n < \infty$, is defined as $C^0(\cdot\,; (S^n U)_+)$, where $(S^n U)_{m+n} = U_m$, and $(S^n U)_+$ is the non-negative part of $S^n U$.

For $U \in$ Grad Z Ens, we shall in the sequel write LU for $L(U_+)$. Also, put

$$L(U, n) = L(S^n U)\ .$$

An immediate consequence of Definition 2.2 and Theorem 2.1 is the corepresentability of the cochain functors, i.e., the existence of an equivalence, natural in X and U,

$$C^n(X; U) \cong \mathrm{Hom}_{\mathrm{CSS}}(X, L(U,n)) . \tag{3}$$

Note that for the fundamental cochain from (2) we have

$$e_U \in C^0(LU; U) .$$

With the usual notation $f^\#$ for the mapping

$$f^\#: \; C^n(Y; U) \to C^n(X; U)$$

induced by a CSS-mapping $f: X \to Y$, the isomorphism (3) may be described as follows: given $x \in C^n(X; U)$; the corresponding CSS-mapping $\tilde{x}: X \to L(U,n)$ is the unique CSS-mapping with the property

$$x = (\tilde{x})^\#(e_{(U,n)}) \tag{4}$$

with $e_{(U,n)} \in C^0(L(S^n U)_+; S^n U) = C^n(L(U,n); U)$, the fundamental cochain.

From the right adjointness of L to $\square$ we obtain two propositions.

PROPOSITION 2.3. *Let* $U \in$ Grad Ens. *Then* LU *is injective in the sense that for* Y *a sub-*CSS*-complex of the* CSS*-complex* X, *any* CSS*-mapping* $f: Y \to LU$ *can be extended to* X.

PROPOSITION 2.4. *Let* $U, V \in$ Grad Ens. *Then* $L(U \times V) \cong LU \times LV$.

The following is a well-known and easy-to-see fact of corepresentable functors: Given a functor transformation τ

$$\tau: \; \textstyle\prod_i \mathrm{Hom}(\cdot\,; LU_i) \to \mathrm{Hom}(\cdot\,; LV);$$

then there is a unique mapping

$$\tilde{\tau}: \; \prod LU_i \to LV \tag{5}$$

inducing τ.

Assume now that $U \in$ Grad Z Ab. Then the usual operations on cochains (natural in X),

$$-, + : \; C^0(X; U) \times C^0(X; U) \to C^0(X; U),$$

induce a CSS-map

$$\tilde{-}, \tilde{+} : \; LU \times LU \to LU \tag{6}$$

according to (5). In fact, LU is a CSS-abelian group with this structure. (One might, equivalently, define $\tilde{+}$ as the functor L acting on

$+: U \times U \to U$ under the equivalence in Proposition 2.4.) We define the coboundary

$$\delta\colon\ C^n(X; U) \to C^{n+1}(X; U)$$

by the usual formula. According to (5) this gives a CSS-mapping

$$\tilde{\delta}\colon\ L(U,n) \to L(U,n+1)\,.$$

Since δ is a homomorphism with respect to $+$, $\tilde{\delta}$ is a homomorphism with respect to $\tilde{+}$. From $\delta\delta = 0$ follows $\tilde{\delta}\tilde{\delta} = 0$, where we write 0 for the constant map $\tilde{0}$ mapping everything to the base point of $LU(n+2)$. (One easily sees that if $\bar{0}$ denotes the graded zero-group, then $L\bar{0}$ consists of exactly one simplex in each dimension, i.e., $L\bar{0}$ is geometrically a point. By functoriality $L\bar{0}$ maps into LU for $U \in$ Grad Ab; the image is then called the base point.)

We note that by the definitions of $\tilde{+}$ and $\tilde{\delta}$, if $x, y \in C^n(X; U)$, then

$$(8) \qquad \begin{aligned} (x+y)^{\sim} &= \tilde{x} \,\tilde{+}\, \tilde{y}\colon\ X \to L(U,n)\,,\\ (\delta x)^{\sim} &= \tilde{\delta} \circ \tilde{x}\colon\ X \to L(U,n+1)\,. \end{aligned}$$

The definition of Eilenberg–MacLane complexes takes the following form.

DEFINITION 2.5. Let $K(U,n)$ denote the kernel of $\tilde{\delta}\colon L(U,n) \to L(U,n+1)$, i.e., let the sequence

$$(9) \qquad 0 \to K(U,n) \xrightarrow{\ i\ } L(U,n) \xrightarrow{\ \tilde{\delta}\ } L(U,n+1)$$

be exact in CSSAb. Denote by $z_{(U,n)}$ the cocycle

$$z_{(U,n)} = i^{\#} e_{(U,n)} \in C^n(K(U,n); U)\,,$$

where $e_{(U,n)}$ is the fundamental cochain in (2).

That $z_{(U,n)}$ is a cocycle is seen by means of the obvious identities in (10) below, which together with those in (8) will be in constant use. Let $f\colon X \to Y$ be a CSS-mapping

$$(10) \qquad \begin{aligned} (f^{\#} y)^{\sim} &= \tilde{y} \circ f \quad \text{for} \quad y \in C^n(Y; U)\,,\\ \tilde{e}_{(U,n)} &= id_{L(U,n)}\,,\\ \tilde{z}_{(U,n)} &= i\,,\\ \tilde{\delta}^{\#} z_{(U,n)} &= \delta e_{(U,n-1)}\,. \end{aligned}$$

Denoting the cocycles in $C^n(X; U)$ by $Z^n(X; U)$, we get by (8) that $K(U,n)$ is a corepresenting object for the functor $Z^n(\cdot\,; U)$. For a cocycle $x \in C^n(X; U)$, the mapping $\tilde{x}: X \to L(U,n)$ factors through the subcomplex $K(U,n)$ and $\tilde{x}^{\#} z_{(U,n)} = x$.

From $\tilde{\delta}\tilde{\delta} = 0$ it follows that $\tilde{\delta}$ in (9) factors over $K(U, n+1)$ so that we have the fundamental exact sequence

$$\begin{array}{ccc} 0 \longrightarrow K(U,n) & \xrightarrow{\iota} & L(U,n) \\ & & \downarrow{\scriptstyle\tilde{\delta}} \\ & & K(U,n+1)\,. \end{array} \tag{11}$$

We do not claim that $\tilde{\delta}$ is onto (although it will actually be so if $S^n U$ is non-negatively graded). We claim, however, that $\tilde{\delta}$ is a Kan-fibering with $K(U,n)$ as fiber, in a sense which we shall now recall. In [3], a subcomplex $\Lambda[n]$ of $\Delta[n]$ is defined. Geometrically, $\Lambda[n]$ consists of all $(n-1)$-faces of $\Delta[n]$ except one. The following is the usual definition of Kan-properties for complexes and fiberings except that we do not require the fibre map to be onto.

Definition 2.6. A CSS-complex X has the *Kan-property* if, for all $q \geqq 0$, any CSS-map $\Lambda[q] \to X$ can be extended to a CSS-map $\Delta[q] \to X$. A map $p: X \to Y$ is a *Kan-fibering* if for all $q \geqq 0$ the commutative diagram (full arrows)

$$\begin{array}{ccc} \Lambda[q] & \xrightarrow{f} & X \\ {\scriptstyle\iota}\downarrow & \nearrow{\scriptstyle h} & \downarrow{\scriptstyle p} \\ \Delta[q] & \xrightarrow[g]{} & Y \end{array} \tag{12}$$

can be completed by a map $h: \Delta[q] \to X$ (dotted arrow) so that the triangles commute.

Proposition 2.7 *For arbitrary U, n, the mapping $\tilde{\delta}$ in* (11) *is a Kan-fibering. Furthermore, $K(U,n)$ and $L(U,n)$ have the Kan-property.*

Proof. We interpret the maps f and g in (12) (with p as $\tilde{\delta}$) as a n-cochain on $\Lambda[q]$, respectively an $(n+1)$-cocycle on $\Delta[q]$, by the fundamental corepresentation. We define relative cochain complexes in the usual way so that the horizontal sequences in the following diagram are exact:

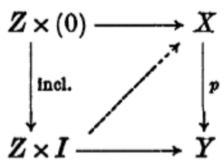
$$\begin{array}{ccccccc} 0 \to C^n(\Delta,\Lambda; U) & \to & C^n(\Delta; U) & \xrightarrow{\iota^{\#}} & C^n(\Lambda; U) & & \\ \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\ 0 \to C^{n+1}(\Delta,\Lambda; U) & \to & C^{n+1}(\Delta; U) & \xrightarrow{\iota^{\#}} & C^{n+1}(\Lambda; U) & \to & 0 \\ \delta\downarrow & & \delta\downarrow & & & & \\ 0 \to C^{n+2}(\Delta,\Lambda; U) & \to & C^{n+2}(\Delta; U)\,. & & & & \end{array}$$

We are given $g \in Z^{n+1}(\Delta; U)$, $f \in C^n(\Lambda; U)$ with $\delta f = i^{\#} g$. Now it is well known (or easily seen) that the chain complex in the left column is exact. Then a simple diagram chase gives $h \in C^n(\Delta; U)$ with $\delta h = f$, $i^{\#} h = g$. Then $\bar{h}\colon \Delta[q] \to L(U,n)$ proves the first part of the proposition. The second is trivial in view of Proposition 2.3. The argument for the third part is much similar to that of the first: a diagram chase in chain complexes for Δ, Λ and (Δ, Λ).

We recall a few concepts from CSS-homotopy theory (Kan [4]). Let I denote $\Delta[1]$. Geometrically, this is the unit interval. The complex I has two 0-dimensional simplexes, (0) and (1), and one non-degenerate 1-simplex t, with $\partial_0 t = (1)$, $\partial_1 t = (0)$. Two CSS-maps $f, g\colon X \to Y$ are *homotopic* if there is a CSS-map $h\colon X \times I \to Y$ with

$$h \mid X \times (0) = f, \qquad h \mid X \times (1) = g \,.$$

We write $f \simeq g$ if f is homotopic to g; if Y has the Kan-property, then $\simeq$ is an equivalence relation. The quotient set of $\mathrm{Hom}_{\mathrm{CSS}}(X, Y)$ is in this case denoted $\mathrm{Hom}_{[\mathrm{CSS}]}(X, Y)$ or $[X, Y]$. Let $p\colon X \to Y$ be a CSS-map. It is said to have the *homotopy lifting property* if for all Z the commutative diagram (full arrows)



can be completed by a map $Z \times I \to X$ (dotted arrow) so that the triangles commute. If p is a Kan-fibering, it has the homotopy lifting property. The cone TX on a CSS-complex X is $X \times I / X \times (1)$. It contains X as a subcomplex. If it contains X as a retract (i.e. the injection has a left inverse), X is called *contractible*. A cone is acyclic. Therefore any contractible complex is acyclic.

We have in particular

PROPOSITION 2.8. *For any* $U \in \mathrm{Grad\ Ens}$, *LU is contractible (and thus acyclic).*

PROOF. From Proposition 2.3 we get that if LU is a subcomplex of Y, then LU is a retract of Y. In particular, LU is a retract of TLU.

Obviously, if Y is a contractible Kan-complex, $[X, Y]$ consists of one element only. In particular, $[X, LU]$ is trivial. We shall study $[X, KU]$;

this set is in general non-trivial. The abelian group structure of KU $(=K(U,0))$ makes $[X,KU]$ into an abelian group.

THEOREM 2.9 (Eilenberg–MacLane). *There is an equivalence* χ

$$\chi\colon\ [X,K(U,n)] \to H^n(X;\,U)\,,$$

natural in X *and* U. *For* $f\colon X \to K(U,n)$, $\chi([f])$ *is given by* $f^*\hat{z}_{(U,n)}$, *where* $\hat{z}_{(U,n)}$ *denotes the cohomology class of the fundamental cocycle* $z_{(U,n)}$.

PROOF. First we prove that χ is well defined. Let $\tilde{a}\simeq\tilde{b}$. We shall prove $a \sim b$ (a cohomologous to b). We obviously have $(\tilde{a}\,\tilde{-}\,\tilde{b})\simeq 0$. Since the 0 map can be lifted over $\tilde{\delta}$, we get by the homotopy lifting property for $\tilde{\delta}$ that there is an $h\colon X \to L(U,n-1)$ with

$$\tilde{a}\,\tilde{-}\,\tilde{b} = \tilde{\delta}\circ h\,.$$

Therefore, writing z for $z_{(U,n)}$ and e for $e_{(U,n-1)}$

$$a-b = (a-b)^{\sim\#}z = (\tilde{a}\,\tilde{-}\,\tilde{b})^{\#}z = (\tilde{\delta}\circ h)^{\#}z = h^{\#}\tilde{\delta}^{\#}z = h^{\#}\delta e \sim 0\,.$$

For $\hat{x}\in H^n(X;\,U)$, let x be a representing cocycle. Then $\chi([\tilde{x}])=\hat{x}$, so that χ is onto. By the definition of $\tilde{+}$, the equivalence

$$\mathrm{Hom}_{\mathrm{CSS}}(X,K(U,n)) \to Z^n(X;\,U)$$

is a homomorphism, and χ is induced by this equivalence; so χ is a homomorphism. Thus, to prove χ one-one, it suffices to prove the kernel to be zero. Let $f\colon X \to K(U,n)$ be such that $f^{\#}z=\delta b$. Then $f=\tilde{\delta}\circ\tilde{b}$, that is, f factors over the contractible $L(U,n-1)$. Hence $f\simeq 0$.

3. An exact sequence of operations.

It is well known how the corepresentability of the cohomology functor (as expressed by the Eilenberg–MacLane Theorem 2.9) makes it easy to handle cohomology operations. Here we shall use the corepresentability of the cochain functor to investigate cochain operations, and to get a connection between cochain- and cohomology operations. The case of operations in one variable is the most important. In this form (with Z_2-coefficients) the theorem may be found in [7] and [8]; the two-variable case is treated in [9]. Here we perform the generalization to an arbitrary finite number of variables and almost arbitrary coefficients. In the subsequent paper [6], we shall prove a corresponding theorem (the one-variable case) for certain general cohomology theories.

Let

$$U_1,\ldots,U_m, V \in \text{Grad Ab}\,.$$

DEFINITION 3.1. A locally defined cohomology operation λ of type $(S^{n_1}U_1,\ldots,S^{n_m}U_m;\ V)$ and degree q is a functor transformation

$$\lambda:\ \prod_{i=1}^{m} H^{n_i}(\cdot\,;U_i) \to H^p(\cdot\,;V)\,,$$

$p=\sum n_i+q$, with the property that for any complex X,

$$\lambda(X)(\hat{x}_1,\ldots,\hat{x}_m) = 0$$

if at least one of the arguments $\hat{x}_i \in H^{n_i}(X;U_i)$ is zero. Locally defined cochain operations are defined similarly, by replacing the letter H by the letter C everywhere. We do not require additivity neither for cohomology nor for cochain operations.

DEFINITION 3.2. An (everywhere defined) cohomology operation λ of type $(U_1,\ldots,U_m;\ V)$ and degree q is a family of locally defined cohomology operations

$$\lambda_{\langle n_1,\ldots,n_m\rangle} \quad \text{of type} \quad (S^{n_1}U_1,\ldots,S^{n_m}U_m;\ V)$$

of degree q, one for each m-tuple $\langle n_1,\ldots,n_m\rangle$ of integers. Everywhere defined cochain operations are defined similarly, using locally defined cochain operations.

Note that the zero condition implies that one may relativise a cohomology or cochain operation in any one of the variables; given λ as in Definition 3.1, one may thus for $A\subset X$ define

$$\lambda:\quad H^{n_1}(X;U_1)\times\ldots\times H^{n_{m-1}}(X;U_{m-1})\times H^{n_m}(X,A;U_m) \to H^p(X,A;V)\,.$$

Since the functor $H^{n_1}(\cdot\,;U_1)\times\ldots\times H^{n_m}(\cdot\,;U_m)$ is corepresented by $K(U_1;n_1)\times\ldots\times K(U_m;n_m)$, there is a $1-1$ correspondence between locally defined cohomology operations λ of type $(S^{n_1}U_1,\ldots,S^{n_m}U_m;\ V)$ and degree q and cohomology classes

$$\lambda \in H^p\big((K(U_1;n_1),*)\times\ldots\times(K(U_m,n_m),*);\ V\big)\,,$$

$p=\sum n_i+q$. Similarly for locally defined cochain operations λ and p-cochains on $L(U_1,n_1)\times\ldots\times L(U_m;n_m)$ (coefficients in V). It is now legitimate to speak about the *set* of (everywhere defined) cochain operations of type $(U_1,\ldots,U_m;\ V)$. Actually the set is in Grad Z Ab, letting grading q consist of the cochain operations of degree q. We shall make it into a chain complex by introducing a differential ∇ of degree $+1$.

DEFINITION 3.3. Denote by $\mathcal{O}(U_1,\ldots,U_m;\ V)$ the set of everywhere defined cochain operations of type $(U_1,\ldots,U_m;\ V)$. For

$$\theta \in \mathcal{O}(U_1,\ldots,U_m;\, V)$$

of degree q, $\nabla\theta \in \mathcal{O}(U_1,\ldots,U_m;\, V)$ is the cochain operation of degree $q+1$ given by

$$(\nabla\theta)(x_1,\ldots,x_m) = \delta\theta(x_1,\ldots,x_m) + \sum_{i=1}^{m} (-1)^{p_i}\,\theta(x_1,\ldots,\delta x_i,\ldots,x_m)\,,$$

where $p_i = 1+q+\sum_{j=1}^{i-1} g(x_j)$. (We use here and everywhere the notation $g(x)$ for that integer for which $x \in C^{g(x)}(X;\, U)$.)

Obviously $\nabla\nabla = 0$, and ∇ is a homomorphism with respect to the abelian group structure on $\mathcal{O}$. Denote by $Z\mathcal{O}(U_1,\ldots,U_m;\, V)$ the kernel of ∇. We are going to assign a cohomology operation $\varepsilon(\theta)$ or $\hat{\theta}$ to every $\theta \in Z\mathcal{O}(U_1,\ldots,U_m;\, V)$. For each m-tuple of integers $\langle n_1,\ldots,n_m\rangle$, evaluate θ on $(\mathrm{proj}_1^{\#} z_{n_1},\ldots,\mathrm{proj}_m^{\#} z_{n_m})$, where z_{n_i} denotes the fundamental cocycle on $K(U_i;\, n_i)$ and proj_i is the projection onto the i'th factor. This gives a family λ (indexed by m-tuples of integers) of cocycles. The corresponding family of cohomology classes defines the cohomology operation $\varepsilon(\theta)$ (of the same type and degree as θ).

A cohomology operation of the form $\varepsilon(\theta)$, $\theta \in Z\mathcal{O}$, will be multistable in the following obvious sense:

Definition 3.4. A cohomology operation λ of type $(U_1,\ldots,U_m;\, V)$ and degree q is called multistable if for any pair (X,A), inclusion $i: A \to X$, and any integer r, $1 \leqq r \leqq m$,

$$\delta^*\lambda(i^*\hat{x}_1,\ldots,\hat{y}_r,\ldots,i^*\hat{x}_n) = (-1)^s\lambda(\hat{x}_1,\ldots,\delta^*\hat{y}_r,\ldots,\hat{x}_n)\,,$$

where $\hat{x}_i \in H^{n_i}(X;\, U_i)$ for $i \neq r$, $\hat{y}_r \in H^{n_r}(A;\, U_r)$ and $s = q+\sum_{i=1}^{r-1} n_i$.

By corepresentability, λ determines, and is determined by, a family of cohomology classes (with coefficients in V) on $(K(U_1,n_1),*)\times\ldots\times(K(U_m,n_m),*)$, indexed by all m-tuples of integers. We can give an alternative description of multistability in terms of this family. Let $1 \leqq r \leqq m$. Denote by σ_r the additive relation

$$\begin{aligned}
&C^*\big(K(U_1;n_1)\times\ldots\times(K(U_r;n_r),*)\times\ldots;\, V\big) \xrightarrow{(1\times\ldots\times\bar{\delta}\times\ldots\times 1)^{\#}}\\
&C^*\big(K(U_1,n_1)\times\ldots\times(L(U_r,n_r-1),*)\times\ldots\times K(U_m,n_m);\, V\big) \xleftarrow{\delta}\\
&C^*\big(K(U_1,n_1)\times\ldots\times(L(U_r,n_r-1),*)\times\ldots;\, V\big) \xrightarrow{(1\times\ldots\times i\times\ldots\times 1)^{\#}}\\
&C^*\big(K(U_1,n_1)\times\ldots\times(K(U_r,n_r-1),*)\times\ldots;\, V\big)\,.
\end{aligned}$$

Since $L(U_r,n_r-1)$ is contractible to its base point, $*$, the chain complex in the middle is totally acyclic, and σ_r induces a homomorphism $\hat{\sigma}_r$ ("the r'th suspension") on cohomology level,

$$\hat{\sigma}_r\colon\ H^t\big(K(U_1; n_1)\times\ldots\times(K(U_r; n_r),*)\times\ldots; V\big)$$
$$\to H^{t-1}\big(K(U_1; n_1)\times\ldots\times(K(U_r; n_r-1),*)\times\ldots; V\big)\,.$$

A family of cohomology classes $\hat{b}_{\bar{n}}$ ($\bar{n}$ running through m-tuples of non-negative integers) corresponds to a multistable operation λ if and only if the s'th suspension of $\hat{b}_{n_1,\ldots,n_m}$ is equal to

$$(-1)^s\hat{b}_{n_1,\ldots,n_r-1,\ldots,n_m} \quad \text{with} \quad s = q+\textstyle\sum_{i=1}^{r-1} n_i\,,$$

where again q is the degree of λ. This is easily seen. "Multistability" for an operation in one variable is the usual stability.

Let us denote the set of multistable cohomology operations of type $(U_1,\ldots,U_m;\ V)$ by $\mathcal{A}(U_1,\ldots,U_m;\ V)$. We recall Serre's Theorem on cohomology of Eilenberg–MacLane spaces. It contains at least this much: If V is a field and $U\in$ Grad Ab, then $H^t\big((K(U;n),*);\ V\big)$ is 0 for $t<n$ and consists of products (with one or more factors) of classes of the form $\lambda(\hat{z})$, where $\hat{z}\in H^n\big((K(U;n),*);\ U\big)$ is the basic class and $\lambda\in\mathcal{A}(U;\ V)$.

Using this theorem, one may describe $\mathcal{A}(U;V)$ as the inverse limit,

$$\lim_{\leftarrow} H^{m+q}\big((K(U,m),*);\ V\big)$$

under $\hat{\sigma}$.

To do the same for the many-variable case, one has to use the Eilenberg–Zilber–Künneth map ψ (exterior cup product)

$$H^*\big((K(U_1,;n_1),*);\ V\big)\otimes\ldots\otimes H^*\big((K(U_m;n_m),*);\ V\big)$$
$$\xrightarrow{\ \psi\ } H^*\big((K(U_1;n_1),*)\times\ldots\times(K(U_m;n_m),*);\ V\big)$$

This is an isomorphism if V is a field, U_i finitely generated in each dimension. One may then give a third description of multistability: a cohomology operation λ in m variables of degree q is multistable if the corresponding family $\{\hat{b}_{\bar{n}}\}$ of cohomology classes ($\bar{n}$ running over m-tuples of integers) has the property that for $1\leq r\leq m$

$$(1\otimes\ldots\otimes\hat{\sigma}\ldots\otimes 1)\,\hat{b}_{(n_1,\ldots,n_m)} = (-1)^s\hat{b}_{(n_1,\ldots,n_r-1,\ldots,m)}\,,$$

where $s=q+\sum_{i=1}^{r-1}n_i$. In this case, we may therefore describe the set of multistable cohomology operations by

$$\mathcal{A}(U_1,\ldots,U_m;\ V)\cong\mathcal{A}(U_1;\ V)\otimes\ldots\otimes\mathcal{A}(U_m;\ V)\,.$$

We now prove a lemma on cohomology operations in one variable.

LEMMA 3.5. *Let $V\in$ Grad Ab be finitely generated. Then the sequence*

$$Z\mathcal{O}(U;\ V)\xrightarrow{\ \varepsilon\ }\mathcal{A}(U;\ V)\to 0$$

is exact.

PROOF. One might, of course, prove this as in [8]. We shall, however, present a slightly more conceptual proof, using very little about the cohomology of Eilenberg–MacLane spaces: The suspension

$$(1)\qquad \hat{\sigma}\colon\ H^{t+n}\big((K(U,n),*);\ V\big) \to H^{t+n-1}\big((K(U,n-1),*);\ V\big)$$

is an epimorphism for n large.

The proof that ε is onto the set of stable cohomology operations now goes as follows. Let λ be a stable cohomology operation of type $(U;\ V)$ and degree q. Denote for short $K(U,n)$ by K_n, $L(U,n)$ by L_n, the fundamental cocycle on K_n by z_n. If we can find a family of cochains $a_t \in C^{t+q}(L_t,*;\ V)$ (indexed by integers) such that

$$(2)\qquad i^{\#}a_{t+1} \in \lambda\{z_{t+1}\},$$

$$(3)\qquad \delta a_{t+1} = (-1)^q \tilde{\delta}^{\#} a_{t+2}$$

(i and $\tilde{\delta}$ as in (9) of Section 2), then, by corepresentability, this family determin a cochain operation with the required properties as is easily seen. To construct the family, choose N large. Choose a cocycle in $\lambda\{z_N\}$ and extend it to a cochain a_N on $(L_N,*)$. (Recall that $\tilde{\delta}\colon L_{N-1} \to L_N$ factors through $K_N \subseteq L_N$.) Then $(-1)^q\tilde{\delta}^{\#}a_N$ is a cocycle on $(L_{N-1},*)$, hence a coboundary; choose

$$a_{N-1} \in C^{N-1+q}(L_{N-1},*) \quad \text{with} \quad \delta a_{N-1} = (-1)^q \tilde{\delta}^{\#} a_N\,.$$

Then $(-1)^q\tilde{\delta}^{\#}a_{N-1}$ is a cocycle on $(L_{N-2},*)$, hence a coboundary; choose

$$a_{N-2} \in C^{N-2+q}(L_{N-2},*) \quad \text{with} \quad \delta a_{N-2} = (-1)^q \tilde{\delta}^{\#} a_{N-1}\,.$$

And so forth. The cochains constructed so far satisfy (2) and (3) as is easily seen using the stability of λ. We shall now define a_t for $t > N$ and redefine a_N. Put b_N equal to $i^{\#}a_N$. Suppose that for some $M \geqq N$ we have defined a_t for all $t < M$ and a cochain b_M such that

$$\begin{array}{lll} (\mathrm{i}_M) & i^{\#}a_t \in \lambda\{z_t\} & \text{for } t<M\,, \\ (\mathrm{ii}_M) & b_M \in \lambda\{z_M\}, & \\ (\mathrm{iii}_M) & \delta a_t = (-1)^q\tilde{\delta}^{\#}a_{t+1} & \text{for } t<M-1\,, \\ (\mathrm{iv}_M) & \delta a_{M-1} = (-1)^q\tilde{\delta}^{\#}b_M\,. & \end{array}$$

(Note that (ii_M) can be deduced from (i_M) and (iv_M) since λ is stable.) We are going to define

$$a_M \in C^{q+M}(L_M,*) \quad \text{and} \quad b_{M+1} \in Z^{q+M+1}(K_{M+1},*)$$

so that $(\mathrm{i}_{M+1})-(\mathrm{iv}_{M+1})$ hold. Choose a $b'_{M+1} \in \lambda\{z_{M+1}\}$. Then there is an $a_M{}' \in C^{M+q}(L_M,*)$ with

$$(4) \qquad \delta a'_M = (-1)^q \tilde{\delta}^\# b'_{M+1} .$$

Then $i^\# a_M'$ is a cocycle on K_M. Since M is large enough, we get by (1) that the class of the cocycle $b_M - i^\# a_M'$ is in the image of $\hat{\sigma}$, that is, we can find

$$w \in C^{M-1+q}(K_M, *), \quad z \in Z^{M+1+q}(K_{M+1}, *), \quad r' \in C^{M+q}(L_M, *)$$

so that

$$b_M - i^\# a_M' = i^\# r' + \delta w, \qquad \delta r' = \tilde{\delta}^\# z .$$

Extend w to $v \in C^{M-1+q}(L_M, *)$. Then with $r = r' + \delta v$ we have

$$(5) \qquad b_M - i^\# a_M' = i^\# r, \qquad \delta r = \tilde{\delta}^\# z .$$

Put

$$(6) \qquad a_M = a_M' + r, \qquad b_{M+1} = b'_{M+1} + (-1)^q z .$$

Then by (5)

$$(\mathrm{i}_{M+1}) \qquad i^\# a_M = b_M ,$$

and by (4), (5), and (6)

$$(\mathrm{iv}_{M+1}) \qquad \delta a_M = (-1)^q (\tilde{\delta}^\# b'_{M+1} + (-1)^q \tilde{\delta}^\# z) = (-1)^q \tilde{\delta}^\# b_{M+1} .$$

Finally (iii_{M+1}) easily follows from (iv_M) and (i_{M+1}). This completes the proof of Lemma 3.5.

Recalling the definition of $\mathcal{O}(U_1, \ldots, U_m; V)$ and ∇ from Definition 3.3, and the definition of ε from $Z\mathcal{O}$ to $\mathcal{A}(U_1, \ldots, U_m; V)$, we are now able to state and prove the main theorem of this section. The restrictive hypothesis on the U_i and V may be unnecessary (see the remarks after the proof).

THEOREM 3.6. *Let $U_1, \ldots, U_m \in$ Grad Ab be finitely generated in each dimension. Let V be a field. Then the sequence*

$$\mathcal{O}(U_1, \ldots, U_m; V) \xrightarrow{\nabla} Z\mathcal{O}(U_1, \ldots, U_m; V) \xrightarrow{\varepsilon} \mathcal{A}(U_1, \ldots, U_m; V) \to 0$$

is exact, and $\mathcal{A}(U_1, \ldots, U_m; V) \cong \mathcal{A}(U_1; V) \otimes \ldots \otimes \mathcal{A}(U_m; V)$.

PROOF. One easily sees that $\varepsilon \circ \nabla = 0$. Also, in Lemma 3.4, we already proved ε to be onto if $m = 1$. Because the exterior cup product is an isomorphism, we get that ε is onto, as well as the last statement of the theorem. The hard part of the proof is to prove

$$\operatorname{Ker} \varepsilon \subseteq \operatorname{Im} \nabla .$$

It will be necessary for this purpose to generalize the definitions of $\mathcal{O}$, ∇, and ε (Definition 3.3 etc.) slightly and prove a lemma concerning the generalizations $\nabla_{m,p,q}$ and $\varepsilon_{m,p,q}$ of ∇ and ε, respectively. The proof of Theorem 3.6 is completed by putting $p=q=0$ in this lemma, Lemma 3.10.

We consider in what follows a set of graded abelian groups

$$U_1,\ldots,U_m, V_1,\ldots,V_p, W_1,\ldots,W_q \in \text{Grad Ab};$$

assume each is finitely generated in each dimension; further, let V be a field. Let K, I, and J denote an m-tuple, a p-tuple, and a q-tuple of nonnegative integers, $K=(k_1,\ldots,k_m)$, $I=(i_1,\ldots,i_p)$, and $J=(j_1,\ldots,j_q)$, respectively.

Definition 3.7. A locally defined Q-operation θ of type

$$(U_1,\ldots,U_m;\ V_1,\ldots,V_p;\ W_1,\ldots,W_q;\ I,J,K;\ V)$$

and degree t is a functor transformation (no additivity required)

$$C^{k_1}(\cdot\,;U_1)\times\ldots\times C^{k_m}(\cdot\,;U_m)\times C^{i_1}(\cdot\,;V_1)\times\ldots\times \\ \times C^{i_p}(\cdot\,;V_p)\times Z^{j_1}(\cdot\,;W_1)\times\ldots\times Z^{j_q}(\cdot\,;W_q)\to C^r(\cdot\,;V)$$

where $r=\sum k+\sum i+\sum j+t$, satisfying the following zero conditions:

$$\theta(x_1,\ldots,x_m,c_1,\ldots,c_p,y_1,\ldots,y_q) = 0$$

if:

(i) at least one of the variables is 0,

or if:

(ii) at least one of $c_1,\ldots,c_p$ is a cocycle.

Again, we collect such locally defined Q-operations to get everywhere defined ones. Note, however, that we only collect "over K":

Definition 3.8. An everywhere defined Q-operation θ of type

$$(U_1,\ldots,U_m;\ V_1,\ldots,V_p;\ W_1,\ldots,W_q;\ I,J;\ V)$$

and degree t is a collection of locally defined Q-operations θ_K of type

$$(U_1,\ldots,U_m;\ V_1,\ldots,V_p;\ W_1,\ldots,W_q;\ I,J,K;\ V)$$

and degree t, one for each m-tuple K of integers.

Definition 3.9. Denote by

$$Q(U_1,\ldots,U_m;\ V_1,\ldots,V_p;\ W_1,\ldots,W_q;\ I,J;\ V)$$

or short $Q^{m,p,q}$ the Z-graded vector space over V, having in degree t the set of everywhere defined Q-operations θ of degree t and type

$$(U_1,\ldots,U_m;\ V_1,\ldots,V_p;\ W_1,\ldots,W_q;\ I,J;\ V)\,.$$

Let $\nabla_{m,p,q}$, or just ∇, denote the endomorphism of $Q^{m,p,q}$ of degree $+1$ given by

$$\begin{aligned}\nabla\theta(x_1,\ldots,x_m;\,c_1,\ldots,c_p;\,y_1,\ldots,y_q) &= \delta\theta(x_1,\ldots,x_m;\,c_1,\ldots,c_p;\,y_1,\ldots,y_q)+\\ &\quad+\sum_{s=1}^{m}(-1)^{r_s}\theta(x_1,\ldots,\delta x_s,\ldots,x_m;\,c_1,\ldots,c_p;\,y_1,\ldots,y_q)\,,\end{aligned}$$

where $r_s=g(\theta)+\sum_{j=1}^{s-1}g(x_j)+1$.

One easily sees that $\nabla\nabla=0$ and that the definition generalizes Definition 3.3 (putting $p=q=0$).

We also generalize ε. Let $ZQ^{m,p,q}$ denote the kernel of ∇. The homomorphism $\varepsilon_{m,p,q}$, or just ε,

$$\begin{aligned}\varepsilon\colon\ ZQ^{m,p,q} \to \mathcal{A}(U_1,\ldots,U_m;\ V)&\otimes H^*(L(V_1,i_1),K(V_1,i_1))\otimes\ldots\otimes\\ &\otimes H^*(L(V_p;\,i_p),K(V_p;\,i_p))\otimes H^*(K(W_1;\,j_1))\otimes\ldots\otimes H^*(K(W_q;\,j_q))\end{aligned}$$

is given on a $\theta\in ZQ^{m,p,q}$ in the following manner: Evaluate θ on

$$(\mathrm{proj}^{\#}z_N,\ldots,\mathrm{proj}^{\#}e_{i_1},\ldots,\mathrm{proj}^{\#}e_{i_p},\ \mathrm{proj}^{\#}z_{j_1},\ldots,\mathrm{proj}^{\#}z_{j_q})\,,$$

where N is large; hereby is determined (a cocycle and thus) a cohomology class on

$$\begin{aligned}(K(U_1,N),*)\times\ldots\times(K(U_m,N),*)&\times(L(V_1,i_1),K(V_1,i_1))\times\ldots\times\\ &\times(L(V_p,i_p),K(V_p,i_p))\times(K(W_1,j_1),*)\times\ldots\times(K(W_q,j_q),*)\,.\end{aligned}$$

By the Künneth–Eilenberg–Zilber Theorem and the Serre Theorem one gets an element in the said tensorproduct.

Denote $\mathcal{A}(U_1,\ldots,U_m;\ V)$ shortly by $\mathcal{A}^m$. By induction in m we shall prove

LEMMA 3.10. *Any sequence of the form*

$$\begin{aligned}Q^{m,p,q}\xrightarrow{\ \nabla\ } ZQ^{m,p,q}\xrightarrow{\ \varepsilon\ }\mathcal{A}^m&\otimes H^*(L(V_1,i_1),K(V_1,i_1))\otimes\\ &\otimes\ldots\otimes H^*(K(W_1,j_1),*)\otimes\ldots\to 0\end{aligned}$$

is exact.

PROOF. Let $m=0$. If $\theta\in ZQ^{0,p,q}$, then the cochain

(7) $$\theta(\mathrm{proj}^{\#}e_{i_1},\ldots,\mathrm{proj}^{\#}e_{i_p},\mathrm{proj}^{\#}z_{j_1},\ldots,\mathrm{proj}^{\#}z_{j_q})$$

is a cocycle on $(L_{i_1}, K_{i_1}) \times \ldots \times (K_{j_1}, *) \times \ldots \times (K_{j_q}, *)$, and if $\varepsilon\theta = 0$, it is a coboundary on this relative complex, e.g. equal to δb. Then b, by corepresentability, determines a $\bar{b} \in Q^{0,p,q}$, and it is obvious that $\nabla \bar{b} = \theta$. Also, clearly ε is onto in this case. This proves the case $m = 0$.

Now, let $\theta \in ZQ^{m,p,q}$ have $\varepsilon\theta = 0$. For each integer k, one gets a $\theta_k \in Q^{m-1,p,q+1}$ by requiring that the m'th coordinate on which θ acts is a cocycle of dimension k,

$$\begin{aligned}\theta_k &\in Q^{m-1,p,q+1} \\ &= Q(U_1, \ldots, U_{m-1};\, V_1, \ldots, V_p;\, U_m, W_1, \ldots, W_q;\, I, (k, J);\, V)\,,\end{aligned}$$

and clearly $\nabla_{m-1,p,q+1}\theta_k = 0$, $\varepsilon\theta_k = 0$. Using an inductive hypothesis for $m-1$, we find to each integer k an element $d_k \in Q^{m-1,p,q+1}$ such that

$$\nabla_{m-1,p,q+1} d_k = \theta_k\,.$$

The corepresenting cochain of d_k can be extended from

$$(L_{n_1}, *) \times \ldots \times (L_{n_{m-1}}, *) \times (L_{i_1}, K_{i_1}) \times \ldots \times (K_k, *) \times (K_{j_1}, *) \times \ldots \times (K_{j_q}, *)$$

to

$$(L_{n_1}, *) \times \ldots \times (L_{n_{m-1}}, *) \times (L_{i_1}, K_{i_1}) \times \ldots \times (L_k, *) \times (K_{j_1}, *) \times \ldots \times (K_{j_q}, *)\,;$$

doing this for all k, we get a family of cochains which defines a cochain operation $d \in Q^{m,p,q}$. One easily sees that if x_k is a cocycle, then

$$\begin{aligned}&\nabla_{m,p,q} d(x_1, \ldots, x_k, c_{i_1}, \ldots, c_{i_p}, y_{j_1}, \ldots, y_{j_q}) \\ &\qquad\qquad = \theta(x_1, \ldots, x_k, c_{i_1}, \ldots, c_{i_p}, y_{j_1}, \ldots, y_{j_q})\,.\end{aligned}$$

We therefore have

$$\psi = \theta - \nabla_{m,p,q} d \in ZQ^{m,p,q}\,. \tag{8}$$

The value of ψ is zero if the m'th variable is a cocycle. Thus, for each integer k, ψ defines, by restricting the m'th variable to be of dimension k, an operation

$$\begin{aligned}\psi &\in Q^{m-1,p+1,q} \\ &= Q(U_1, \ldots, U_{m-1};\, U_m, V_1, \ldots, V_p;\, W_1, \ldots, W_q;\, (k, I), J;\, V)\,.\end{aligned}$$

Again one sees that $\nabla_{m-1,p+1,q}\psi = 0$. Let $\varepsilon(\psi)$ be expressed

$$\textstyle\sum_\nu \hat{\alpha}_\nu \otimes \hat{\varrho}_\nu \otimes \hat{\beta}_\nu \otimes \hat{\gamma}_\nu, \qquad \nu \in N_k\,,$$

where $\hat{\alpha}_\nu \in \mathfrak{A}^{(m-1)}$, $\hat{\varrho}_\nu \in H^*(L_k, K_k)$, $\hat{\beta}_\nu \in H^*(L_{i_1}, K_{i_1}) \otimes \ldots \otimes H^*(L_{i_p}, K_{i_p})$, and $\hat{\gamma}_\nu \in H^*(K_{j_1}, *) \otimes \ldots \otimes H^*(K_{j_q}, *)$. By the Serre Theorem and the long exact sequence for the triple $(L_k, K_k, *)$ we get that every $\hat{\varrho}_\nu$ is of the form either or

$$\hat{\varrho}_\nu = \delta^* \hat{a}(\hat{z}_k), \qquad \nu \in N_k' ,$$

$$\hat{\varrho}_\nu = \delta^*(\hat{b}_1(\hat{z}_k) \cdot \ldots \cdot \hat{b}_r(\hat{z}_k)), \qquad \nu \in N_k - N_k' ,$$

where $\hat{a}$ and $\hat{b}_i$ are stable cohomology operations. Using Lemma 3.5 we find cochain operations "representing" these, that is, $a, b_i \in Z\mathcal{O}(U_m; V)$ with $\varepsilon a = \hat{a}$, $\varepsilon b_i = \hat{b}_i$. Also use the induction hypothesis (for $m-1,0,0$)) to find cochain operations $\alpha_\nu \in Z\mathcal{O}(U_1, \ldots, U_{m-1}; V)$ with $\varepsilon\alpha_\nu = \hat{\alpha}_\nu$. For each $\nu \in N_k'$, define the operation $G_\nu \in Q^{m,p,q}$ by means of the following family of cochains

$$\begin{array}{ll} (-1)^t \alpha_\nu(\mathrm{proj}^\# e_{n_1}, \ldots, \mathrm{proj}^\# e_{n_{m-1}}) \times a(e_{n_m}) \times \beta_\nu \times \gamma_\nu & \text{for } n_m > k , \\ 0 & \text{for } n_m \leqq k , \end{array}$$

$\times$ designating exterior cup-product and t being the integer

$$t = (\textstyle\sum_{i=1}^{m-1} n_i)\big(g(a) + g(\beta_\nu) + g(\gamma_\nu) + \sum_{s=1}^{p} i_s + \sum_{s=1}^{q} j_s + \\ + \big(n_m g(\beta_\nu) + g(\gamma_\nu) + \sum_{s=1}^{p} i_s + \sum_{s=1}^{q} j_s\big) .$$

One easily sees that $\nabla_{m,p,q} G_\nu$ has the property that it is zero if a cocycle is inserted at the m'th place; also, it is zero if a cochain of dimension $\neq k$ is inserted at the m'th place. For each $\nu \in N_k - N_k'$, define the operation $G_\nu \in Q^{m,p,q}$ by means of the following family of cochains

$$\begin{array}{ll} (-1)^t \alpha_\nu(\mathrm{proj}^\# e_{n_1}, \ldots, \mathrm{proj}^\# e_{n_{m-1}}) \times [b_1(e_{n_m}) \cdot \ldots \cdot b_r(e_{n_m})] \times \beta_\nu \times \gamma_\nu & \text{for } n_m = k , \\ 0 & \text{for } n_m \neq k , \end{array}$$

where t is the integer

$$t = (\textstyle\sum_{i=1}^{m-1} n_i)\big(\sum_{i=1}^{r} g(b_i) + g(\beta_\nu) + g(\gamma_\nu) + \sum_{s=1}^{p} i_s + \sum_{s=1}^{q} j_s\big) .$$

Again, we get zero if we insert a cocycle at the m'th place in $\nabla_{m,p,q} G_\nu$, and one easily sees that in the $(m-1, p+1, q)$-sequence

$$\varepsilon\big((\nabla_{m,p,q} G_\nu)_{k'}\big) = 0 \quad \text{if } k' \neq k .$$

It is now clear that a suitable linear combination P of such G_ν's will have the properties

(i) $P \in Q^{m,p,q}$,
(ii) $(\nabla_{m,p,q} P)_k \in Q^{m-1,p+1,q}$,
(iii) $\varepsilon\big((\nabla_{m,p,q}(P) + \psi)_k\big) = 0$ in the $(m-1, p+1, q)$-sequence.

If we use the exactness of the $(m-1, p+1, q)$-sequence (which we have by induction), we find operations $E_k \in Q^{m-1,p+1,q}$ with

$$\nabla_{m-1,p+1,q} E_k = (\nabla_{m,p,q} P + \psi)_k .$$

Together the E_k's determine an operation $E \in Q^{m,p,q}$. Since E is zero if a cocycle is inserted on the m'th place, one easily gets for a k-dimensional x_m that

$$\begin{aligned}\nabla_{m,p,q}E(x_1,\ldots,x_m,c_1,\ldots,c_p,y_1,\ldots,y_q) \\ &= \nabla_{m-1,p+1,q}E_k(x_1,\ldots,x_m,c_1,\ldots,c_p,y_1,\ldots,y_q) \\ &= (\nabla_{m,p,q}P+\psi)(x_1,\ldots,y_q)\,.\end{aligned}$$

This proves that ψ is in the image of $\nabla_{m,p,q}$, and the induction step for exactness in the middle is performed. That ε is onto follows easily from Lemma 3.5, and from the assumptions we made on $U_1,\ldots,U_m,V_1,\ldots,V_p,W_1,\ldots,W_q,V$, which guarantee that the exterior cup product is an isomorphism. The proof of Lemma 3.10 is complete.

We note the following: If V is finitely generated, the proof works with no further restrictive assumptions on U and V in the one-variable case

$$\mathcal{O}(U;V) \xrightarrow{\nabla} Z\mathcal{O}(U;V) \longrightarrow \mathcal{a}(U;V) \to 0\,. \tag{9}$$

This is the sequence used for constructing secondary (and higher) cochain operations yielding higher order cohomology operations; see [8] and the subsequent paper [6]. The one-variable case will in [6] be generalized to "extraordinary cochain theories". The sequence in Theorem 3.6 for two variables was the main tool in constructing the secondary cohomology product [5]; the many-variable theorem we have presented here should make it possible to get a different approach to the secondary product in many variables considered by Schweitzer [12]. However, a lot of computation probably has to be done for this purpose.

REFERENCES

1. E. H. Brown, *Note on an invariant of Kervaire*, Michigan Math. J. 12 (1965), 23–24.
2. E. H. Brown and F. P. Peterson, *The Kervaire invariant of (8k+2)-manifolds*, Amer. J. Math. 88 (1966), 815–826.
3. A. Douady, *Les complexes d'Eilenberg-MacLane*, Exposé 8, Seminaire H. Cartan, Paris, 1958–59.
4. D. M. Kan, *On the homotopy relation for* c.s.s. *maps*, Bol. Soc. Mat. Mexicana 2 (1957), 75–81.
5. A. Kock and L. Kristensen, *A secondary product structure in cohomology theory*, Math. Scand. 17 (1965), 113–149.
6. A. Kock, L. Kristensen, and I. Madsen, *Cochain functors for general cohomology theory* II, Math. Scand. 20 (1967), 151–176.
7. L. Kristensen, *On secondary cohomology operations*, Coll. on algebraic topology, Aarhus, 1962, 16–21.
8. L. Kristensen, *On secondary cohomology operations*, Math. Scand. 12 (1963), 57–82.

9. L. Kristensen, *On a Cartan formula for secondary cohomology operations*, Math. Scand. 16 (1965), 97–115.
10. L. Kristensen and I. Madsen, *On evaluation of higher order cohomology operations*, Math. Scand. 20 (1967), 114–130.
11. M. E. Mahowald, *Some Whitehead products in S^n*, Topology 4 (1965), 17–26.
12. P. A. Schwoitzer, *Secondary cohomology operations induced by the diagonal mapping*, Topology 3 (1965), 337–355.
13. H. Toda, *Composition methods in homotopy groups of spheres* (Annals of Mathematics Studies 49), Princeton, 1962.

UNIVERSITY OF AARHUS, DENMARK