

Integration of 1-forms and connections

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Introduction

We shall present a geometric/combinatorial version of the following general wishes: 1) closed group-valued 1-forms locally have primitives; 2) flat (curvature-free) connections in groupoids locally have trivializations; 3) spaces with flat (= curvature free) and symmetric (= torsion free) affine connections are locally affine spaces. In the presentation here, each of two last stages presupposes the preceding one.

For the case of affine connection in combinatorial terms (“formation of infinitesimal parallelograms”), we solve a problem left open in [15] p. 48: when can parallelogram formation be extended to formation of infinitesimal parallelopipeda? - like in an affine space? The classical answer is: when the affine connection is flat and symmetric. In our geometric/combinatorial version, it is a consequence of Theorem 3.7 below.

The solutions we give do not depend on the real numbers; they may be reformulated (when coordinatized) into statements about existence of suitable “formal power series”, without any discussion of convergence. Such reformulations work, when coordinatized, over any field (or even local ring) of characteristic 0. But largely, our exposition is coordinate free.

We shall (at least in the present version) freely use notation and concepts from [7] and [15].

Via well adapted models of synthetic differential geometry, as constructed by Dubuc, (see [7]), the results can be interpreted in the category of smooth manifolds in the classical sense (see e.g. [11]). But some of them apply in other categories, e.g. in some categories coming from algebraic geometry. We shall consider the category of formal manifolds,

in the sense of [7] I.17. The main thing is that the objects M which we consider come equipped with a reflexive symmetric relation \sim , (preserved by the morphisms). For schemes M in algebraic geometry, such \sim was introduced by French algebraic geometry (notably Grothendieck) in the 1960s, via what was called *the first neighbourhood of the diagonal*, $M_{(1)} \subseteq M \times M$.

Part of the notions and proofs we develop in the present paper are phrased entirely in terms of this relation \sim and is purely combinatorial.¹ But to be specific, we consider (formal) manifolds only.

We call \sim the *(first order) neighbour* relation, so $x \sim y$ is read “ x and y are neighbours”, or even (first order) *infinitesimal* neighbours. The set of neighbours of x , we denote $\mathfrak{M}(x)$, the (first order) *monad* of x .

Note that the relation \sim is not assumed to be transitive. The transitive closure of \sim is an equivalence relation, \sim_∞ , namely x and z satisfy $x \sim_\infty z$, if for some $k \in \mathbb{N}$, we have $x \sim_k z$; this in turn means that there is a chain (“ k -path”) $x \sim y_1 \sim y_2 \sim \dots y_{k-1} \sim z$. Therefore, we call the equivalence classe *(infinitesimal) path components*. Our theory deals with such path components, or, equivalently, with a path connected M (= equivalence class for \sim_∞). The equivalence class of x is denoted $\mathfrak{M}_\infty(x)$ (the ∞ -*monad* around x). The notion of ‘local’ is, for simplicity, taken to refer to formally open subsets, i.e. subsets which are closed under the relation \sim_∞ .

1 Group valued 1-forms

1.1 Basic theory of group valued 1-forms

The following Subsection depends on the axiomatics of synthetic differential geometry; the reader who wants to go straight to the combinatorics, may skip this, and take the conclusion Proposition 1.1, and in more general form, Proposition 1.2, as an axiom.

Let M be a manifold and G a group (not necessarily commutative, multiplication denoted $*$, unit by 1). Recall (from [15], say) that a G -valued 1-form is a map $\omega : M_{(1)} \rightarrow G$ with $\omega(x, x) = 1$ for all $x \in M$ (and with $\omega(y, x) = \omega(x, y)^{-1}$; this can often be *deduced*, see [15] Proposition

¹It is worth investigating what the present theory of affine connections has to do with the theory of “edge symmetric double groupoids with connections” of [3].

6.1.3). It is *closed* if

$$\omega(x, y) * \omega(y, z) = \omega(x, z), \quad (1)$$

whenever x, y and z are mutual neighbours. In particular, for a closed 1-form ω , we have for mutual neighbours x, y, z that $\omega(x, y) * \omega(y, z)$ is independent of y . We may ask whether this independence of y also applies if we do not assume that $x \sim z$. We shall prove in the context of synthetic differential geometry, that for closed forms, this independence indeed obtains, under the auxiliary assumption that G is (isomorphic to) a matrix group in the following sense: there exists an associative unitary algebra $(W, *)$, such that G is a subgroup of the multiplicative monoid, and such that W is a KL vector space, in the sense of [15] 1.3. (Think of $(W, *)$ as a matrix algebra.) So for $x \sim y$, we have $\omega(x, y) \sim 1$, so it is of the form $1 + d$ for some $d \in D(W)$ (= the set of \sim -neighbours of $0 \in W$, or $\mathfrak{M}(0)$). Therefore, choosing a coordinate chart $U \rightarrow M$ around x and y , where U is a formally open subset of a KL vector space V , and identifying points in the image of the charts by their coordinates in the KL vector space V , the function ω may in U be expressed in the form

$$\omega(x, y) = 1 + \Omega(x; y - x)$$

for a unique function $\Omega : U \times V \rightarrow W$, linear in the second argument (using the KL property). Recall that $x \sim y$ in U means that $y - x \in D(V)$, (the first order infinitesimal neighbourhood of $0 \in V$). Thus the relation between ω and Ω may equally be expressed that for $x \in U$ and $d \in D(V)$, we have $\omega(x, x + d) = 1 + \Omega(x; d)$.

The following calculation is basically identical to some that occurs in the proof of Proposition 6.2.5 in [15] (where the present Ω is denoted $l\omega$, and the present $(W, *)$ is denoted (A, \cdot)). But note that in loc.cit., it is assumed that x, y, z are mutual neighbours, whereas we here do not assume that $x \sim z$, but only that $x \sim y \sim z$. Thus we have $y = x + d_1$ and $z = y + d_2 = x + d_1 + d_2$ with d_1 and d_2 in $D(V)$; but we are not assuming that $d_1 + d_2 \in D(V)$.

So the x, y , and z considered are of the form $x, x + d_1$, and $x + d_1 + d_2$, respectively, with d_1 and d_2 in $D(V)$. We calculate for such $(d_1, d_2) \in D(V) \times D(V)$ the expression for $\omega(x, y) * \omega(y, z)$ in terms of Ω :

$$\begin{aligned} \omega(x, y) * \omega(y, z) &= (1 + \Omega(x; d_1)) * (1 + \Omega(x + d_1; d_2)) \\ &= 1 + \Omega(x; d_1) + \Omega(x + d_1; d_2) + \Omega(x; d_1) * \Omega(x + d_1; d_2). \end{aligned}$$

By Taylor expansion, $\Omega(x + d_1; d_2) = \Omega(x; d_2) + d\Omega(x; d_1, d_2)$; substituting this in the two places where $\Omega(x + d_1; d_2)$ occurs, allows us to continue

$$= 1 + \Omega(x; d_1) + \Omega(x; d_2) + d\Omega(x; d_1, d_2) + \\ + \Omega(x; d_1) * \Omega(x; d_2) + \Omega(x; d_1) * d\Omega(x; d_1, d_2)$$

The last term here contains d_1 in a bilinear way, so it vanishes. So we are left with

$$1 + \Omega(x; d_1) + \Omega(x; d_2) + d\Omega(x; d_1, d_2) + \Omega(x; d_1) * \Omega(x; d_2),$$

so (using $\Omega(x; d_1) + \Omega(x; d_2) = \Omega(x; d_1 + d_2)$), we conclude

$$\omega(x, y) * \omega(y, z) = 1 + \Omega(x; d_1 + d_2) + d\Omega(x; d_1, d_2) + \Omega(x; d_1) * \Omega(x; d_2). \quad (2)$$

Proposition 1.1 [*Quadrangle Law*] *If ω is a closed G -valued form, then for $x \sim y \sim z$, we have that $\omega(x, y) * \omega(y, z)$ is independent of y .*

(Note that we cannot shortcut the conclusion of the Proposition by saying: “in fact, $\omega(x, y) * \omega(y, z)$ equals $\omega(x, z)$ ”; for, $\omega(x, z)$ only makes sense if $x \sim z$.)

Proof. We pick a coordinate chart U as above, in particular, $y = x + d_1$ and $z = x + d_1 + d_2$. In terms of these coordinates, we have derived the expression (2) for $\omega(x, y) * \omega(y, z)$. If $d_1 + d_2 \in D(V)$, we have $\omega(x, z) = 1 + \Omega(x; d_1 + d_2)$, so if further ω is closed, we therefore have, by subtracting from (2), that

$$d\Omega(x; d_1, d_2) + \Omega(x; d_1) * \Omega(x; d_2) = 0. \quad (3)$$

For fixed x , the function $d\Omega(x; v_1, v_2) + \Omega(x; v_1) * \Omega(x; v_2)$ is a bilinear function $V \times V \rightarrow W$. By the equation (3), this function vanishes when d_1, d_2 , and $d_1 + d_2$ are in $D(V)$. We leave to the reader to prove that if d_1 and d_2 are in $D(V)$, then $d_1 + d_2 \in D(V)$ iff $d_1 - d_2 \in D(V)$, i.e. iff $d_1 \sim d_2$, (use the characterization of $D(V)$ in terms of symmetric bilinear $V \times V \rightarrow R$, Proposition 1.2.12 in [15]), or again, iff $(d_1, d_2) \in \tilde{D}(2, V)$ (as defined in [15] 1.2). So it follows (Proposition 1.3.3 in [15]) that the expression in (2) only depends on $d_1 + d_2$. For x, y, z , this says that $\omega(x, y) * \omega(x, z)$ does not depend on y , (in coordinates: it does not depend on d_1), but only on x and z . And this assertion does not depend on the choice of chart. This proves the Proposition.

(The converse is also true: if $\omega(x, y) * \omega(y, z)$ is independent of y , then ω is closed. We leave this as an exercise.)

The reason for the name “quadrangle law” is that the conclusion may expressed by saying that given a \sim -quadrangle, meaning four points x, y_1, y_2, z with $x \sim y_1 \sim z$ and $x \sim y_2 \sim z$, we have (for ω closed) that $\omega(x, y_1) * \omega(y_1, z) = \omega(x, y_2) * \omega(y_2, z)$. This equality we shall express as an equality of two “path integrals”, or “curve integrals” of the 1-form ω along the periphery of the quadrangle.

We shall, more generally, describe path integrals of a G -valued 1-forms ω along “paths” of arbitrary finite length. We consider the formal (infinitesimal) substitute of the notion of *path* \underline{x} , for which the task is to describe the “path integral” $\int_{\underline{x}} \omega \in G$:

We define an n -*path* \underline{x} in a manifold M to be an $n + 1$ -tuple (x_0, x_1, \dots, x_n) of points in M with $x_i \sim x_{i+1}$ for $i = 0, \dots, n - 1$. The point x_0 is the *domain* of \underline{x} , and the point x_n is the *codomain* of \underline{x} . If ω is a G -valued 1-form on M , we define the “path integral” $\int_{\underline{x}} \omega$ by

$$\int_{\underline{x}} \omega := \omega(x_0, x_1) * \omega(x_1, x_2) * \dots * \omega(x_{n-1}, x_n).$$

So for $n = 1$, $\int_{\underline{x}} \omega = \omega(x_0, x_1)$.

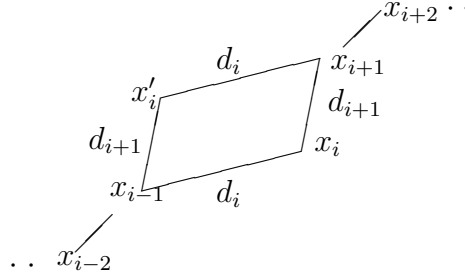
Proposition 1.2 *If ω is a closed G -valued 1-form on a manifold M , then $\int_{\underline{x}} \omega$ only depends on the domain and the codomain of the path \underline{x} .*

Note that for $n = 2$, this is a restatement of the Proposition 1.1.

Proof. As in the proof of the Proposition 1.1, we pick an arbitrary chart U containing all the x_i s of the path; so the path (say, an n -path) may be presented with x_0 , and a sequence $\underline{d} = d_1, d_2, \dots, d_n$ (with $d_i \in D(V)$) with $x_i = x_{i-1} + d_i$ for $i = 1, \dots, n$. From Proposition 1.1 follows that

$$\int_{\underline{x}} \omega = \int_{\underline{x}'} \omega,$$

where \underline{x}' is obtained from \underline{x} by swapping the i th and $(i + 1)$ st of the d_j s ($i = 1, \dots, n - 1$), so as to obtain a new point x'_i (this x'_i is something that depends on the chart)



We can thus swap any two consecutive entries in the sequence of d_j , without changing the value of the integral; and since neighbour transpositions generate the whole symmetric group S_n of permutations σ of n letters, it follows that (for closed ω)

$$\int_{\underline{x}} \omega = \int_{\sigma(\underline{x})} \omega, \quad (4)$$

where $\sigma(\underline{x})$ replaces the $x_i = x_0 + \sum_{j=1}^i d_j$ in the original \underline{x} by $x'_i := x_0 + \sum_{j=1}^i d_{\sigma(j)}$. For fixed x_0 , we therefore have a map which is invariant under the $n!$ permutations of the n input entries (d_1, d_2, \dots, d_n) . By the “Symmetric Functions Property” in its geometric manifestation, [5] Theorem 2.1, it follows that (4), as a function of the d_i s, factors (in fact uniquely) across the addition map $D(V)^n \rightarrow D_n(V)$, i.e. it depends only of the sum $\sum d_j$, not on the individual d_j 's. Equivalently, $\int_{\underline{x}} \omega$ only depends on x_0 and x_n . This is now a statement which does not mention any particular chart. This proves the Proposition.

There is a similar result for 1-forms with values in (the additive group of) a vector space, - say, in the space of scalars R . The proof is simpler, but similar. It is sketched in [5], and it was one of the motivations for that paper.

1.2 Primitives of closed group-valued 1-forms

Let M be a manifold and $G = (G, *)$ a group. If $f : M \rightarrow G$ is a function, we get a G -valued 1-form² df as follows: Let $x \sim y$ in M . Then we put

$$df(x, y) := f(x)^{-1} * f(y).$$

²sometimes called the *Darboux derivative* of f ; it is f^* applied to the Maurer-Cartan form $a^{-1}b$ on G .

This is clearly a closed form. If ω is any G -valued 1-form on M , and $\omega = df$ for some $f : M \rightarrow G$, we say that f is a *primitive* of ω . So a necessary condition for ω to have a primitive is that ω is closed. If $U \subseteq M$ is “formally open”, (meaning: $x \in U$ and $x \sim y$ implies $y \in U$), then we may have a function $f : U \rightarrow G$ satisfying $df(x, y) = \omega(x, y)$ for $x \sim y$ in U , a primitive of ω on U . In global terms, it may be that M can be covered by such U_i s, and possessing primitives on each U_i but with obstructions to patching these “partial” primitives together to a global function $f : M \rightarrow G$. However, on the formal level, we have the following construction. For $x \in M$, let $\mathfrak{M}_\infty(x)$ be the set of points $y \in M$ for which there exists an n -path (for some n) with x as domain and y as codomain. This is clearly a formally open subset of M . Now the following is an easy Corollary of Proposition 1.2

Corollary 1.3 *Let ω be a closed G -valued 1-form on M . Then for each x_0 , there exists a unique partial primitive f for ω , defined on $\mathfrak{M}_\infty(x_0)$ and with $f(x_0) = 1$.*

Proof. Let $y \in \mathfrak{M}_\infty(x_0)$, so there exists (for some n) an n -path \underline{x} from x_0 to y , say $(x_0, x_1, \dots, x_{n-1}, y)$. We put $f(y) := \int_{\underline{x}} \omega$. By Proposition 1.2, this is, for given n , independent of the choice of the path. It may be that y is the codomain of a shorter path, say of length $m < n$; such path \underline{z} may be augmented by a $n-m$ copies of y in the codomain end, to provide an n -path $\underline{z}' = (\underline{z}, y, \dots, y)$; but $\int_{\underline{z}} \omega = \int_{\underline{z}'} \omega$, because $\omega(y, y) = 1$. So f is well defined. Furthermore $df = \omega$. For, if $y \sim z$ and if y can be reached from $x = 0$ by an n -path \underline{x} , then we may use \underline{x} and the $n+1$ -path (\underline{x}, z) to describe $f(y)$ and $f(z)$ respectively; and these two paths show that $f(z) = f(y) * \omega(y, z)$, or equivalently, $\omega(y, z) = f(y)^{-1}f(z)(= df(y, z))$. So the constructed f is indeed a primitive of ω . The uniqueness of f follows easily by induction in n .

Note that the paths in M form a category, by concatenation of paths; and that $\int \omega$ takes composition in this category to multiplication $*$ in G .

In the following, we assume that the manifold M is path connected, meaning that any two points in M can be connected by an n -path, for some n ; equivalently, for all $x \in M$, we have $\mathfrak{M}_\infty(x) = M$. This is a strong smallness condition; in fact, if \sim is trivial in the sense that $x \sim y$ implies $x = y$, then only one-point spaces are path connected! However,

in well adapted toposes, and in algebraic geometry, the infinitesimal neighbour relation \sim is not trivial.

More importantly, though, is the fact that the infinitesimal constructions form a blueprint of what kind of approximations can be made in the physical world, where one considers small steps as infinitesimal (building a round chimney out of square bricks, or forming Riemann sums). But it goes also, and primarily, the other way: from the small steps in the real world, one gets the geometric idea for rigorous \sim -infinitesimal notions - out of which even may grow rigorous analytic calculations (say in the form of power series). Often, the calculations is all you are presented with, as if they had dropped from the skyes.

2 Connections in groupoids

The content of the present Section is presented in more detail in [12].

We consider a groupoid $\Phi \rightrightarrows M$, where M is equipped with a reflexive symmetric relation \sim . Recall from [6], [21] or [15] that a connection in such groupoid may be defined as a map $\nabla : M_{(1)} \rightarrow \Phi$ with $\nabla(x, x) = 1_x$ and $\nabla(y, x) = \nabla(x, y)^{-1}$.

The connection ∇ is called *flat* (or *curvature-free*) if

$$\nabla(x, y) \cdot \nabla(y, z) = \nabla(x, z), \quad (5)$$

whenever $x \sim y$, $y \sim z$ and $x \sim z$, in analogy with (1). (We compose from left to right in Φ .) In fact (1) may be seen as the special case where the groupoid $\Phi \rightrightarrows M$ is $M \times M \times G$, and the connection is given by $\nabla(x, y) := (x, y, \omega(x, y))$ (a “constant” groupoid with vertex group G). For a groupoid which is locally of this form, one may locally choose such a trivialization, and in terms of that, one can encode the connection by a G -valued 1-form, which is closed iff the connection is flat. Therefore, for a flat connection ∇ in such a groupoid, the Proposition 1.1 implies that

$$\nabla(x, y_1) \cdot \nabla(y_1, z) = \nabla(x, y_2) \cdot \nabla(y_2, z), \quad (6)$$

for any \sim quadrangle (x, y_1, y_2, z) (meaning that $x \sim y_i \sim z$ for $i = 1$ and $i = 2$); and this statement does not depend of the choice of the local trivializations.

There is a more general notion of non-holonomous³ connection in such a groupoid; it is a law which to an n -path \underline{x} in M associates an arrow $\nabla(\underline{x}) : x_0 \rightarrow x_n$ in Φ ; such laws are called *non-holonomous* connections (of order n), cf. considered in [6] and [21]; se also [10]. Any connection ∇ in $\Phi \rightrightarrows M$ gives rise to such non-holonomous connection of order n , namely: to $\underline{x} = (x_0, \dots, x_n)$, one associates the composite arrow in Φ ,

$$x_0 \xrightarrow{\nabla(x_0, x_1)} x_1 \xrightarrow{\nabla(x_1, x_2)} x_2 \cdots \xrightarrow{\nabla(x_{n-1}, x_n)} x_n.$$

This non-holonomous connection is denoted $\nabla * \nabla * \dots * \nabla$ (n times), or ∇^{*n} , cf. [21]. In case $\Phi \rightrightarrows M$ is the groupoid $M \times M \times G \rightrightarrows M$, ∇ may be identified with a G -valued 1-form, and $\nabla^{*n}(\underline{x})$ may be identified with $\int_{\underline{x}} \omega$.

We assume, as in the beginning of Section 1, that G admits an (auxiliary) multiplication preserving embedding into an algebra $(W, *)$ (short: “ G is a matrix group”).

The following⁴ is now an immediate generalization of Proposition 1.2.

Proposition 2.1 *Assume that, locally, $\Phi \rightrightarrows M$ admits some isomorphisms (over M) with groupoids of the form $M \times G \times M$ for some matrix group G ; then if ∇ is flat, $\nabla^{*n}(\underline{x})$ only depends on x_0 and x_n .*

Proof. The auxiliary isomorphism allows us to translate the data of ∇ into a G -valued 1-form ω , which is closed iff ∇ is flat. Then Proposition 1.2 shows the independence.

Note that such an auxiliary isomorphism of Φ with $M \times G \times M$ is not intrinsic to the geometry; but since the conclusion of the Proposition does not mention this auxiliary isomorphism, the conclusion is intrinsic to ∇ and $\Phi \rightrightarrows M$.

The law ∇^{*n} satisfying the conclusion of the Proposition is then what [6] and [21] would call a *holonomous* n th order connection in the groupoid, meaning that its value on an n -path only depends on the endpoints of the path.

³More completely: “not-necessarily-holonomous” connections.

⁴I believe that it was first proved in [21], Theorem 7.

It is clear that (whether ∇ is flat or not), the construction provides a functor from the category of paths in M to the category (groupoid) $\Phi \rightrightarrows M$. Thinking of the category of paths as a formal version of the category of (Moore-) paths in M , this functor is in terminology from [21] (see also 5.8 in [15]), the *path connection* given by ∇ .

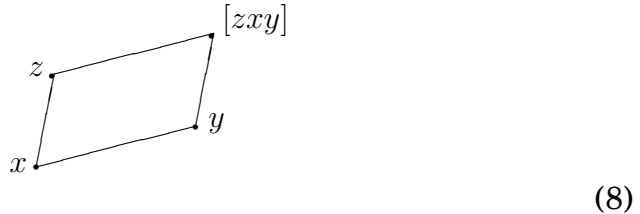
3 Affine connections

Affine connections, in the combinatorial sense of [8], may be seen (Subsection 3.1 below) as a particular case of groupoid valued connections, as discussed in the previous Section.

An affine connection is a certain structure λ on a set M , equipped with a symmetric reflexive relation \sim . Namely λ is a partially defined ternary operation, $(x, y, z) \mapsto [zxy]$ on M , which is defined whenever $x \sim y$ and $x \sim z$.⁵ The axioms are: a book-keeping axiom, and three equational axioms. The book-keeping axiom is that for all such x, y, z :

$$[zxy] \sim y \text{ and } [zxy] \sim z, \tag{7}$$

which may be depicted by



in which the line segments display the \sim relation. The equational axioms are two unit laws and one inversion law: the unit laws are

$$[zxx] = z, \tag{9}$$

$$[xxy] = y. \tag{10}$$

⁵The notation λ for the ternary operation was used in [8] and [15]; what presently is denoted $[zxy]$ was in loc. cit. denoted $\lambda(x, y, z)$; the discrepancy in the ordering of the arguments will not be relevant presently, since we here hardly ever supply the symbol λ with arguments. Essentially, the $[zxy]$ -notation goes back to [20].

and the inversion law is

$$[[zxy]yx] = z. \quad (11)$$

The geometric meaning is that $[zxy]$ is the result of translating z by that parallel translation which takes x to y . This process is in many models for the theory asymmetric in y and z (z is “passive” (being moved), y is “active” (is the mover)); we shall here be interested in the case where further the symmetry law holds:

$$[zxy] = [yxz], \quad (12)$$

in which case we call the affine connection *symmetric* (or *torsion free*) (and then the two unit laws of course are equivalent).

However, even without symmetry, the two unit laws and the inversion law suffice to construe an affine connection as a connection in the groupoid theoretic sense, as described in see Subsection 3.1 below.

An affine connection λ may be used for the following construction. Given two paths \underline{y} and \underline{z} , with common domain x , say $\underline{y} = (x, y_1, \dots, y_n)$ and $\underline{z} = (x, z_1, \dots, z_m)$, we may form a 2-dimensional $m \times n$ “grid” $u_{i,j}$ by induction: We put $u_{0,0} := x$ and $u_{0,j} := y_j$, $u_{i,0} := z_i$, and

$$u_{i+1,j+1} := [u_{i+1,j}u_{i,j}u_{i,j+1}]. \quad (13)$$

We call x the *domain* of the grid, $u_{m,n}$ the *codomain*. We have, by the book-keeping laws for λ , that $u_{i+1,j} \sim u_{i,j} \sim u_{i,j+1}$. Note that $u_{1,1} = [z_1, x, y_1]$. For this construction, we did not assume the symmetry law for λ . But if we also have symmetry of λ , it is clear that the construction of the 2-dimensional grid is likewise symmetric, in the sense that the grid, obtained by interchanging \underline{y} and \underline{z} , is the transpose of the original grid (the j, i entry in the transposed grid equals the i, j entry in the original); so in particular, the codomain of the grid “spanned by” \underline{y} and \underline{z} equals the codomain of the grid “spanned by” \underline{z} and \underline{y} .

3.1 Affine connections as groupoid connections

Affine connections may be seen as a particular case of groupoid valued connections in the sense of Section 2. Namely, for any manifold M , we

have the groupoid $GL(M) \rightrightarrows M$, where an arrow $x \rightarrow y$ is a bijection $\mathfrak{M}(x) \rightarrow \mathfrak{M}(y)$ taking x to y .⁶

More explicitly, for $x \sim y$ in M , the map $z \mapsto [zxy]$ defines a map $\widehat{\lambda}(x, y) : \mathfrak{M}(x) \rightarrow \mathfrak{M}(y)$, by the (first) book-keeping law (7), and it takes x to y , by the unit law (10); it is a bijection with inverse $\widehat{\lambda}(y, x) : \mathfrak{M}(y) \rightarrow \mathfrak{M}(x)$ by the inversion law (11). It takes x to y , by the other unit law (9). We shall also denote $\widehat{\lambda}$ by ∇ (if λ is understood), to conform with the notation of Section 2,

$$\nabla(x, y) := z \mapsto [zxy].$$

This viewpoint was likewise introduced in [8], see also [15] 2.3. The further requirement for λ , namely that $z \sim [zxy]$, we have here taken as a further book-keeping law (the second in (7)), even though it in synthetic differential geometry follows from general principles.

An affine connection λ is called *flat*⁷ if the corresponding groupoid valued connection $\widehat{\lambda}$ is flat (curvature free). Thus flatness implies by the Quadrangle Law (Proposition 1.1) that transport of any $z \sim x$ around the two 2-paths in an arbitrary quadrangle with first vertex x yield the same result. - For the case where λ is symmetric, we have some particular quadrangles which deserve the name *parallelograms*, namely quadrangles of the form $x, y, z, [zxy]$, as displayed in the picture (8); it deserves the name: the *parallelogram spanned by y and z with x understood from the context*; x is called the *domain* or the *base* of the parallelogram; the *codomain* of the parallelogram is $[zxy] = [yxz]$.

We consider henceforth an affine connection λ which is both symmetric and flat.

The equation for moving $z \sim x_0$ (using $\widehat{\lambda}$) around the two 2-paths from x_0 to $[x_2x_0x_2]$ in such parallelogram gives same result, by flatness:

$$[[zx_0x_1]x_1[x_1x_0x_2]] = [[zx_0x_2]x_2[x_2x_0x_1]] \quad (14)$$

and is in simplified notation the equation (17) below.

⁶This groupoid is, for suitable notion of manifold, isomorphic to the locally constant groupoid consisting of fibrewise linear isomorphisms $T_x(M) \rightarrow T_y(M)$, see Theorem 4.3.4 in [15], whence the choice of the acronym “GL”.

⁷Sometimes, e.g. in [1], “flat” means what in our terminology is “flat plus symmetric”.

The local triviality assumptions of Proposition 2.1 are valid for the groupoid $GL(M)$, if M is a manifold, (using charts from a vector space) and imply the following for a flat λ : for any path $x \sim y_1 \sim y_2 \sim \dots \sim y_n$, and any $z \sim x$, the result u_n of “iterated transport of z along the path”

$$[[[[z, x, y_1], y_1, y_2], y_2, y_3] \dots, y_{n-1}, y_n] \quad (15)$$

is independent of the intermediate points y_1, \dots, y_{n-1} (so ∇^{*n} is holonomous, in the terminology applicable for general groupoid valued connections).

Suppose we are given two paths with domain x , say \underline{y} and \underline{z} , as above, and the resulting grid $\underline{y} \times \underline{z}$, with entries $u_{i,j}$ as constructed from it, as in (13). Using flatness of the affine connection, we can then prove

Proposition 3.1 *The point $u_{m,n}$ only depends on x , y_n , and z_m .*

Proof. We have to prove that the point $u_{m,n}$ is independent of the choice of the paths \underline{y} and \underline{z} . By symmetry, it suffices to prove that for fixed \underline{z} , it is independent of the choice of the path \underline{y} . This follows by induction in the length m of the path \underline{z} . For $n = 1$, this is a consequence of the flatness of λ : the u_n in (15) above is independent of the choice of \underline{y} , as we observed; and this u_n is the one that in the grid $\underline{y} \times \underline{z}$ appears as $u_{1,n}$. The result now follows by applying the induction hypothesis to z, z_m and $u_{1,n}$, using the bijection of paths from x to y_n on the one hand, and paths from z to $u_{i,n}$ (using transport along xz) on the other.

Assume now that M is path connected. For given $x \in M$, we have therefore an everywhere defined binary operation $+_x$ given as follows: $z +_x y$ is the codomain of the grid given by a path from x to z and a path from x to y . This value does not depend on the paths chosen, by Proposition 3.1. Also, if $z \sim x \sim y$, we have

$$z +_x y = [zxy]. \quad (16)$$

Since M is assumed path connected, the transitive closure \sim_∞ of \sim is the trivial relation: for all x and y in M , we have $x \sim_\infty y$. The following is then almost immediate:

Proposition 3.2 *For fixed x , the binary operation $+_x$ is commutative, and has x as a unit. The ternary operation $(x, y, z) \mapsto z +_x y$ defines a symmetric affine connection, with respect to the trivial neighbour relation \sim_∞ on M , and it extends the given λ .*

Note that we have not yet asserted associativity of $+_x$. This will be proved in Theorem 3.6 below.

The triviality of the relation \sim_∞ means that we can forget about it, in particular, the book-keeping laws (7) are trivially satisfied. We have by (16) in fact extended the given affine connection λ , and may use the same notation $[zxy]$ for this extended and everywhere defined operation. If we need to distinguish, we call the original connection the *small* (or \sim -restricted) one, the new extended we call the *big* (or unrestricted) one, and similarly for parallelograms.

3.2 The Cube Lemma

We come to the combinatorial core of this Section. We still consider a (\sim -restricted) affine connection λ which is both symmetric and flat. By symmetry of λ , we have a well defined notion of parallelogram, spanned by two neighbours x_1 and x_2 of x_0 , and, more generally, we have a well defined 2-dimensional grid spanned by two paths with common domain x_0 .

We now consider the case of *three* neighbours of x_0 , and, more generally, of three paths with common domain x_0 .

Given a point x_0 , and three neighbour points x, y, z of it. Let us name these three points x_1, x_2 , and x_4 , in some order.⁸ We get three parallelograms with base x_0 : 1) the one spanned x_1 and x_2 , 2) the one spanned x_1 and x_4 , and 3) the one spanned by x_2 and x_4 . These parallelograms appear in the following picture as faces adjacent to 0 of the displayed cube (the point marked “7” will be argued after the calculation); for simplicity we have written k for x_k ($k = 0, 1, 2, 4$), and omitted commas.

Moving 4 along the two paths from 0 to $[102] = [201]$ give the same result, by the flatness of λ :

$$[[401]1[102]] = [[402]2[201]]; \quad (17)$$

similarly moving 2, (or by renaming the three variables x, y, z , i.e. by permuting the indices 1,2,4)

$$[[204]4[401]] = [[201]1[104]]; \quad (18)$$

⁸The reason for choosing the name x_4 , rather than x_3 , will be given later.

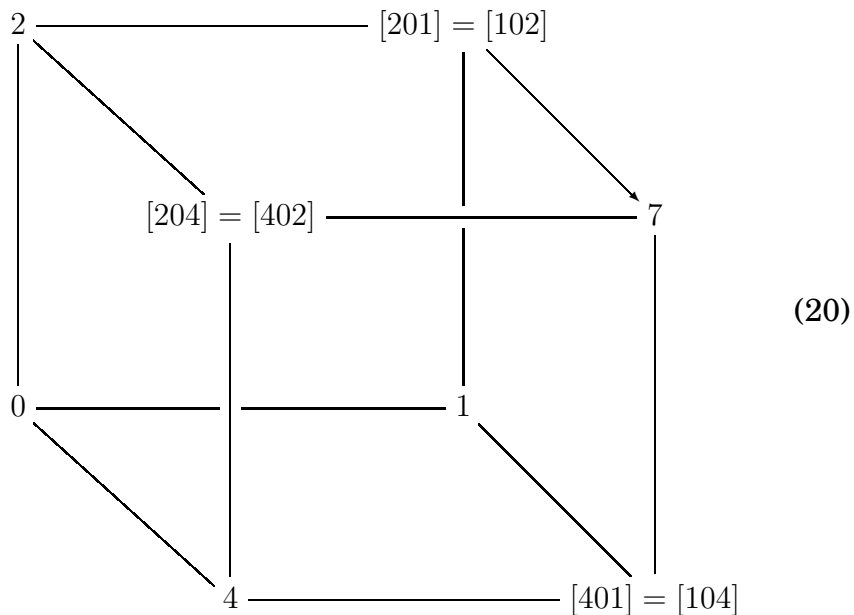
and similarly, moving 1

$$[[102]2[204]] = [[104]4[402]]. \quad (19)$$

The left hand side of (17) equals the right hand side of (18), by symmetry of λ ; the left hand side of (18) equals the right hand side of (19), by symmetry of λ ; and the left hand side of (19) equals the right hand side of (17), by symmetry of λ . Note that we have only been using flatness w.r.to parallelograms (“weak flatness”). We conclude:

Lemma 3.3 (Cube Lemma) *Assume that λ is a symmetric and (weakly) flat affine connection. Then all six expressions appearing in the equations (17), (18) and (19) are equal.*

This equal value is the point named “7” in the following picture.



(The naming of x_0 by 0, and of x , y , and z by 1, 2, 4 (in some order) is a mnemotechnic device, with the purpose that the remaining points in the cube may be named 3, 5, 6, 7 in such a way that $[pqr] = p - q + r$; thus $[204] = 6$, and $[623] = 7$. The recipe for the naming is: consider the coordinate set of a vertex of the unit cube in \mathbb{Z}^3 as a number in digital notation; then write this number in decimal notation (just for compactness); e.g. the coordinate set of the point $[204]$ is 110 which

is digital notation for the number which in decimal notation is 6. We invite the reader to write on the cube, writing the “3” for $[102](= [201])$ etc.; “7” is then the equal value of any of the expressions in (17), (18) and (19).)

(On the other hand, it is easy to see that if for a symmetric affine connection, the conclusion of the Cube Lemma holds, then this connection is weakly flat, i.e. the conclusion of the Quadrangle Law (Proposition 1.1) holds if the quadrangle is a parallelogram.)

3.3 Three dimensional grid

In the following Subsection, we shall strengthen the conclusion of Proposition 3.2 by adding a flatness assertion:

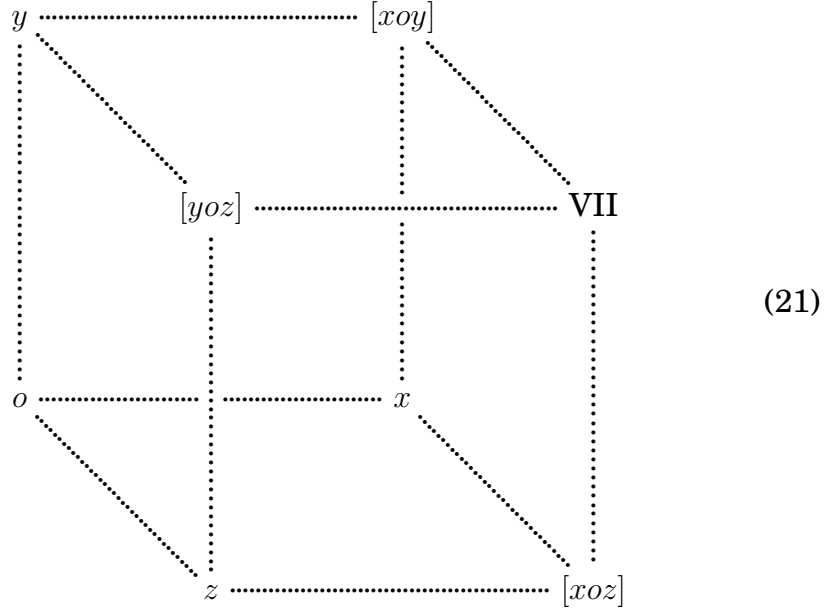
Proposition 3.4 *If λ is a symmetric flat (\sim -restricted) affine connection λ on a path connected M , then the extension of λ to an unrestricted affine connection is symmetric, and flat with respect to (big) parallelograms.*

Proof. Only the flatness remains to be proved. The crux is to use the Cube Lemma for the restricted λ to build a 3-dimensional grid (or “big cube”) $\underline{x} \times \underline{y} \times \underline{z}$ out of three paths \underline{x} , \underline{y} , \underline{z} with common domain, say o , and lengths n , m , and k , respectively. The i, j, l entry $w_{i,j,l}$ in the desired 3-dimensional grid is constructed by induction, using the given affine connection λ . The initial conditions are $w_{0,0,0} = o$, $w_{i,0,0} = x_i$, $w_{0,j,0} = y_j$, $w_{0,0,l} = z_l$. The codomains of the three paths are denoted x , y , and z , respectively, thus $x = x_n$, $y = y_m$, $z = z_k$.

The induction step uses crucially the Cube Lemma: $w_{i+1,j+1,l+1}$ is the last vertex in the cube generated by $0 := w_{i,j,k}$, $1 := w_{i+1,j,k}$, $2 := w_{i,j+1,k}$, $4 := w_{i,j,k+1}$, as in the figure (20) (thus, $3 = [w_{i,j,k}, w_{i+1,j,k}, w_{i,j+1,k}]$ etc.)

The codomain $w_{n,m,k}$ of the 3-dimensional grid is the point VII in the figure below.

Each of the six faces of the big cube is a 2-dimensional grid, and each of them can be seen as the witness of a parallelogram for the unrestricted connection which we have constructed; thus the face containing o, x, y is a grid constructing $[xoy]$ for the unrestricted connection. Similarly, the face containing the vertices $x, [xoy], [xoz]$ is a grid constructing $[[xoy]x[xoz]]$; this is in the figure named “VII”. It also appears as a construction of other combinations of o, x, y, z , like $[[xoy]y[yoz]]$:



The reader will observe that, except for the naming of the vertices, the cube in this figure looks like the cube in the previous one. But note the difference: in the previous one (the “small” cube), the lines indicate the \sim relation, and the argument was that there were several constructions leading to the same result, which we then were allowed to give a name (choosing “7” for this name). In the present “big” cube, the lines indicate paths (therefore displayed as “dotted” lines), where the three lines out of o are arbitrary paths, and the rest of the cube is constructed canonically as the grid which these three paths generate. The argument is now that the vertex VII (Roman notation for 7) is *constructed* (as the last vertex $w_{n,m,k}$ of the grid), and we give *interpretations* of it in terms of the unrestricted connection. More explicitly, the expressions in these equations (with suitable renaming) express the various ways we may see the, apriori existing, VII.

There are some interpretations of VII, available in the big cube, whose analogs are not available for 7 in the small cube. These interpretations are based on the fact that one can concatenate paths. Thus, VII is the common value of the three expressions in (23):

Proposition 3.5 *[Cancellation law] For any o, x, y, z on M , we have*

$$[[xoy]oz] = [[xoy]y[yoz]] = [xo[yoz]]. \quad (22)$$

In particular, we have the associative law

$$[[xoy]oz] = [xo[yoz]]. \quad (23)$$

Proof. The middle expression in (22) is the VII in the cube. To construct the right hand side, we must pick two paths: from o to x , and from o to $[yoz]$. For the first, we pick the path \underline{x} , already used, and which appears in the grid as the path of points

$$o, w_{1,0,0}, w_{2,0,0}, \dots, w_{n,0,0}.$$

For the second, we pick the concatenation of two of the paths that appear as edges in the cube; explicitly, the concatenated path is

$$o, w_{0,1,0}, \dots, w_{0,2,0}, \dots, w_{0,m,0}, w_{0,m,1}, \dots, w_{0,m,k}.$$

It is clear that the 2-grid (of size $n \times (m + k)$) (constructed using the original (small) connection) has as its codomain $w_{n,m,k}$, which is also the codomain of the 3-grid used for VII.

(Note that we cannot state the equation (22) for a general affine connection, since we do not have $[xoy] \sim o$, in general, so the \sim -restriction for forming $[[xoy]oz]$ may not hold.)

Therefore, we have that for fixed $o \in M$, the binary operation $(x, y) \mapsto [xoy]$ is associative: both $[[xoy]oz]$ and $[xo[yoz]]$ equal $[[xoy]y[yoz]]$. If M is path connected, this is an everywhere defined binary operation $M \times M \rightarrow M$. It makes good sense to denote this binary operation on M by $x +_o y$:

$$x +_o y := [xoy].$$

Theorem 3.6 *For each $o \in M$ (assumed path connected), the binary operation $(x, y) \mapsto x +_o y$ makes M into an abelian group. For any other $o' \in M$, the bijection $z \mapsto [zoo']$ is a group isomorphism.*

Proof. We have already (Proposition 3.2) that the operation $+_o$ is commutative, and also that o is a unit. We just proved that it is associative. For existence of inverses, we have that $[oxo]$ will serve:

$$[xo[oxo]] = [[xoo]o[oxo]] = [[xoo]xo] = [xxo] = o$$

the second equality sign by the cancellation law, and the first and the two last equality signs by the unit law.

For the last assertion of the Theorem, we calculate:

$$[[xoo']o'[yoo']] = [[xoo']o'[o'oy]] = [[xoo']oy]$$

by symmetry and cancellation; and we continue:

$$= [[o'ox]oy] = [o'o[xoy]] = [[xoy]oo']$$

by symmetry, the associative law, and by symmetry again. So if f denotes the bijection considered, the total equation says that $[f(x)o'f(y)] = f([xoy])$, and the Theorem is proved.

The Theorem here is really classical, going back to Prüfer [P], who considered a ternary operation λ satisfying similar equations as ours; but globally defined. He denoted by $(zx^{-1}y)$ what we denote $[zxy]$; out of which he derives abelian group structures like $x +_o y$. In fact, his theory is, just as ours, an equational presentation of the affine core of the theory of abelian groups, with this ternary (but globally defined) operation as (the only) generator.

[P] calls a set with such a ternary operation a *Schar*; such structures, or generalizations thereof, have been discovered independently by many authors, and under many names: by Baer, Certainé, Vagner, Lawson, and others, including myself; see in particular Lawson's "Generalised Heaps as Affine Structures", in Hollings and Lawson Wagner's *Theory of Generalised Heaps*, Springer 2017, Cham), [18].

In fact, a "Schar" or "heap" should just be termed: an *affine space over the ring \mathbb{Z} of integers of scalars*.

What distinguishes our equations from Prüfer's is that they admit restriction by a (reflexive symmetric) "neighbour" relation \sim . Some of the equations, valid in the unrestricted theory, do not make sense under such a restriction, see the remark after Proposition 22.

Theorem 3.6 may be reformulated, using known properties of affine spaces in general:

Theorem 3.7 *Consider a manifold M equipped with a symmetric and flat connection λ . Then every ∞ -monad \mathfrak{M} in M carries canonically structure of an affine space over \mathbb{Z} , with $\lambda(x, y, z) = y - x + z$, for any $y \sim x \sim z$ in M .*

Our version is a “formally local” one, i.e. for the notion of “local” derived from the notion “formally open”. So in be coordinatized form, it gives only formal power series solutions, not anything about convergence. On the other hand, our result is canonical, whereas the classical result expresses that charts *exist* with certain properties, not the naturality of such charts.

If we replace the ring \mathbb{Z} by the ring of reals \mathbb{R} , the (real) local result is a version of a Theorem of Chern 1952, as quoted in [1] as “p. 108” in these Chern notes..

We shall sketch in the following Subsection how scalars (from \mathbb{R} , say) may be introduced in the \mathbb{Z} -affine structure that we have constructed from λ .

The conclusion of the (classical) result, as rendered synthetically in Theorem 3.7.4 in [15] (= Corollary 3 in [8] = Theorem 2.3 in [9]) imply this result, but note that in in these formulations, an abelian group (in fact, a vector space) is *apriori given* (the vector space on which the manifold M is locally modelled), whereas in our formulation above, the abelian group is *constructed* (in the form of the affine structure on the ∞ -monad). Furthermore, in loc. cit. we *assume* that a certain closed (group valued) 1-form is exact, which we here have *proved* in Corollary 1.3 to be the case.

So what the present paper adds to this classical theory is that it *derives* the global ternary operation, and the needed equations, out of infinitesimal data, namely the \sim -restricted (thus partially defined) affine connection.

3.4 Affine combinations with scalars

We take the notion of *affine space* over a commutative ring R as meaning: “affine combinations with coefficients from R , may be formed” (recall that an affine combination is a linear combination where the sum

of the coefficients is 1). If we take R to be \mathbb{Z} , we get the notion of “Schar”, “heap”, “commutative pregroup” etc. To have more general coefficients (say \mathbb{Q}), so that we e.g. can form the affine combination “midpoint”, $\frac{1}{2}x + \frac{1}{2}y$, it suffices that we can form binary affine combinations, like $(1-t)x + ty$, for any $t \in R$, satisfying suitable compatibilities. We shall only be sketchy here.

Let R denote any commutative ring containing the rational numbers \mathbb{Q} . We assume that we can form binary affine combinations with scalars from R , like $(1-t)x + ty$, for $x \sim y$. Such kind of structure we do have in the following two cases: 1) manifolds over R (meaning: we can locally use charts from a KL vector space over R); and 2) general affine schemes M over R , see [17]. The latter is purely formal, but the theory developed presently does not allow for this level of generality, since the groupoid $GL(M)$ is not necessarily locally constant, (say, if M has singularities). For the manifold case, we have the technique of coordinate charts available, and hence the use of encoding the given affine connection in terms of Christoffel symbols:

$$\lambda(x, y, z) = [zxy] = z - x + y + \Gamma(x; z - x, y - x)$$

with Γ bilinear in the arguments after the semicolon; and for $x \sim y$, the expression $(1-t)x + ty$ turns out not to depend on Γ at all. See [15], 2.3 for details. In fact, the monads $\mathfrak{M}(x)$ and $\mathfrak{M}(y)$ carry an action by the multiplicative monoid of R , and the map $\hat{\lambda} : \mathfrak{M}(x) \rightarrow \mathfrak{M}(y)$ induced by the affine connection λ preserves the action, Proposition 2.3.7 in loc. cit.

To indicate how the action of scalars extend to the whole of M (assumed path connected), I shall just indicate how to form $(1-t)x + tu$ in case where x and u are “second order” neighbours, i.e. in the case where there exists a 2-path $x \sim y \sim u$. Then u is of the form $[zxy]$ for some (unique) z (take $z = [uyx]$). Then we have (for λ symmetric, equivalently, $\Gamma(x; -, -)$ symmetric bilinear):

Proposition 3.8 *Let $u = \lambda(x, y, z)$. Then $\lambda(x, y_t, z_t)$ only depends on t and on u .*

Proof. In a coordinatized situation, let $y = x + d_1$ and $u = y + d_2$. Then $z = x + d_2 - \Gamma(d_1, d_2)$, where Γ denotes the Christoffel symbol at the

point x . Then $u = \lambda(x, y, z)$. The calculation for equation (3.3) in [16] gives that

$$\lambda(x, y_t, z_t) = x + td_1 + td_2 - t\Gamma(d_1, d_2) + \Gamma(td_1, td_2),$$

and using that λ is assumed symmetric, we have that Γ is a symmetric bilinear form, so $\Gamma(v_1, v_2) = \frac{1}{2}\Gamma(v_1 + v_2, v_1 + v_2)$ for any pair of vectors v_1 and v_2 in V ; thus with $y = x + d_1, z = x + d_2 - \Gamma(d_1, d_2)$,

$$\lambda(x, y_t, z_t) = x + t \cdot (d_1 + d_2) + \frac{t^2 - t}{2}\Gamma(d_1 + d_2, d_1 + d_2).$$

This clearly only depends on t and $d_1 + d_2$, i.e. on t and $u = x + d_1 + d_2$ as asserted.

This means that we can define $(1 - t)x + tu$ as $[x, (1 - t)x + ty, (1 - t)x + tz]$, independent of the “interpolating” point y .

Equational and foundational aspects of partially defined structures, like affine connection, with \sim_2 (like the above calculation) rather than \sim_1 , may be found in [2].

Combining the (sketched) possibility of affine combinations with scalar coefficients with Theorem 3.7, we can state the following

Theorem 3.9 *Every flat and symmetric affine connection locally comes about from an actual affine structure, canonically constructed.*

(“locally” in the sense of “formally local”, i.e. on each ∞ -monad).

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The cubes were made with Paul Taylor's "Diagrams" package.

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