Generators and Relations for $\Delta$ as a Monoidal 2-Category

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Introduction

By $\Delta$, we understand the category of finite ordinals $0, 1, 2, \ldots$, and their order preserving maps. It is the category which defines the notion of simplicial set. As a category, it is well known to be generated by face- and degeneracy operators $d_i : n \to n + 1$, $s_j : n + 2 \to n + 1$ (for each $n$), and a certain list of equations (for each $n$) for its relations, cf e.g. [6]. This category carries structure of monoidal (=tensored) category with $n \otimes m = n + m$ (ordinal sum formation). As a monoidal category, it has a much smaller presentation, namely generated by $d_0 : 0 \to 1$, $s_0 : 2 \to 1$, and three equations, namely the three equations which define the notion of monoid (unit- and associative law); this was proved by Lawvere [4].

However, as observed by Street [8], being a category of order preserving maps between ordered sets, the category is enriched over partially ordered sets, in the sense that its hom-sets carry a partial order (compatible with the composition, and also with the $\otimes$). In fact it is a 2-category, cf e.g. [1], [2] for these notions; in fact, it is a strict monoidal 2-category. We give generators and relations for its 2-dimensional structure. The result (Theorem 2.1 below) is:

- one generator $\lambda$, namely the inequality ("2-cell") $d_1 \leq d_0$ (as maps $1 \to 2$)
- two relations: both $d_0 \ast \lambda : 0 \to 2$ and $\lambda \ast s_0 : 1 \to 1$ are identity 2-cells.

The conventions are that we compose from left to right, for the horizontal composition $\ast$ (and also for the vertical composition $\cdot$ occurring later on); and that $d_i : n \to n + 1$ omits $i \in n + 1$, where $n + 1 = \{0, 1, \ldots, n\}$. Note that the 2-cell $d_0 \ast \lambda$ has the same map $0 \to 2$ for its vertical source and target, but we are presenting $\Delta$ as a category enriched over categories (a 2-category), rather than enriched over partially ordered sets; thus we do not apriori assume that a 2-cell with its source and target coinciding is an identity 2-cell.

I want to acknowledge correspondence with Ross Street; in fact a result about a universal property of $\Delta$ as a monoidal 2-category appears in his [8]. It is not exactly the same universal property as the one given here (cf. remark at the end of the paper); the one given here was proposed by Ross Street when he was refereeing [3]. Since the proof I was able to give for it turned out to be a little involved, I decided to publish it separately.
1. The comparison functors

Let $\Delta'$ denote a (strictly) monoidal 2-category with a given object $T$, given 1-cells $y : I \to T$ and $m : T \otimes T \to T$ ($I$ denoting the (strict) unit object), and a given 2-cell $\lambda : y \otimes T \to T \otimes y$, such that $y$ and $m$ make $T$ into a monoid object, and such that $\lambda$ satisfies the axioms

\begin{align*}
(1) \quad y \circ \lambda &= y \otimes y \\
(2) \quad \lambda \circ m &= T;
\end{align*}

and which is universal with these data. (We have here used the convention that the identity 1-cell of a 0-cell (object) $A$ is also denoted $A$; and that the identity 1-cell of a 1-cell (arrow) $f$ is also denoted $f$.) The monoidal 2-category $\Delta$ carries such $T$, $y$, $m$, $\lambda$, satisfying the axioms, namely $T = 1$, $y$ and $m$ the unique maps $0 \to 1$ and $2 \to 1$, respectively, and with $\lambda$ the inequality $d_1 \leq d_0$ (as maps $1 \to 2$). By the universal property of $\Delta'$, there is a monoidal 2-functor $P : \Delta' \to \Delta$ taking $T$ to 1, $y$ and $m$ to $0 \to 1$ and $2 \to 1$, respectively, and $\lambda$ to $d_1 \leq d_0$.

Since it is clear that the 2-dimensional structure on $\Delta'$ does not impose any constraints on the 1-dimensional structure, it follows that the underlying monoidal 1-category of $\Delta'$ is the free monoidal category with a monoid object $T$, $y$, $m$; and therefore, by [4], it is (isomorphic to) $\Delta$, so on the underlying monoidal 1-categories, $P : \Delta' \to \Delta$ is an isomorphism. Our theorem is that $P$ is also an isomorphism of the 2-dimensional structure, Theorem 2.1 below.

Since $P$ is an isomorphism on the 1-dimensional structure, we may identify $\Delta'$ and $\Delta$ in so far as this structure goes. This gives us two sets of notations for 0- and 1-cells of these categories. For instance, the map (face operator) $d_1 : 2 \to 3$ which omits 1 $\in \{0, 1, 2\} = 3$ may also be denoted $T \otimes y \otimes T$, or just $TyT$. As in [3], we let $\otimes$ bind more strongly than the horizontal composition $\circ$ or the vertical composition $\cdot$, and sometimes we omit $\otimes$ (but we never omit $\circ$ or $\cdot$). Also $T \otimes T = TT$ may be denoted $T^2$, and similarly for $T^3$, etc. To fit with the conventions of [3], we need to have $yT : T \to T^2$ match $d_1 : 1 \to 2$ (which omits the last element $1 \in 2$), so it is most natural in the graphics to think of the elements 0 and 1 of 2 as displayed graphically in the order: 1,0 , with the largest element leftmost; similarly for the higher $n$’s.

The strategy is now as follows: we shall provide an inverse $J : \Delta \to \Delta'$ for $P$. On 0- and 1-cells, $J$ will be the identity, of course. On 2-cells, it will be constructed as a splitting of $P$, $P(J(\alpha)) = \alpha$, and commuting with $\otimes$ and $\cdot$. The construction of this splitting will occupy the rest of the present section, whereas Section 2 will prove that $J$ is an onto map, implying hat $J$ is a two-sided inverse for $P$ which is thus an isomorphism.

Let $f$ and $g$ be maps $n \to m$ in $\Delta$. If $f \leq g$ (i.e. $f(i) \leq g(i) \forall i \in n$), we say that $\text{dist}(f, g) = p$ if $\sum_{i \in n} g(i) - f(i) = p$. If $f \leq g$ and $\text{dist}(f, g) = 1$, we say that the
pair \((f, g)\) is an elementary move out of \(f\) with result \(g\). The unique element \(i \in \mathbb{n}\) with \(f(i) \neq g(i)\) (in fact with \(f(i) + 1 = g(i)\)) is called the element moved by the elementary move. Clearly, an elementary move out of \(f\) moving \(i\) is possible (and then unique) iff \(i\) is the largest element in the set (interval) \(f^{-1}(f(i))\), and \(f(i)\) is not the largest element in \(m\).

It is clear that if \(\text{dist}(f, g) = p\), then we can interpolate a chain consisting of \(p\) elementary moves between \(f\) and \(g\). Such chain is not unique, but at least we have the following "confluence lemma":

**Proposition 1.1** Let \(f : n \to m\). If \(g\) and \(h\) are results of two distinct elementary moves out of \(f\), there is a \(k : n \to m\) which is the result of an elementary move out of \(g\) and also the result of an elementary move out of \(h\). (Such \(k\) is unique.)

**Proof.** Let \(i\) and \(j\) be the elements moved by the elementary moves \((f, g)\) and \((f, h)\), respectively. We have \(i \neq j\) since \(g\) and \(h\) are distinct. We may assume \(i > j\). Because of the elementary move \((f, g)\), \(i\) is the largest element in \(f^{-1}(f(i)) = f^{-1}(h(i))\). This set differs from \(h^{-1}(h(i))\) at most by the presence or not of \(j\), which is the only element where \(f\) and \(h\) differ. Since \(i > j\), the status of \(i\) as largest in \(f^{-1}(f(i))\) is not affected by the presence or not of \(j\), so \(i\) is the largest element also in \(h^{-1}(h(i))\). So an elementary move out of \(h\) moving \(i\) is possible, with result the map \(k\) given by

\[
k(i) = f(i) + 1; k(j) = f(j) + 1; k(x) = f(x) \quad \text{for all other } x.
\]

Also \(g(i) = f(i) + 1 > f(i) \geq f(j)\), so \(g(i) \neq f(j) = g(j)\). Also \(f(i) \neq f(j)\), for \(f(i) = f(j)\) and \(i > j\) would contradict \(j\) being largest in \(f^{-1}(f(j))\) (as witnessed by the elementary move \((f, h)\)). Since \(f\) and \(g\) differ only at \(i\), \(f^{-1}(f(j)) = g^{-1}(g(j))\) \((i\) belongs to neither); since \(j\) is the largest in \(f^{-1}(f(j))\), it is largest in \(g^{-1}(g(j))\). So an elementary move out of \(g\) moving \(j\) is possible, and it is clear that the result of the move is again \(k\).

It is clear that any elementary move \((f, g)\) in \(\Delta\) can be written uniquely

\[
(f' \otimes f_0 \otimes f'', f' \otimes g_0 \otimes f'')
\]

with \((f_0, g_0)\) an elementary move of maps \(n_0 \to 2\), for suitable \(n_0, f', f''\): if \(i\) is the element moved by the elementary move \((f, g)\), \(f''\) is the restriction of \(f\) to the set of \(j\)'s with \(f(i) > f(j)\), and \(f'\) (essentially) the restriction to the set of \(j\)'s with \(f(j) > f(i) + 1\). And it is clear that there is exactly one elementary move \(L_{n,i}\) between maps \(n \to 2\) which moves \(i\), namely

\[
L_{n,i} = (f_0, g_0)
\]

with

\[
f_0(j) = 0 \text{ for } j \leq i, \quad 1 \text{ otherwise},
\]

\[
g_0(j) = 0 \text{ for } j < i, \quad 1 \text{ otherwise}.
\]

We associate to this elementary move \(L_{n,i}\) the 2-cell in \(\Delta'\)

\[
\lambda_{n,i} := (T^{n-1-i} \otimes \lambda \otimes T^i) * (m' \otimes m'')
\]
where \( m' : n - i \to 1 \) and \( m'' : i + 1 \to 1 \) are the unique such maps. And to the elementary move (4), we associate \( f^i \odot \lambda_{n,i} \odot f'' \). This defines our map \( J : \Delta \to \Delta' \) on all elementary moves \( f \leq g \) in \( \Delta \). It is clear that \( J \), so far as it is defined by now, respects the domain/codomain book-keeping, and (hence) splits \( P \). In this and similar considerations, the reader may find the following kind of diagram (illustrating, in this case, \( \lambda_{n,i} \)) useful (vertical concatenation corresponds to \(*\), horizontal concatenation to \( \odot \)). Note that \( \lambda \) itself is \( \lambda_{1,0} \).

To extend \( J \) to yield a 2-cell \( J(f, g) : f \Rightarrow g \) in \( \Delta' \) for any inequality \( f \leq g \) in \( \Delta \), we decompose \( f \leq g \) into a chain of elementary moves \( f = h_0 \leq h_1 \leq \ldots \leq h_p = g \), and we would then like to put \( J(f, g) \) equal the \( \cdot \)-composite of the \( J(h_q, h_{q+1}) \)'s. Since the chain decomposition is not unique, there is a question about well-definedness. Because of the confluence Lemma (Proposition 1.1), it easily follows that it suffices to see that

\[
J(f \leq g) \cdot J(g \leq k) = J(f \leq h) \cdot J(h \leq k)
\]

where \( f, g, h, \) and \( k \) are as in the Proposition. Also, let \( i \) and \( j \) be as in its proof.

It is clear that if \( \phi \) is a 2-cell in \( \Delta \) which is an elementary move, then so is \( \phi \odot f \) and \( f \odot \phi \) for any 1-cell \( f \) in \( \Delta \); and it is clear from the description of \( J \) on elementary moves that

\[
J(\phi \odot f) = J(\phi) \odot f, \quad J(f \odot \phi) = f \odot J(\phi).
\]

Using such reductions, the problem of proving (6) gets reduced to the case where \( f : n + 1 \to 3 \), and the elements \( i \) and \( j \) moved are the last and first elements \( n \) and \( 0 \). Thus

\[
(f, g) = L_{n,n-1} \odot T \mapsto (\lambda \odot T^{n-1} \ast T \odot m^{(n)}) \odot T
\]

\[
(g, k) = T \odot L_{n,0} \mapsto T \odot (T^{n-1} \odot \lambda \ast m^{(n)} \odot T)
\]

\[
(f, h) = y \odot L_{n+1,0} \mapsto y \odot (T^n \odot \lambda \ast m^{(n+1)} \odot T)
\]

\[
(h, k) = L_{n+1,n} \odot y \mapsto (\lambda \odot T^n \ast T \odot m^{(n+1)}) \odot y,
\]
where \(m^{(q)} : q \to 1\) is the unique such map (thus \(m^{(2)} = m, m^{(1)} = T, m^{(0)} = y\).

We now calculate \(\lambda T^n * T^{n+1} \lambda * Tm^{(n+1)} T\) by the 2-category rules for the *-composite of two 2-cells. On the one hand, we get, using bifunctorality of \(\otimes\),

\[
(\lambda T^{n-1} T * Tm^{(n)} T) \cdot (T^n \lambda * Tm^{(n)} T) = J(f, g) * J(g, k);
\]

calculating this the other way round, we get

\[
(y T^{n+1} * T^{n+1} \lambda * Tm^{(n+1)} T) \cdot (\lambda T^n * T^{n+2} \lambda * Tm^{(n+1)} T) =
(y T^{n+1} * T^{n+1} \lambda * Tm^{(n+1)} T) \cdot (T^{n+1} \lambda * \lambda T^{n+1} * Tm^{(n+1)} T).
\]

But the two factors here are, by bifunctorality of \(\otimes\) seen to be, respectively, \(y \otimes (T^n \lambda * m^{(n+1)} T)\) and \((\lambda T^n * Tm^{(n+1)}) \otimes y\), i.e. \(J(f, h)\) and \(J(h, k)\), respectively. This proves the commutativity (6) for the special case considered, but as argued, this case suffices.

This now means that we have a map \(J : \Delta \to \Delta'\) which is the identity on 0- and 1-cells, and preserves the book-keeping and the compositions \(\cdot\) and \(\otimes\) on 2-cells; and \(J\) followed by \(P\) is the identity. So to see that \(J\) is a two-sided inverse for \(P\), it suffices to see that every 2-cell in \(\Delta'\) is in the image of \(J\). We turn to this task:

2. 2-cells in the image of \(J\)

To see that \(J\) is surjective on 2-cells, we use 'structural induction': any 2-cell in \(\Delta'\) is given by an expression built by means of \(\otimes\), \(*\), and \(\cdot\) , starting with \(\lambda\) (and identity 2-cells). If we assume that every 2-cell which can be built by an expression of length \(< n\) is in the image of \(J\), and \(\phi\) is an expression of length \(n\), there are four cases to consider:

1) \(n = 1\), in which case \(\phi = \lambda\) or is an identity 2-cell. In both cases, \(\phi\) is in the image of \(J\).

2) \(\phi = \phi' \otimes \phi''\), with \(\phi'\) and \(\phi''\) expressions of length \(< n\). Then the 2-cells given by these expressions are in the image of \(J\), and since \(J\) commutes with \(\otimes\), the 2-cell given by the expression \(\phi\) is in the image of \(J\).

3) \(\phi = \phi' \cdot \phi''\) - the argument is similar.

4) We finally consider the case \(\phi = \phi' * \phi''\). Using the rules of 2-categories, this may be written as a \(\cdot\) -composite of expressions of form \(f * \phi''\) and \(\phi' * g\) with \(f\) and \(g\) 1-cells, and each of these expressions is strictly shorter than \(\phi = \phi' * \phi''\) unless one of the \(\phi'\)’s is of length 1, that is, unless \(\phi' = \lambda\) or \(\phi'' = \lambda\). So this leaves us with proving

**Proposition 2.1** 1) For any 1-cell \(f, f * \lambda\) is in the image of \(J\). 2) For any 1-cell \(g, \lambda * g\) is in the image of \(J\).

**Proof.** 1) \(f\) is of form \(m^{(k)} : k \to 1\) for some \(k\); the cases \(k = 0\) and \(k = 1\) yield \(y * \lambda = y \otimes y\) (by (1)) and \(\lambda\), respectively, and both of these are in the image of \(J\). For \(k = n > 1\), we shall prove the equation

\[
(\lambda T^{n-1} * m^{(1)} \otimes m^{(n)}) \cdot (\lambda T^{n-2} * m^{(2)} \otimes m^{(n-1)}) \cdot \ldots \cdot (T^{n-1} \lambda * m^{(n)} \otimes m^{(1)}).
\]
Each of the factors on the right hand side are values of $J$ on elementary moves, cf. formula (5). To prove (7), we observe that the two terms to be compared have domain-1-cell of form
\[ f \ast yT \]
(in fact $f = m^{(n)}$). We prove

**Lemma 1** Assume that there are given 2-cells $\alpha, \beta : f \ast yT \Rightarrow g : T^n \to T^2$. If $\alpha \ast m = \beta \ast m$, then $\alpha = \beta$.

**Proof.** It suffices to prove that $\alpha$ can be reconstructed from $\alpha \ast m$. Let us calculate $\alpha \ast \lambda T \ast Tm$ in two ways by the 2-category rules for the $\ast$-composite of two 2-cells. Since the domain 1-cell of $\lambda T \ast Tm$ is $yT^2 \ast Tm = m \ast yT$, one way of rewriting $\alpha \ast \lambda T \ast Tm$ gives

\[ (\alpha \ast m \ast yT) \cdot (g \ast \lambda T \ast Tm); \]

since the codomain 1-cell of $\lambda T \ast Tm$ is $TyT \ast Tm = id$, the other way of rewriting gives

\[ (f \ast yT \ast \lambda T \ast Tm) \cdot \alpha, \]

which is $\alpha$, by (1). Since (8)=(9), $\alpha$ can be reconstructed from $\alpha \ast m$, using the expression (8).

Using the lemma, we may prove (7) by proving that we get the same 2-cell by $\ast$-composing either side of (7) on the right by $m$. We have $m^{(n)} \ast \lambda \ast m$ the identity 2-cell of $m^{(n)}$, by (2). But applying $- \ast m$ to each of the factors on the right of (7) yields an identity 2-cell, since $(m^{(j)} \otimes m^{(n-j+1)}) \ast m = m^{(n+1)}$, which when applied on the right of $T^{j-1} \lambda T^{n-j}$ yields an identity 2-cell, essentially by (2).

**Proof of 2.** We may assume that $g : 2 \to n$ is monic, for, if $g$ identifies the two elements in the domain, it is of form $m \ast g'$, and hence $\lambda \ast g = \lambda \ast m \ast g'$ which is an identity 2-cell by (2). Also, by bifunctorality of $\otimes$, it is easy to see that we only need to consider the case where $g$ preserves first and last element, so is of form $g = Ty^kT = T \otimes s \otimes \ldots \otimes y \otimes T$ ($k$ $y$'s). We may rewrite this as

\[ g = TyT \ast TyT^2 \ast \ldots \ast TyT^k. \]

Consider the horizontal composite

Since each 1-cell in the top row composes with its 2-cell neighbour to the right to give an identity 2-cell (by (1)), the one way of rewriting this as a vertical composite of 2-cells yields only one 2-cell factor, namely
\[ \lambda \ast TyT \ast TyT^2 \ast \ldots \ast TyT^k, \]
i.e. \( \lambda \ast g \). The other way of rewriting it yields a \( k \)-composite of \( k + 1 \) factors; for instance, the one involving the 2-cell \( \lambda T^p \) \( (p = 0, 1, \ldots, k) \) from the above horizontal composite is

\[
T y \ast T y T \ast \ldots \ast T y T^{p-1} \ast \lambda T^p \ast y T^{p+2} \ast \ldots \ast y T^{k+1}.
\]

Using the rules of monoidal 2-categories, this is analyzed as

\[
y^{k-p} \otimes \lambda \otimes y^p,
\]

which is in the image of \( J \) since \( \lambda \) is. This proves part 2) of the Proposition.

From this follows that \( J \) is a two-sided inverse for \( P \) (on the level of 2-cells; on the level of 0- and 1-cells, \( J \) and \( P \) are already known to be mutually inverse). Thus \( P \) is an isomorphism.

From this follows that the monoidal category \( \Delta \) has the same universal property as \( \Delta' \); thus

**Theorem 2.1** The (strict) monoidal 2-category \( \Delta \) is freely generated by the 0-cell \( \mathbf{1} \) \((=T)\), the 1-cells \( y : 0 \to \mathbf{1} \) \((or \ y : I \to T)\), \( m : 2 \to \mathbf{1} \) \((or \ m : T \otimes T \to T)\), and the 2-cell

\[
\lambda : y \otimes T \Rightarrow T \otimes y : T \to T \otimes T,
\]

modulo the relations: \( y \) and \( m \) make \( T \) into a monoid; and \( \lambda \) satisfies (1) and (2), i.e. \( y \ast \lambda = y \otimes y \) and \( \lambda \ast m = T \).

**Remark** The cocompletion constructions of [3], \( T, y, m \), on the 2-category \( \text{Cat} \) of categories, or on the 2-category \( \text{Pos} \) of partially ordered sets, say, are, (modulo coherent isomorphism, in the former case) monads, thus monoids in the monoidal 2-category \( C \) of endo-2-functors on \( \text{Cat} \) \((resp. \ \text{Pos})\), where the (strictly associative) \( \otimes \) on \( C \) is composition of functors. But, as pointed out in [3], a special feature of cocompletion constructions is the presence of a natural transformation \( \lambda : y T \Rightarrow Ty \) \((in \ standard \ right-to-left \ notation): \)

\[
\lambda_{C} : T(yC) \Rightarrow yTC
\]

for each category \((respectively \ poset) \ C)\), satisfying (1) and (2), at least in the \( \text{Pos} \)-case; in the \( \text{Cat} \)-case, the coherent isomorphisms must be taken into account. Thus by the universal property of Theorem 2.1, there is a monoidal 2-functor \( \Delta \to C \) with \( 1 \mapsto T \), extending the monoidal 1-functor \( \Delta \to C \) arising from the 1-dimensional aspect of the monad \( T, y, m \) \((according \ to \ [4])\). In [8], a similar extension is also carried out, but only for the sub-2-category of \( \Delta \), consisting of the non-empty ordinals, and the last-element preserving maps.

The endofunctor \(- \otimes T\) on \( \Delta \) is itself \((the \ restriction \ to \ \Delta \subseteq \text{Pos} \ of)\) a cocompletion monad, namely ”freely adding a smallest element \( \bot \)."
References


