A general algebra/geometry duality, and synthetic scheme theory*

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Let $\mathbb{T}$ be an algebraic theory, and $FP\mathbb{T}$ the category of finitely presented $\mathbb{T}$-algebras. There is a full and faithful functor

$$\mathbb{T} \xhookrightarrow{F} FP\mathbb{T}^{op}$$

given by $n \mapsto F(n)$, the free algebra in n generators, and this functor is known to have the following universal property: if $\mathcal{E}$ is a cartesian closed category with finite inverse limits, and $A : \mathbb{T} \to \mathcal{E}$ is a finite-product preserving functor (in other words, an algebra for $\mathbb{T}$), then A extends in an essentially unique way over F to a finite limit preserving functor $\overline{A} : FP\mathbb{T}^{op} \to \mathcal{E}$. Furthermore, for any functor $B : FP\mathbb{T}^{op} \to \mathcal{E}$, any natural transformation

$$B \circ F \xRightarrow{t} A$$

extends uniquely to a natural transformation

$$B \xRightarrow{\bar{t}} \overline{A}.$$

Let us now further assume that $\mathcal{E}$ is Cartesian closed, and R is a $\mathbb{T}$-algebra in $\mathcal{E}$. Let R-Alg denote the comma-category $R \downarrow \mathbb{T}\text{-Alg}(\mathcal{E})$ of $\mathbb{T}$-algebras under R in $\mathcal{E}$. We may in $\mathcal{E}$ form the internal hom-object

$$\underline{\mathrm{Hom}}_{R\text{-Alg}}(A_1, A_2)$$

whenever A_1 and $A_2 \in R$-Alg.

Let $A \in R$-Alg. In particular, we have $\overline{A} : FP\mathbb{T}^{op} \to \mathcal{E}$. Using the above mentioned universal property of $\overline{A}$, we may construct a map, natural in $B \in FP\mathbb{T}$

$$\underline{\mathrm{Hom}}_{R\text{-Alg}}(R^{\overline{R}(B)}, A) \xrightarrow{\nu_{(B,A)}} \overline{A}(B)$$

(note that, for any X, $R^X \in R$-Alg in a natural way). It suffices to give $\nu_{F(n),A}$ which is a map

$$\underline{\mathrm{Hom}}_{R\text{-Alg}}(R^{R^n}, A) \longrightarrow A^n;$$

*The content of this note was part of an exposition at the 17th PSSL, Sussex Nov. 1980; it is a retyping (Jan. 2009) of an article that appeared in *Prépublications Math., U. Paris Nord* 23 (1981), 33-34. It is quoted as entry [38] in the author's 1981 book "Synthetic Differential Geometry".

in naive description, it sends ϕ to $(\phi(\text{proj}_1), \ldots, \phi(\text{proj}_n))$.

We say that R is a *model for synthetic scheme theory* if $\nu_{B,A}$ is an isomorphism for any $A \in R$-Alg, and any $B \in FP\mathbb{T}$.

Theorem *Let $R \in FP\mathbb{T}$ be the generic $\mathbb{T}$-algebra (= the forgetful functor). Then it is a model for synthetic scheme theory.*

The reason for the name is that if R is such, then the canonical map η into the double dual

$$M \longrightarrow \underline{\text{Hom}}_{R\text{-Alg}}(R^M, R)$$

is an isomorphism for any $M = \overline{R}(B)$, with $B \in FP\mathbb{T}$; namely η is inverse to $\nu_{B,R}$. So internally in $\mathcal{E}$, the "affine scheme" M can be recovered from its "*internal* coordinate ring" R^M. Also, the functor $FP\mathbb{T} \to R$-Alg given by $B \mapsto R^{\overline{R}(B)}$ preserves finite colimits, and R^{R^n} is internally the free R-algebra in n generators.

If $\mathbb{T}$ is the theory of commutative rings, and R is a model for synthetic scheme theory, it is automatically a model for synthetic differential geometry.