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Generators and Relations for Δ as a Monoidal 2-Category

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Introduction

By Δ, we understand the category of finite ordinals $\underline{0}, \underline{1}, \underline{2}, \ldots$, and their order preserving maps. It is the category which defines the notion of simplicial set. As a category, it is well known to be generated by face- and degeneracy operators $d_i : \underline{n} \to \underline{n+1}$, $s_j : \underline{n+2} \to \underline{n+1}$ (for each n), and a certain list of equations (for each n) for its relations, cf e.g. [6]. This category carries structure of monoidal (=tensored) category with $\underline{n} \otimes \underline{m} = \underline{n+m}$ (ordinal sum formation). As a *monoidal* category, it has a much smaller presentation, namely generated by $d_0 : \underline{0} \to \underline{1}$, $s_0 : \underline{2} \to \underline{1}$, and three equations, namely the three equations which define the notion of monoid (unit- and associative law); this was proved by Lawvere [4].

However, as observed by Street [8], being a category of order preserving maps between ordered sets, the category is enriched over partially ordered sets, in the sense that its hom-sets carry a partial order (compatible with the composition, and also with the $\otimes$). In fact it is a 2-category, cf e.g. [1], [2] for these notions; in fact, it is a strict monoidal 2-category. We give generators and relations for its 2-dimensional structure. The result (Theorem 2.1 below) is:

- one generator λ, namely the inequality ("2-cell") $d_1 \leq d_0$ (as maps $\underline{1} \to \underline{2}$)
- two relations: both $d_0 * \lambda : \underline{0} \to \underline{2}$ and $\lambda * s_0 : \underline{1} \to \underline{1}$ are identity 2-cells.

The conventions are that we compose from left to right, for the horizontal composition $*$ (and also for the vertical composition $\cdot$ occurring later on); and that $d_i : \underline{n} \to \underline{n+1}$ *omits* $i \in \underline{n+1}$, where $\underline{n+1} = \{0, 1, \ldots, n\}$. Note that the 2-cell $d_0 * \lambda$ has the same map $\underline{0} \to \underline{2}$ for its vertical source and target, but we are presenting Δ as a category enriched over categories (a 2-category), rather than enriched over partially ordered sets; thus we do not apriori assume that a 2-cell with its source and target coinciding is an identity 2-cell.

I want to acknowledge correspondence with Ross Street; in fact a result about a universal property of Δ as a monoidal 2-category appears in his [8]. It is not exactly the same universal property as the one given here (cf. remark at the end of the paper); the one given here was proposed by Ross Street when he was refereeing [3]. Since the proof I was able to give for it turned out to be a little involved, I decided to publish it separately.

1. The comparison functors

Let Δ' denote a (strictly) monoidal 2-category with a given object T, given 1-cells $y : I \to T$ and $m : T \otimes T \to T$ (I denoting the (strict) unit object), and a given 2-cell $\lambda : y \otimes T \to T \otimes y$,

such that y and m make T into a monoid object, and such that λ satisfies the axioms

$$y * \lambda = y \otimes y \tag{1}$$

$$\lambda * m = T; \tag{2}$$

and which is universal with these data. (We have here used the convention that the identity 1-cell of a 0-cell (object) A is also denoted A; and that the identity 1-cell of a 1-cell (arrow) f is also denoted f.) The monoidal 2-category Δ carries such T, y, m, λ, satisfying the axioms, namely $T = \underline{1}$, y and m the unique maps $\underline{0} \to \underline{1}$ and $\underline{2} \to \underline{1}$, respectively, and with λ the inequality $d_1 \leq d_0$ (as maps $\underline{1} \to \underline{2}$). By the universal property of Δ', there is a monoidal 2-functor $P : \Delta' \to \Delta$ taking T to $\underline{1}$, y and m to $\underline{0} \to \underline{1}$ and $\underline{2} \to \underline{1}$, respectively, and λ to $d_1 \leq d_0$.

Since it is clear that the 2-dimensional structure on Δ' does not impose any constraints on the 1-dimensional structure, it follows that the underlying monoidal 1-category of Δ' is the free monoidal category with a monoid object T, y, m; and therefore, by [4], it is (isomorphic to) Δ, so on the underlying monoidal 1-categories, $P : \Delta' \to \Delta$ is an isomorphism. Our theorem is that P is also an isomorphism of the 2-dimensional structure, Theorem 2.1 below.

Since P is an isomorphism on the 1-dimensional structure, we may identify Δ' and Δ in so far as this structure goes. This gives us two sets of notations for 0- and 1-cells of these categories. For instance, the map (face operator) $d_1 : \underline{2} \to \underline{3}$ which omits $1 \in \{0, 1, 2\} = \underline{3}$ may also be denoted $T \otimes y \otimes T$, or just TyT. As in [3], we let $\otimes$ bind more strongly than the horizontal composition $*$ or the vertical composition $\cdot$, and sometimes we omit $\otimes$ (but we never omit $*$ or $\cdot$). Also $T \otimes T = TT$ may be denoted T^2, and similarly for T^3, etc. To fit with the conventions of [3], we need to have $yT : T \to T^2$ match $d_1 : \underline{1} \to \underline{2}$ (which omits the last element $1 \in \underline{2}$), so it is most natural in the graphics to think of the elements 0 and 1 of $\underline{2}$ as displayed graphically in the order: 1,0 , with the largest element leftmost; similarly for the higher $\underline{n}$'s.

The strategy is now as follows: we shall provide an inverse $J : \Delta \to \Delta'$ for P. On 0- and 1-cells, J will be the identity, of course. On 2-cells, it will be constructed as a splitting of P, $P(J(\alpha)) = \alpha$, and commuting with $\otimes$ and $\cdot$. The construction of this splitting will occupy the rest of the present section, whereas Section 2 will prove that J is an onto map, implying hat J is a two-sided inverse for P which is thus an isomorphism.

Let f and g be maps $\underline{n} \to \underline{m}$ in Δ. If $f \leq g$ (i.e. $f(i) \leq g(i)\ \forall\ i \in \underline{n}$), we say that $dist(f, g) = p$ if $\sum_{i \in \underline{n}} g(i) - f(i) = p$. If $f \leq g$ and $dist(f, g) = 1$, we say that the pair (f, g) is an *elementary move out of* f *with result* g. The unique element $i \in \underline{n}$ with $f(i) \neq g(i)$ (in fact with $f(i) + 1 = g(i)$) is called the *element moved* by the elementary move. Clearly, an elementary move out of f moving i is possible (and then unique) iff i is the largest element in the set (interval) $f^{-1}(f(i))$, and $f(i)$ is not the largest element in $\underline{m}$.

It is clear that if $dist(f, g) = p$, then we can interpolate a chain consisting of p elementary moves between f and g. Such chain is not unique, but at least we have the following "confluence lemma":

Proposition 1.1 *Let* $f : \underline{n} \to \underline{m}$*. If* g *and* h *are results of two distinct elementary moves out of* f*, there is a* $k : \underline{n} \to \underline{m}$ *which is the result of an elementary move out of* g *and also the result of an elementary move out of* h*. (Such* k *is unique.)*

Proof. Let i and j be the elements moved by the elementary moves (f, g) and (f, h), respectively. We have $i \neq j$ since g and h are distinct. We may assume $i > j$. Because of the elementary move (f, g), i is the largest element in $f^{-1}(f(i)) = f^{-1}(h(i))$. This set differs from $h^{-1}(h(i))$ at most by the presence or not of j, which is the only element where f and h differ. Since $i > j$, the status of i as largest in $f^{-1}(f(i))$ is not affected by the presence or not of j, so i is the largest element also in $h^{-1}(h(i))$. So an elementary move out of h moving i is possible, with result the map k given by

$$k(i) = f(i) + 1; k(j) = f(j) + 1; k(x) = f(x) \quad \text{for all other } x. \tag{3}$$

Also $g(i) = f(i) + 1 > f(i) \geq f(j)$, so $g(i) \neq f(j) = g(j)$. Also $f(i) \neq f(j)$, for $f(i) = f(j)$ and $i > j$ would contradict j being largest in $f^{-1}(f(j))$ (as witnessed by the elementary move (f, h)). Since f and g differ only at i, $f^{-1}(f(j)) = g^{-1}(g(j))$ (i belongs to neither); since j is the largest in $f^{-1}(f(j))$, it is largest in $g^{-1}(g(j))$. So an elementary move out of g moving j is possible, and it is clear that the result of the move is again k.

It is clear that any elementary move (f, g) in Δ can be written uniquely

$$(f' \otimes f_0 \otimes f'', f' \otimes g_0 \otimes f'') \tag{4}$$

with (f_0, g_0) an elementary move of maps $\underline{n_0} \to \underline{2}$, for suitable $\underline{n_0}, f', f''$: if i is the element moved by the elementary move (f, g), f'' is the restriction of f to the set of j's with $f(i) > f(j)$, and f' (essentially) the restriction to the set of j's with $f(j) > f(i) + 1$. And it is clear that there is exactly one elementary move $L_{n,i}$ between maps $\underline{n} \to \underline{2}$ which moves i, namely

$$L_{n,i} = (f_0, g_0)$$

with

$$f_0(j) = 0 \text{ for } j \leq i, \ 1 \text{ otherwise,}$$
$$g_0(j) = 0 \text{ for } j < i, \ 1 \text{ otherwise.}$$

We associate to this elementary move $L_{n,i}$ the 2-cell in Δ'

$$\lambda_{n,i} := (T^{n-1-i} \otimes \lambda \otimes T^i) * (m' \otimes m'') \tag{5}$$

where $m' : \underline{n-i} \to \underline{1}$ and $m'' : \underline{i+1} \to \underline{1}$ are the unique such maps. And to the elementary move (4), we associate $f' \otimes \lambda_{n,i} \otimes f''$. This defines our map $J : \Delta \to \Delta'$ on all elementary moves $f \leq g$ in Δ. It is clear that J, so far as it is defined by now, respects the domain/codomain book-keeping, and (hence) splits P. In this and similar considerations, the reader may find the following kind of diagram (illustrating, in this case, $\lambda_{n,i}$) useful (vertical concatenation corresponds to $*$, horizontal concatenation to $\otimes$). Note that λ itself is $\lambda_{1,0}$.

To extend J to yield a 2-cell $J(f,g) : f \Rightarrow g$ in Δ' for any inequality $f \leq g$ in Δ, we decompose $f \leq g$ into a chain of elementary moves $f = h_0 \leq h_1 \leq \ldots \leq h_p = g$, and we would then like to put $J(f,g)$ equal the $\cdot$ -composite of the $J(h_q, h_{q+1})$'s. Since the chain decomposition is not unique, there is a question about well-definedness. Because of the confluence Lemma (Proposition 1.1), it easily follows that it suffices to see that

$$J(f \leq g) \cdot J(g \leq k) = J(f \leq h) \cdot J(h \leq k) \tag{6}$$

where f, g, h, and k are as in the Proposition. Also, let i and j be as in its proof.

It is clear that if ϕ is a 2-cell in Δ which is an elementary move, then so is $\phi \otimes f$ and $f \otimes \phi$ for any 1-cell f in Δ; and it is clear from the description of J on elementary moves that

$$J(\phi \otimes f) = J(\phi) \otimes f, \;\; J(f \otimes \phi) = f \otimes J(\phi).$$

Using such reductions, the problem of proving (6) gets reduced to the case where $f : \underline{n+1} \to \underline{3}$, and the elements i and j moved are the last and first elements n and 0. Thus

$$\begin{aligned}
(f,g) &= L_{n,n-1} \otimes T \mapsto (\lambda \otimes T^{n-1} * T \otimes m^{(n)}) \otimes T \\
(g,k) &= T \otimes L_{n,0} \mapsto T \otimes (T^{n-1} \otimes \lambda * m^{(n)} \otimes T) \\
(f,h) &= y \otimes L_{n+1,0} \mapsto y \otimes (T^{n} \otimes \lambda * m^{(n+1)} \otimes T) \\
(h,k) &= L_{n+1,n} \otimes y \mapsto (\lambda \otimes T^{n} * T \otimes m^{(n+1)}) \otimes y,
\end{aligned}$$

where $m^{(q)} : \underline{q} \to \underline{1}$ is the unique such map (thus $m^{(2)} = m$, $m^{(1)} = T$, $m^{(0)} = y$).

We now calculate $\lambda T^n * T^{n+1}\lambda * Tm^{(n+1)}T$ by the 2-category rules for the $*$-composite of two 2-cells. On the one hand, we get, using bifunctorality of $\otimes$,

$$(\lambda T^{n-1}T * Tm^{(n)}T) \cdot (T^n\lambda * Tm^{(n)}T) = J(f,g) * J(g,k);$$

calculating this the other way round, we get

$$(yT^{n+1} * T^{n+1}\lambda * Tm^{(n+1)}T) \cdot (\lambda T^n * T^{n+2}y * Tm^{(n+1)}T) =$$
$$(yT^{n+1} * T^{n+1}\lambda * Tm^{(n+1)}T) \cdot (T^{n+1}y * \lambda T^{n+1} * Tm^{(n+1)}T).$$

But the two factors here are, by bifunctorality of $\otimes$ seen to be, respectively, $y \otimes (T^n\lambda * m^{(n+1)}T)$ and $(\lambda T^n * Tm^{(n+1)}) \otimes y$, i.e. $J(f,h)$ and $J(h,k)$, respectively. This proves the commutativity (6) for the special case considered, but as argued, this case suffices.

This now means that we have a map $J : \Delta \to \Delta'$ which is the identity on 0- and 1-cells, and preserves the book-keeping and the compositions $\cdot$ and $\otimes$ on 2-cells; and J followed by P is the identity. So to see that J is a two-sided inverse for P, it suffices to see that every 2-cell in Δ' is in the image of J. We turn to this task:

2. 2-cells in the image of J

To see that J is surjective on 2-cells, we use 'structural induction': any 2-cell in Δ' is given by an expression built by means of $\otimes$, $*$, and $\cdot$, starting with λ (and identity 2-cells). If we assume that every 2-cell which can be built by an expression of length $< n$ is in the image of J, and ϕ is an expression of length n, there are four cases to consider:

1) $n = 1$, in which case $\phi = \lambda$ or is an identity 2-cell. In both cases, ϕ is in the image of J.
2) $\phi = \phi' \otimes \phi''$, with ϕ' and ϕ'' expressions of length $< n$. Then the 2-cells given by these expressions are in the image of J, and since J commutes with $\otimes$, the 2-cell given by the expression ϕ is in the image of J.
3) $\phi = \phi' \cdot \phi''$ - the argument is similar.
4) We finally consider the case $\phi = \phi' * \phi''$. Using the rules of 2-categories, this may be written as a $\cdot$ -composite of expressions of form $f * \phi''$ and $\phi' * g$ with f and g 1-cells, and each of these expressions is strictly shorter than $\phi = \phi' * \phi''$ unless one of the ϕ's is of length 1, that is, unless $\phi' = \lambda$ or $\phi'' = \lambda$. So this leaves us with proving

Proposition 2.1 *1) For any 1-cell f, $f * \lambda$ is in the image of J. 2) For any 1-cell g, $\lambda * g$ is in the image of J.*

Proof. 1) f is of form $m^{(k)} : \underline{k} \to \underline{1}$ for some k; the cases $k = 0$ and $k = 1$ yield $y * \lambda = y \otimes y$ (by (1)) and λ, respectively, and both of these are in the image of J. For $k = n > 1$, we shall prove the equation

(7) $$m^{(n)} * \lambda =$$
$$(\lambda T^{n-1} * m^{(1)} \otimes m^{(n)}) \cdot (T\lambda T^{n-2} * m^{(2)} \otimes m^{(n-1)}) \cdot \ldots \cdot (T^{n-1}\lambda * m^{(n)} \otimes m^{(1)}).$$

Each of the factors on the right hand side are values of J on elementary moves, cf. formula (5). To prove (7), we observe that the two terms to be compared have domain-1-cell of form

$$f * yT$$

(in fact $f = m^{(n)}$). We prove

Lemma 1 *Assume that there are given 2-cells* $\alpha, \beta : f * yT \Rightarrow g : T^n \to T^2$. *If* $\alpha * m = \beta * m$, *then* $\alpha = \beta$.

Proof. It suffices to prove that α can be reconstructed from $\alpha * m$. Let us calculate $\alpha * \lambda T * Tm$ in two ways by the 2-category rules for the $*$-composite of two 2-cells. Since the domain 1-cell of $\lambda T * Tm$ is $yT^2 * Tm = m * yT$, one way of rewriting $\alpha * \lambda T * Tm$ gives

$$(\alpha * m * yT) \cdot (g * \lambda T * Tm); \tag{8}$$

since the codomain 1-cell of $\lambda T * Tm$ is $TyT * Tm = id$, the other way of rewriting gives

$$(f * yT * \lambda T * Tm).\alpha, \tag{9}$$

which is α, by (1). Since (8)=(9), α can be reconstructed from $\alpha * m$, using the expression (8).

Using the lemma, we may prove (7) by proving that we get the same 2-cell by $*$-composing either side of (7) on the right by m. We have $m^{(n)} * \lambda * m$ the identity 2-cell of $m^{(n)}$, by (2). But applying $- * m$ to each of the factors on the right of (7) yields an identity 2-cell, since $(m^{(j)} \otimes m^{(n-j+1)}) * m = m^{(n+1)}$, which when applied on the right of $T^{j-1}\lambda T^{n-j}$ yields an identity 2-cell, essentially by (2).

Proof of 2). We may assume that $g : \underline{2} \to \underline{n}$ is monic, for, if g identifies the two elements in the domain, it is of form $m * g'$, and hence $\lambda * g = \lambda * m * g'$ which is an identity 2-cell by (2). Also, by bifunctorality of $\otimes$, it is easy to see that we only need to consider the case where g preserves first and last element, so is of form $g = Ty^kT = T \otimes y \otimes \ldots \otimes y \otimes T$ (k y's). We may rewrite this as

$$g = TyT * TyT^2 * \ldots * TyT^k. \tag{10}$$

Consider the horizontal composite

Since each 1-cell in the top row composes with its 2-cell neighbour to the right to give an identity 2-cell (by (1)), the one way of rewriting this as a vertical composite of 2-cells yields only one 2-cell factor, namely

$$\lambda * TyT * TyT^2 * \ldots * TyT^k,$$

i.e. $\lambda * g$. The other way of rewriting it yields a $\cdot$-composite of $k+1$ factors; for instance, the one involving the 2-cell λT^p ($p = 0, 1, \ldots, k$) from the above horizontal composite is

$$Ty * TyT * \ldots * TyT^{p-1} * \lambda T^p * yT^{p+2} * \ldots yT^{k+1}. \tag{11}$$

Using the rules of monoidal 2-categories, this is analyzed as

$$y^{k-p} \otimes \lambda \otimes y^p, \tag{12}$$

which is in the image of J since λ is. This proves part 2) of the Proposition.

From this follows that J is a two-sided inverse for P (on the level of 2-cells; on the level of 0- and 1-cells, J and P are already known to be mutually inverse). Thus P is an isomorphism.

From this follows that the monoidal category Δ has the same universal property as Δ'; thus

Theorem 2.1 *The (strict) monoidal 2-category Δ is freely generated by the 0-cell $\underline{1}$ (=T), the 1-cells $y : \underline{0} \to \underline{1}$ (or $y : I \to T$), $m : \underline{2} \to \underline{1}$ (or $m : T \otimes T \to T$), and the 2-cell*

$$\lambda : y \otimes T \Rightarrow T \otimes y : T \to T \otimes T,$$

modulo the relations: y and m make T into a monoid; and λ satisfies (1) and (2), i.e. $y * \lambda = y \otimes y$ and $\lambda * m = T$.

Remark The cocompletion constructions of [3], T, y, m, on the 2-category **Cat** of categories, or on the 2-category **Pos** of partially ordered sets, say, are, (modulo coherent isomorphism, in the former case) monads, thus monoids in the monoidal 2-category C of endo-2-functors on **Cat** (resp. **Pos**), where the (strictly associative) $\otimes$ on C is composition of functors. But, as pointed out in [3], a special feature of cocompletion constructions is the presence of a natural transformation $\lambda : yT \Rightarrow Ty$ (in standard right-to-left notation:

$$\lambda_{\mathbf{C}} : T(y_{\mathbf{C}}) \Rightarrow y_{T\mathbf{C}}$$

for each category (respectively poset) **C**), satisfying (1) and (2), at least in the **Pos**-case; in the **Cat**-case, the coherent isomorphisms must be taken into account. Thus by the universal property of Theorem 2.1, there is a monoidal 2-functor $\Delta \to \mathcal{C}$ with $1 \mapsto T$, extending the monoidal 1-functor $\Delta \to \mathcal{C}$ arising from the 1-dimensional aspect of the monad T, y, m, (according to [4]). In [8], a similar extension is also carried out, but only for the sub-2-category of Δ, consisting of the non-empty ordinals, and the last-element preserving maps.

The endofunctor $- \otimes T$ on Δ is itself (the restriction to $\Delta \subseteq$ **Pos** of) a cocompletion monad, namely "freely adding a smallest element $\perp$".

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