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FORMAL MANIFOLDS AND SYNTHETIC THEORY OF JET BUNDLES

by Anders KOCK

In the present Note, we aim to set up a framework in which the theory of jets can be treated from the viewpoint of synthetic differential geometry, in the sense of several of the articles of [21]. The advantage the synthetic viewpoint has here is that jets become *representable* : a k-jet at x is not an *equivalence class* of maps, but *is* a map, defined on what we shall call the *k-monad around* x, $\mathfrak{M}_k(x)$.

The content of sections 3-5 on groupoids is essentially due to C. Ehresmann and his followers, like P. Libermann, Kumpera, ... Ehresmann's observation that the jet-notion naturally leads to «differentiable categories» and in particular «differentiable groupoids» (= category- and groupoid-objects in the category of smooth manifolds) forced him to become a category theorist and provided a certain completion of the Lie-Klein programme that types of geometries are distinguished by their Lie-groups or better their Lie-*groupoids*.

In Section 4, we give some ideas on how sheaves naturally occur in the synthetic setting; Section 6 contains scattered remarks on possible applications of synthetic jet theory.

1. FORMAL MANIFOLDS.

Let R be a ring object in a topos $\mathcal{E}$, in which we shall work as if it were the category of sets. R is to be thought of as the numerical line.

For $k \geq 0$, $n \geq 0$ natural numbers, we put

$$D_k(n) = \{ (x_1, \dots, x_n) \in R^n \mid \text{any product of } k+1 \text{ or more of the } x_i\text{'s is } 0 \}.$$

It is an example of an *infinitesimal object* ; to give the general definition, we need the notion of *Weil algebra over* R : this is a commutative R-algebra W whose underlying R-module is of the form $R \oplus R^h$ for some natural number h, where $(1, \underline{0})$ is the multiplicative unit, and every element $(0, \underline{v})$ ($\underline{v} \in R^h$) is nilpotent. The *spectrum* jW of the Weil algebra W is the object («set») $[W, R]$ of R-algebra maps $W \to R$. Objects of form jW are called *infinitesimal;* jW has a canonical element called $\underline{0}$, corresponding to the canonical map «projection to the first factor»:

$$W = R \oplus R^h \to R .$$

To get $D_k(n)$ above as a jW, take

$$W = R \otimes (Z[X_1, \dots, X_n]/I) \tag{1.1}$$

where I is the ideal generated by all products of $k+1$ or more of the X_i's.

There is a canonical map $\alpha : W \to R^{jW}$ exponential adjoint of the evaluation map

$$W \times jW = W \times [W, R] \to R.$$

We shall assume that R is of «line type» in a strong sense (cf. [12], strengthening the line type notion of Lawvere and the author [9]), namely we shall assume that α is invertible. This implies in particular:

For any map $f: D_k(n) \to R$ there is a unique polynomium ϕ with coefficients from R, in n variables and of total degree $\leq k$, such that

$$f(x_1, \dots, x_n) = \phi(x_1, \dots, x_n) \quad \forall (x_1, \dots, x_n) \in D_k(n).$$

The heuristics is that

$$\phi(x_1, \dots, x_n) = \sum_{|\alpha| \leq k} \frac{f^{(\alpha)}(\underline{0})}{\alpha !} x^{\alpha}, \tag{1.2}$$

and in fact, derivation of functions can be defined in such a way that (1.2) can be *proved* (provided R contains the rational numbers); see [10] .

We note the following consequence of binomial expansion:

$$u \in D_k(n) \wedge v \in D_m(n) \Rightarrow u + v \in D_{k+m}(n). \tag{1.3}$$

We consider the object

$$D_\infty(n) = \bigcup_{k=1}^{\infty} D_k(n). \tag{1.4}$$

It is a subobject of R^n containing $\underline{0}$. From (1.3) follows that it is stable under addition.

(It is easy to see that $D_\infty(n) = (D_\infty)^n$ where $D_\infty \subset R$ is the «set» of nilpotent elements ; see [10].)

Strengthening the definition from [13] slightly, we say that a map $P \to Q$ is *étale* if for any infinitesimal object K and any commutative square

(1.5)
$$\begin{tikzcd} 1 \arrow[r, "\underline{0}"] \arrow[d] & K \arrow[d] \arrow[dl, dashed] \\ P \arrow[r] & Q \end{tikzcd}$$

there is a unique diagonal fill-in $K \to P$. In particular, a monic map $P \to Q$ is étale if whenever $K \to Q$ maps $\underline{0}$ into P then the whole map factors across P.

DEFINITION 1.1. An object M is called a *formal n-dimensional manifold* if for each $x \in M$ there exists an étale subobject $\mathfrak{M} \to M$ containing x and isomorphic to $D_\infty(n)$ (i.e. there exists a bijective map $D_\infty(n) \to \mathfrak{M}$).

We shall see that such a subobject $\mathfrak{M}$ is unique if it exists ; it will be called the *monad* or *∞-monad* around x, and denoted $\mathfrak{M}(x)$.

To prove the uniqueness, assume that also $\mathfrak{M}' \to M$ is étale, contains x and is isomorphic to $D_\infty(n)$. It suffices by symmetry to prove $\mathfrak{M} \subset \mathfrak{M}'$. Choose a bijection $\phi : D_\infty(n) \to \mathfrak{M}$. We may in fact assume that $\phi(\underline{0}) = x$; for, if not, there is a unique $\underline{v} \in D_\infty(n)$ with $\phi(\underline{v}) = x$. Then replace ϕ by the composite

$$D_\infty(n) \xrightarrow{+\underline{v}} D_\infty(n) \xrightarrow{\phi} \mathfrak{M}$$

which makes sense since, as we have observed, $D_\infty(n)$ is stable under addition.

It now suffices to prove, for each k, that $\phi(D_k(n)) \subset \mathfrak{M}'$. But $\phi(\underline{0}) = x \in \mathfrak{M}'$; thus, since $\mathfrak{M}'$ is étale, $\phi|_{D^k(n)}$ factors through $\mathfrak{M}'$.

PROPOSITION 1.2. *For each n, R^n is a formal n-dimensional manifold. The monad around $\underline{v} \in R^n$ is $\underline{v} + D_\infty(n)$.*

PROOF. Clearly «adding $\underline{v}$» gives a bijection from $D_\infty(n)$ to $\underline{v}+D_\infty(n)$, so it suffices to see that the latter subset is étale. Again, by a parallel-translation argument, it suffices to see that $D_\infty(n) \subset R^n$ is étale. It is easy to see that each Weil algebra W' is a quotient of a Weil algebra W of form (1.1). By the line type assumption, $R^{jW'}$ is a quotient of R^{jW}. This may be read: every jW' is contained in some $D_k(n)$, and every map $jW' \to R$ extends to a map $D_k(n) \to R$. From this follows that to test étaleness of $D_\infty(n) \to R^n$, it suffices to test the étaleness condition (1.5) for objects K of form $D_k(m)$. Now, a map $f: D_k(m) \to R^n$ is given by an n-tuple of polynomials in m variables (like in (1.2)). To say $f(\underline{0}) \in D_\infty(n)$ is to say that the constant terms of these polynomials are nilpotent. The arguments of f range over $D_k(m)$, so are also nilpotent. But putting nilpotent elements as arguments in a polynomial with nilpotent constant term yields a nilpotent value. Thus f factors across $D_\infty(n)$. This proves the proposition.

Keeping track of the degrees involved in the latter argument immediately yields also the following proposition which will be useful later on.

PROPOSITION 1.3. *If $f: D_k(n) \to R^m$ maps $\underline{0}$ to $\underline{0}$, then it factors through $D_k(m) \subset R^m$.*

Formal manifolds have several stability properties. Thus, if P and Q are formal manifolds of dimension p and q, respectively, $P \times Q$ is a formal manifold of dimension $p+q$; the monad around (x, y) is $\mathfrak{M}(x) \times \mathfrak{M}(y)$ which is isomorphic to $D_\infty(p) \times D_\infty(q) = D_\infty(p+q)$.

Slightly less trivial is

PROPOSITION 1.4. *If M is a formal manifold of dimension n, then its tangent bundle M^D is a formal manifold of dimension $2n$.*

PROOF. If $t: D \to M$ belongs to M^D, it factors through the monad $\mathfrak{M}(t(\underline{0}))$. Thus, M^D is covered by $\{\mathfrak{M}(x)^D \mid x \in M\}$. However $\mathfrak{M}(x) \to M$ étale implies $\mathfrak{M}(x)^D \to M^D$ étale. But clearly

$$\mathfrak{M}(x)^D \approx (D_\infty(n))^D \approx D_\infty(n) \times R^n ,$$

the last isomorphism by the line type assumption. So M^D is covered by étale subobjects isomorphic to $D_\infty(n) \times R^n$, which however, as a product of two n-dimensional formal manifolds is a $2n$-dimensional formal manifold. The result now easily follows.

Evidently the result and proof generalize to M^K for any $K = D_n(m)$ as well as to the jet bundles introduced later on.

A bijective map $D_\infty(n) \to \mathfrak{M}(x)$ onto the monad around x will be called a *frame* (or ∞*-frame*) *at* x if it maps $\underline{0}$ to x.

In order not to confuse the monad $\mathfrak{M}(x)$ with the «k-monads» we now introduce, we shall sometimes apply the notation $\mathfrak{M}_\infty(x)$ instead of $\mathfrak{M}(x)$.

Let M be an n-dimensional formal manifold. The *k-monad* around $x \in M$, denoted $\mathfrak{M}_k(x)$, is defined as the image of

$$(1.6) \qquad D_k(n) \hookrightarrow D_\infty(n) \xrightarrow[\approx]{\phi} \mathfrak{M}_\infty(x) \subset M,$$

where ϕ is a frame. It is an easy consequence of Proposition 1.3 that this image does not depend on the choice of frame ϕ. A map $D_k(n) \to M$ which can be written as a composite (1.6) for suitable ∞-frame ϕ around x is called a *k-frame* around x. If we add to our assumptions the following (non-coherent) axiom about R (for any natural number p):

(1.7) An injective linear $R^p \to R^p$ is necessarily bijective.

then one can prove that any injective $D_k(n) \to M$ is in fact a k-frame. (Use inverse function theorem, [10], Theorem 5.6.) Two k-frames around x differ by a $\underline{0}$-preserving bijective $D_k(n) \to D_k(n)$.

From étaleness of $\mathfrak{M}_\infty(x) \subset M$ and Proposition 1.3 follows:

(1.8) any map $D_h(p) \to M$ with $\underline{0} \mapsto x$ and $h \leq k$ factors through $\mathfrak{M}_k(x)$.

Also, let us note that if N is also a formal manifold, then

(1.9) any map $f: \mathfrak{M}_k(x) \to N$ factors through $\mathfrak{M}_k(f(x))$.

It is not surprising that formal manifolds share with R^n all the good infinitesimal properties of the latter. Thus, they are infinitesimally linear, satisfy the (Wraith-)requirement (Axiom 2 of [18] or [11] page 146) allowing one to define Lie bracket of vector fields, etc...

2. JETS AND NEIGHBOURS.

In the following, M is a fixed n-dimensional formal manifold. Let $x \in M$. For any object P, a map $\mathfrak{M}_k(x) \to P$ is called a *k-jet at* x «of a map from M to P ». Also if $\pi : E \to M$ is an arbitrary map, a map $g : \mathfrak{M}_k(x) \to E$ such that $\pi \circ g$ is the inclusion map $\mathfrak{M}_k(x) \to M$ is called a *k-jet at* x « of a section of π ». We can in fact define a functor

$$J^k : \mathcal{E}/M \to \mathcal{E}/M$$

which to $\pi : E \to M$ associates the object $J^k E \to M$ whose fibre over x is the set of k-jets at x of sections of π.

For each natural number k, we define a binary relation $\sim_k$ on M by putting:

$$x \sim_k y \quad \text{if } y \in \mathfrak{M}_k(x)$$

(we say then: x *and* y *are k-neighbours*).

PROPOSITION 2.1. *The relation* $\sim_k$ *is reflexive and symmetric. Also*

$$x \sim_k y \wedge y \sim_h z \Rightarrow x \sim_{k+h} z .$$

PROOF. Reflexivity is clear. To prove symmetry, we identify $\mathfrak{M}_\infty(x)$ with $D_\infty(n)$ via a frame at x, and utilize that $\mathfrak{M}_\infty(x) = \mathfrak{M}_\infty(y)$. Then $x \sim_k y$ means that y is identified with a $y \in D_k(n)$. But $y \in D_k(n) \subset D_\infty(n)$ implies

$$\underline{0} \in \mathfrak{M}_k(y) = y + D_k(n).$$

The third assertion follows similarly by also using (1.3).

We denote by M_k the set

$$M_k = \{ (x, y) \in M \times M \mid x \sim_k y \} .$$

It comes equipped with two projections $M_k \rightrightarrows M$ denoted $proj_1$ and $proj_2$; M_k is called the «k'th neighborhood of the diagonal $M \to M \times M$ ». (The object M_k is in fact classical as a ringed space, Grothendieck/Malgrange, cf. [16] ; it has also been considered synthetically by Joyal.)

Consider the functor $I_k : \mathcal{E}/M \to \mathcal{E}/M$ which to an object $\gamma : G \to M$ associates the upper composite in the diagram below, in which the square is formed as a pullback:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & M_k \arrow[r, "proj_2"] \arrow[d, "proj_1"] & M \\ G \arrow[r, "\gamma"] & M & \end{tikzcd}$$

PROPOSITION 2.2. *The functor $I_k : \mathcal{E}/M \to \mathcal{E}/M$ is left adjoint to J^k.*

PROOF. Let $\pi : E \to M$ be arbitrary. An element in either of the hom-sets (writing G for $\gamma : G \to M$, etc...)

$$hom_{\mathcal{E}/M}(I_k(G), E) \text{ and } hom_{\mathcal{E}/M}(G, J^k E)$$

consists of a law which to each $x \in M$ with $x \sim_k y$, each $g \in G$ over x and each $y \in M$ associates an element in E over y.

From Proposition 2.1 follows that we have a map

$$(2.1) \qquad M_k \times_M M_h \to M_{k+h} : (x, y, z) \mapsto (x, z).$$

Consider the diagram

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & M_k \times_M M_h \arrow[r] \arrow[d] & M_h \arrow[r, "proj_2"] \arrow[d, "proj_1"] & M \\ \cdot \arrow[r] \arrow[d] & M_k \arrow[r, "proj_2"] \arrow[d, "proj_1"] & M & \\ G \arrow[r, "\gamma"] & M & & \end{tikzcd}$$

in which all squares are pullbacks, and where the middle horizontal composite is $I_k(G)$ and the upper horizontal composite therefore $I_h(I_k(G))$. The map (2.1) is compatible with the relevant projections, and hence induces a map (natural in $G \in \mathcal{E}/M$):

$$(2.2) \qquad I_h(I_k(G)) \to I_{k+h}(G).$$

By the adjointness of Proposition 2.2, this gives rise to a natural:

$$(2.3) \qquad J^{k+h}(E) \to J^k(J^h(E)),$$

natural in $E \in \mathcal{E}/M$. We can describe this in direct terms as follows. An element in the left hand side of (2.3), over $x \in M$, is a section

$$\sigma : \mathfrak{M}_{k+h}(x) \to E .$$

An element in the left hand side over x is a section

$$\mathfrak{M}_k(x) \to J^h(E). \qquad (2.4)$$

To σ, the map (2.3) associates that section (2.4) which to $y \in \mathfrak{M}_k(x)$ associates $\sigma|_{\mathfrak{M}_h(y)} : \mathfrak{M}_h(y) \to E$, noting that $\mathfrak{M}_h(y) \subset \mathfrak{M}_{k+h}(x)$. (Under suitable assumptions on E, (2.3) can be shown to be injective, defining the subset of $J_k(J_h(E))$ of «holonomous jets».)

We shall later need «prolongation»: given a section $\xi : M \to E$ of $E \to M$, we get a section $J^k\xi : M \to J^k E$ of $J^k E \to M$, namely

$$(J^k\xi)(x)(y) = \xi(y) \quad \text{for} \quad y \in \mathfrak{M}_k(x). \qquad (2.5)$$

3. GROUPOIDS IN GENERAL.

Recall that a groupoid(-object) is a category(-object) in which every arrow is invertible. Some of the present Section 3 deals with some classical constructions/properties of groupoids which make sense in any sufficiently good category (say, a topos $\mathcal{E}$, but much weaker things will do).

If Φ is a groupoid with M as its «set» of objects. we employ the notations $\alpha, \beta : \Phi \rightrightarrows M$ for source and target, respectively, and $i : M \to \Phi$ for the inclusion of the identity arrows.

Let G be a group(-object), and $P \to M$ a right G-torsor (principal G-bundle) over M. Recall (from [2], or [15] page 25) the construction of a groupoid PP^{-1} with M as its «set» of objects: an arrow $x \to y$ (where $x, y \in M$) is an equivalence class of pairs

$$(q', q) \quad (\text{with } q' \in P_y \text{ and } q \in P_x)$$

modulo the equivalence relation

$$(q', q) \sim (q'g, qg) \quad \text{for all } g \in G .$$

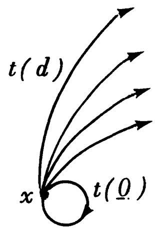
The equivalence class of (q', q) is denoted q'/q.

For a groupoid Φ (with M as its objects), a map $t: D \to \Phi$ is called a *vertical tangent* if

$$\alpha(t(d)) = \alpha(t(\underline{0})) \quad \forall d \in D,$$

and a *deplacement* [15] if further $t(\underline{0}) \in \Phi$ is an identity arrow $x \in M$. So, a deplacement looks like this:

(3.1)



We say t is a deplacement *at* x, *in* the groupoid Φ. They form a sub-vector space of the tangent space $T_x\Phi$.

PROPOSITION 3.1. *The following data are equivalent:*

(i) for each $x \in M$, a deplacement t_x at x.

(ii) a vector field ξ on Φ, which is vertical and right-invariant.

(iii) (if $\Phi = P P^{-1}$) a G-right-invariant vector field $\bar{\xi}$ on P.

The right-invariance of (ii) means: for any composable pair ϕ, ψ of arrows:

$$\xi(\phi, d) \circ \psi = \xi(\phi \circ \psi, d) \qquad \forall d \in D.$$

Note that $\xi(\phi, d)$ and ψ are composable, since, by verticality of ξ, $\alpha(\xi(\phi, d)) = \alpha(\phi)$. (In Remark 6.6, we look at these vector fields from a more categorical viewpoint.)

The equivalence of these data is constructed as follows: Given ξ as in (ii), the deplacement t is obtained by restriction along $i: M \to \Phi$:

$$t_x(d) := \xi(i_x, d).$$

Conversely, given a «deplacement field $\{ t_x \mid x \in M \}$», construct ξ by:

$$\xi(\phi, d) := t_x(d) \circ \phi \quad \text{where } x = \alpha(\phi).$$

Also, given ξ as in (ii), we construct $\bar{\xi}: P \times D \to P$ by:

$$\bar{\xi}(q, d) := \text{that unique } q' \text{ so that } \xi(q/q, d) = q'/q.$$

Conversely, given $\bar{\xi}: P \times D \to P$ as in (iii), we construct ξ as in (ii) by:

$$\xi(q/p, d) := \bar{\xi}(q, d)/p.$$

Generally, a D-deformation of a map $f: X \to Y$ is a map

$$F: X \times D \to Y \quad \text{with} \quad F(x, \underline{0}) = f(x) \quad \forall x \in X.$$

We see that a deplacement field is a D-deformation of $i: M \to \Phi$, having an α-verticality property.

We now assume that Φ satisfies the Wraith requirement (see [18], or [11], page 146) so that Lie brackets of vector fields on Φ can be formed. This will automatically be so if Φ is a formal manifold.

PROPOSITION 3.2. *The Lie bracket* $[\xi, \eta]$ *of two vertical right invariant vector fields* ξ *and* η *on* Φ *is again vertical right invariant.*

PROOF. This is essentially trivial, since

$$[\xi, \eta](-, d'.d'') \quad \text{for} \quad (d', d'') \in D \times D,$$

is defined as the group-theoretic commutator of the infinitesimal transformations $\xi(-, d')$ and $\eta(-, d'')$. Both of these preserve α, so that their commutator does as well, proving verticality. Similarly, since they both are right invariant, so is their commutator.

4. GROUPOIDS OF JETS.

Throughout this section, M is a fixed n-dimensional formal manifold. Let k be a natural number. We associate to M a category $C^k M$ (meaning category-object in $\mathcal{E}$, of course), whose objects are the elements of M, and where an arrow from x to y is a map

$$(4.1) \qquad \mathfrak{M}_k(x) \xrightarrow{f} \mathfrak{M}_k(y) \quad \text{with} \quad f(x) = y.$$

This is the same thing by (1.9) as a k-jet at x of a map from M to itself. The groupoid of invertible arrows of this category is denoted $\Pi^k M$. Domain and codomain are denoted α, β. Thus, for the arrow f in (4.1),

$$\alpha(f) = x, \quad \beta(f) = y.$$

Recall from [13] that we may formulate the étaleness notion (of Section 1, say) as follows: a map $f: N \to M$ is étale if for any infinitesimal object X,

$$\begin{tikzcd} N^X \arrow[r, "f^X"] \arrow[d, "\pi"'] & M^X \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

is a pullback (π being «evaluation at $0 \in X$ »).

PROPOSITION 4.1. *Let $h: N \to M$ be étale. Then N is an n-dimensional formal manifold, and there is a full and faithful functor $\hat{h}: \Pi^k N \to \Pi^k M$ sending the arrow $g: \mathfrak{M}_k(n_1) \to \mathfrak{M}_k(n_2)$ to the arrow*

$$(4.2)\quad \mathfrak{M}_k(h(n_1)) \xrightarrow[\approx]{h^{-1}} \mathfrak{M}_k(n_1) \xrightarrow{g} \mathfrak{M}_k(n_2) \xrightarrow[\approx]{h} \mathfrak{M}_k(h(n_2)).$$

Furthermore, $\hat{h}$ is étale.

PROOF. Choose a frame $D_\infty(n) \to M$ around $h(x)$. On each

$$D_k(n) \subset D_\infty(n),$$

we get a unique lifting over N with $\underline{0} \mapsto x$. So we get a lifting of the whole frame, whose image is an ∞-monad around x, and is mapped bijectively by h to the ∞-monad around $h(x)$. Also h maps the k-monad around x bijectively to the k-monad around $h(x)$, whence (4.2) makes sense, and also it makes clear that $\hat{h}$ is full and faithful. This latter can alternatively be expressed:

$$\begin{tikzcd} \Pi^k N \arrow[r, "\hat{h}"] \arrow[d, "<\alpha, \beta>"'] & \Pi^k M \arrow[d, "<\alpha, \beta>"] \\ N \times N \arrow[r, "h \times h"] & M \times M \end{tikzcd}$$

is a pullback. But h étale implies $h \times h$ étale implies $\hat{h}$ étale (étale maps being stable under pullbacks, evidently).

REMARK. If h is furthermore surjective, then $\hat{h}$ is «an equivalence» since the functor $\hat{h}$ is surjective on objects. But since we cannot split surjections $\hat{h}$ is not an *adjoint* equivalence, in general.

We let G ($= Aut(D_k(n))$) denote the group of $\underline{0}$-preserving invertible maps $D_k(n) \to D_k(n)$.

Closely related to $\Pi^k M$ is a certain right G-torsor over M (= principal fibre bundle with group G), namely the *k-frame* bundle $F_k M$, whose elements are k-frames $\phi : D_k(n) \to M$. The map $\pi : F_k M \to M$ is given by $\pi(\phi) = \phi(\underline{0})$, and the right G-action is evident, since if ϕ is a k-frame at x and $\gamma \in G$, the composite map $\phi \circ \gamma$ is likewise a k-frame at x.

The relationship between $F_k M$ and $\Pi^k M$ is that $\Pi^k M$ arises as $(F_k M)(F_k M)^{-1}$, the latter being a case of the general construction of a groupoid $P P^{-1}$ from a torsor P, described in Section 3. For, if ϕ' is a k-frame at y and ϕ is a k-frame at x, ϕ'/ϕ is identified with the arrow $x \to y$ in $\Pi^k M$ given as the composite

$$\mathfrak{M}_k(x) \xrightarrow{\phi^{-1}} D_k(n) \xrightarrow{\phi'} \mathfrak{M}_k(y).$$

The following proposition, due to P. Libermann [15], Theorem 15.1, is important, but in our context, the proof becomes almost trivial (namely an exponential adjointness).

PROPOSITION 4.2. *There is a natural correspondence between elements in the bundle* $J^k(TM)$ *(where* $TM = (M^D \to M)$ *is the tangent bundle of* M*), and deplacements in* $\Pi^k M$.

PROOF. An element h over $x \in M$ in $J^k TM$ is a section of $M^D \to M$ defined over $\mathfrak{M}_k(x)$, thus, by exponential adjointness, gives a map

$$\mathfrak{M}_k(x) \times D \xrightarrow{h} M \quad \text{with} \quad h(y, \underline{0}) = y.$$

By exponential adjointness once more we get $D \xrightarrow{\hat{h}} M^{\mathfrak{M}_k(x)}$, but since

$$\hat{h}(\underline{0}) = \text{identity map of } \mathfrak{M}_k(x),$$

and the set of those maps $\mathfrak{M}_k(x) \to M$ which map bijectively to some $\mathfrak{M}_k(y)$ is étale, all the values $\hat{h}(d)$ are elements of $\Pi^k M$. - The passage from $\hat{h}$ to h is again just by exponential adjointness.

As a corollary of Proposition 4.2 and Proposition 3.1, we get immediately:

COROLLARY 4.3. *There is a natural correspondence between:*

(i) sections of $J^k TM \to M$;

(ii) deplacement fields on the groupoid $\Pi^k M$;

(iii) vertical right invariant vector fields on the groupoid $\Pi^k M$;

(iv) right G-invariant vector fields on the frame bundle $F^k M$.

In particular, since the set mentioned in (iii) has a natural Lie algebra structure, by Proposition 3.2, we get a natural Lie algebra structure on the set of sections of $J^k TM \to M$.

Classically one can say more, namely: the *sheaf* of (germs of) sections of $J^k TM \to M$ is a sheaf of Lie algebras (and similarly for the data (ii)-(iv) in Corollary 4.3). We can make a similar statement. Consider the full subcategory $i: Et/M \to \mathfrak{S}/M$ having as objects *étale* maps to M. Et/M is closed under finite limits in $\mathfrak{S}/M$. Now $\mathfrak{S}/M$ is a Grothendieck topos and as such carries a canonical Grothendieck topology (= site structure); we equip Et/M with the induced Grothendieck topology and get for general reasons [19] a geometric functor

$$\mathfrak{S}/M = sh(\mathfrak{S}/M) \xrightarrow{i*} sh(Et/M).$$

It sends $E \to M$ to the functor

$$(Et/M)^{op} \to Set \quad \text{given by} \quad hom_{\mathfrak{S}/M}(i(-), E \to M),$$

which is actually a sheaf.

PROPOSITION 4.4. *The object* $i*(J^k TM \to M)$ *carries the structure of a Lie algebra object.*

PROOF. Let $h: N \to M$ be étale. We must give the set $hom_{\mathfrak{S}/M}(N, J^k TM)$ a Lie algebra structure. An element of this set can be identified with a section of the left hand column in the pullback diagram

$$(4.1) \qquad \begin{array}{ccc} P & \xrightarrow{\bar{h}} & J^k TM \\ \downarrow & & \downarrow \\ N & \xrightarrow{h} & M \end{array}$$

However, $P = J^k TN$; for, $h^*TM = TN$, by étaleness, and since h (again by étaleness) maps $\mathfrak{M}_k(z)$ bijectively to $\mathfrak{M}_k(h(z))$, one easily concludes

$J^k h^* TM = h^* J^k TM$. Thus elements in $hom(N, J^k TM)$ are identified with global sections of $P \to N$, i.e. of $J^k TN \to N$. These carry a Lie algebra structure (note that N is a formal manifold if M is).

5. G-STRUCTURES AND LIE EQUATIONS.

Throughout this section, M is a fixed n-dimensional formal manifold. Let H be a subgroup of $G = Aut(D_k(n))$. An *H-structure on M* is a subset S of the frame-bundle $F_k M$ which is stable under the right action of $H \subset G$, and is a H-torsor (= principal H-bundle) over M. The elements of S are called the *admissible frames for the H-structure*.

EXAMPLES. 1° Let $k = 1$. Then G is easily seen to be $GL(n, R)$. Let $H = SL(n, R)$ (matrices of determinant 1). An H-structure on M amounts to a (local) volume notion; the admissible frames are to be thought of as the volume-preserving ones.

2° Let $k = 1$. Let H be the subgroup of those linear $R^n \to R^n$ which map the linear subspace

$$\{ x_{p+1} = \dots = x_n = 0 \}$$

into itself. So H consists of $n \times n$-matrices with 0's in the lower left $(n-p) \times p$ corner. An H-structure is a distribution on M.

The arrows of the groupoid SS^{-1} can be identified with those arrows («the admissible ones for the H-structure»)

$$\mathfrak{M}_k(x) \xrightarrow{\phi} \mathfrak{M}_k(y) \quad \text{in} \quad (F_k M)(F_k M)^{-1} = \Pi^k M,$$

which have the property that if γ is an admissible frame at x, then $\phi \circ \gamma$ is an admissible frame at y. In particular, SS^{-1} is a subgroupoid of $\Pi^k M$. Therefore, the deplacements of x in SS^{-1} form a sub-vector-space of the deplacements of x in $\Pi^k M$. Now by Libermann's bijection (Proposition 4.2), the deplacements of x in SS^{-1} correspond to a subset of $J^k TM$; we denote it $R(S) \subset J^k TM$. Thus, a section

$$\mathfrak{M}_k(x) \to TM = M^D, \quad \text{corresponding to} \quad \mathfrak{M}_k(x) \times D \xrightarrow{\sigma} M,$$

belongs to $R(S)$ iff, for every $d \in D$,

$$\mathfrak{M}_k(x) \xrightarrow{\sigma(-,d)} \mathfrak{M}_k(\sigma(x,d))$$

is admissible (belongs to SS^{-1}).

PROPOSITION 5.1. *$i*(R(S)) \subset i*J^k TM$ is closed under the Lie bracket of Proposition 4.4.*

PROOF. Let $h: N \to M$ be étale. The full and faithful functor $\hat{h}: \Pi^k N \to \Pi^k M$ of Proposition 4.1 pulls the subgroupoid $SS^{-1} \subset \Pi^k M$ back to a subgroupoid of N, which comes from a unique H-structure Σ on N,

$$\hat{h}^{-1}(SS^{-1}) = \Sigma\Sigma^{-1}.$$

Also we have

$$\tilde{h}^{-1}(R(S)) = R(\Sigma) \subset J^k TN$$

(where $\tilde{h}: J^k TN \to J^k TM$ is the map displayed in the pullback diagram (4.1)). Thus, liftings of $h: N \to M$ over $R(S) \to J^k TM \to M$ correspond bijectively to cross-sections of

$$R(\Sigma) \to J^k TN \to N,$$

which by the bijection of Proposition 4.2 and Corollary 4.3 correspond to vertical right-invariant vector fields on $\Sigma\Sigma^{-1} \subset \Pi^k N$. These are stable under Lie bracket, hence carry the desired structure.

A linear Lie equation of order k on M, [15], is now a sub-vector bundle $R \subset J^k TM$, such that $i*R \subset i*J^k TM$ is stable under the Lie bracket. The proposition just proved tells us that H-structures give rise to such.

A solution ([14], page 6) of R is a vector field $\xi: M \to M^D$ such that the k'th prolongation (cf. (2.5)) $J^k\xi: M \to J^k TM$ factors through $R \subset J^k TM$.

The geometric meaning of the solutions of $R(S) \subset J^k TM$, where S is an H-structure on M, is that the infinitesimal transformations $\xi(-,d)$ belonging to ξ preserve the H-structure S:

PROPOSITION 5.2. *The vector field ξ is a solution of $R(S)$ iff, for any $d \in D$, and any $x \in M$,*

$$\xi(-,d)|_{\mathfrak{M}_k(x)}: \mathfrak{M}_k(x) \to \mathfrak{M}_k(\xi(x,d)) \tag{5.1}$$

belongs to SS^{-1}.

PROOF. To say that ξ is a solution means that

$$\xi|_{\mathfrak{M}_k(x)}: \mathfrak{M}_k(x) \to M^D$$

by the bijection of Proposition 4.2 (twisted exponential adjointness) gives a map

$$D \to M^{\mathfrak{M}_k(x)} \tag{5.2}$$

with the property that any d goes to an admissible $\mathfrak{M}_k(x) \to M$. But the value of (5.2) at any $d \in D$ is just (5.1).

6. SIX SCATTERED REMARKS.

REMARK 6.1. Together with the notion of *category-object* C, it is known to be useful to consider the notion of *discrete opfibration* over C (= internal diagram over C, see e.g. [8], 2.14 and 2.15). Such occur in abundance in our context $\mathcal{E}$ (because they occur in Ehresmann's context of differentiable categories). Consider for example, for a formal manifold M, the groupoid $\Pi^1 M$. The tangent bundle $TM = M^D$ is a discrete opfibration over it; the action $M^D \times_M \Pi^1 M \to M^D$ is given by

$$<t, \mathfrak{M}_1(x) \xrightarrow{\phi} \mathfrak{M}_1(\phi(x))> \to \phi \circ t,$$

where t is a tangent vector at x; the composite $\phi \circ t$ makes sense because $t(\underline{0}) = x$ so that t factors through $\mathfrak{M}_1(x)$. Note that we may replace $\Pi^1 M$ by the category $C^1 M$, if we want. The example generalizes in several other directions as well.

Also, note that we may interpret the process described as a process which to any map $\mathfrak{M}_1(x) \to \mathfrak{M}_1(y)$ (taking x to y) constructs a linear map $T_x M \to T_y M$; and this process may, by calculating in coordinates (choosing frames at x and y) be seen to be invertible. So $C^1 M$ is isomorphic to the category of linear maps between the fibres of $TM \to M$.

REMARK 6.2. We note that the jet categories $C^k M$ or $\Pi^k M$ are actually *concrete categories* (relative to our viewing $\mathcal{E}$ as the category of sets). The forgetful functor $\Pi^k M \to \mathcal{E}$ is given by $x \mapsto \mathfrak{M}_k(x)$, for $x \in M$ an object of $\Pi^k M$, i.e. an element of M. For, an arrow $x \to y$ in $\Pi^k M$ is a *map* $\mathfrak{M}_k(x) \to \mathfrak{M}_k(y)$, in $\mathcal{E}$. In this respect, our approach is more concrete than Ehresmann's.

REMARK 6.3. Classification (for each k, n, m) of the equivalence classes of $\underline{0}$-preserving maps $D_k(n) \to D_k(m)$ modulo the equivalence relation given by the evident action of the group $Aut_k(n) \times Aut_k(m)$ is to a certain extent what singularity theory is about, I believe. I hope to be able to utilize/substantiate this remark. It is related to remarks in my paper «On algebraic theories of power series», *Cahiers de Topo. et Géom. Dif.* XVI (1975).

REMARK 6.4. Differentiable groupoids are known to provide a good setting for various connection notions, cf. [4] or [20]. The basic object $Q^k(M, \Phi)$ where Φ is a groupoid-object with M as its object of objects (M a formal manifold) is in our context described as follows:

An element X in $Q^k(M, \Phi)$ *over* $x \in M$ is a law which, to each $y \sim_k x$, associates an arrow f_y in Φ from y to x, and with $f_x = id_x$.
A cross-section of the natural map $Q^k(M, \Phi) \to M$ is a k'th order *connection in* Φ.

REMARK 6.5. Let Φ be as in the preceding remark. The k'th prolongation of Φ (cf. [7, 14, 17]), $J^k\Phi$, is the category with M as its «set» of objects, and where an arrow $x \to y$ is a map

$$s: \mathfrak{M}_k(x) \to \Phi \quad \text{with} \quad \alpha \circ s = \text{inclusion map } \mathfrak{M}_k(x) \to M \ \ (\text{so } s \text{ is a } k\text{-jet of a section of } \alpha: \Phi \to M) \text{ and with } (\beta \circ s)(x) = y.$$

Denote $\beta \circ s$ by $\hat{s}$. If $t: y \to z$ is another arrow in $J^k\Phi$ the composite $t \circ s$ is that map $\mathfrak{M}_k(x) \to \Phi$ which sends

$$x' \sim_k x \quad \text{to} \quad t(\hat{s}(x')).s(x'),$$

the dot denoting the composition in the groupoid Φ. Note that if we employ this process to the obvious codiscrete groupoid with M as object set, we arrive at $C^k M$.

REMARK 6.6. Since we work with category-objects in $\mathcal{E}$, and $\mathcal{E}$ is a topos, we have more freedom in performing genuine category theoretic constructions than when working with «differentiable categories». For instance, we can form functor-categories. To illustrate this, we construct first the category D whose «set» of objects is D, and where, besides the identity arrows, there is exactly one arrow from $\underline{0}$ to each d :

$$(6.1) \qquad \begin{tikzcd}[row sep=small] & & d_0 \\ & & & d_1 \\ \underline{0} \arrow[uurr] \arrow[urrr] \arrow[rrrr] & & & & d_2 \end{tikzcd}$$

(denote the arrow from $\underline{0}$ to d by $\hat{d}$; we require $\hat{\underline{0}} = id_{\underline{0}}$).

Let Φ be a differentiable groupoid with M as its set of objects. A functor from D to Φ is then the same thing as a deplacement. (Compare pictures (3.1) and (6.1).) A functor

$$(6.2) \qquad \Phi \times \mathrm{D} \xrightarrow{\ \xi\ } \Phi$$

with $\xi(-, 0) = id_\Phi$ can be seen to be the same thing as a vertical right invariant vector field on Φ. For, given a functor ξ, construct

$$\hat{\xi} : \Phi \times D \to \Phi \quad \text{by} \quad \hat{\xi}(\phi, d) := \xi(\phi, \hat{d}).$$

Conversely, given a deplacement field $\hat{\xi}$, construct the functor ξ by

$$\xi(\phi, \hat{d}) := \hat{\xi}(z, d) \circ \phi \quad \text{where} \quad \phi : y \to z,$$

$$\xi(\phi, id_d) := \hat{\xi}(z, d) \circ \phi \circ \hat{\xi}(y, d)^{-1}.$$

By exponential adjointness (in the cartesian closed category of category-objects in $\mathcal{E}$), the ξ in (6.2) corresponds to a right-inverse functor to the «evaluation at 0 »-functor π :

$$(6.3) \qquad \Phi^{\mathrm{D}} \xrightarrow{\ \pi\ } \Phi .$$

Recall that a vector field on N is a section of $N^D \to N$. Thus we see that the notion of right-invariant (vertical) vector field on a groupoid Φ is a 2-dimensional lifting of the ordinary notion of vector fields on an object N.

Also, we immediately know that Φ^{D} is a groupoid; so there is a natural groupoid whose «set» of objects is the set of deplacement fields on Φ.

Clearly, since Φ is a groupoid, we may in Φ^{D} replace D by

$$\hat{D} = \text{the groupoid obtained by inverting all arrows in } D.$$

The groupoid $\hat{D}$ looks less ad hoc than the category D : it is the codiscrete groupoid on D (= precisely one arrow between any two objects).

REFERENCES.

1. E.J. DUBUC, Sur les modèles de la Géométrie differentielle synthétique, *Cahiers Topo. et Géom. Diff.* XX-3 (1979), 231-279.
2. C. EHRESMANN, Les connexions infinitésimales dans un espace fibré différentiable, *Colloque Topologie Bruxelles 1950*, CBRM, 29-55.
3. C. EHRESMANN, Calcul des jets, *C.R.A.S. Paris* 233 (1951), 598.
4. C. EHRESMANN, Sur les connexions d'ordre supérieur, *Atti del V Congresso dell'Unione Matem. Ital.* Pavia Torino (1956), 344-346.
5. C. EHRESMANN, Catégories topologiques et catégories différentiables, *Coll. Géom. Diff. Globale Bruxelles 1958*, CBRM, 137-150.
6. C. EHRESMANN, Grupoides diferenciables y pseudogrupos de Lie, *Rev. Un. Mat. Argentina* 19 (1960), 48.
7. C. EHRESMANN, Prolongements des catégories différentiables, *Cahiers Topo. et Géom. Diff.* VI (1964).
8. P. JOHNSTONE, *Topos Theory*, Academic Press, London, 1977.
9. A. KOCK, A simple axiomatics for differentiation, *Math. Scand.* 40 (1977), 183-193.
10. A. KOCK, Taylor series calculus for ring objects of line type, *J. Pure & Appl. Algebra* 12 (1978), 271-293.
11. A. KOCK, On the synthetic theory of vector fields, in Topos theoretic methods in Geometry, *Aarhus Var. Publ. Series* 30 (1979), 139-157.
12. A. KOCK, Properties of well-adapted models for synthetic differential geometry, *J. Pure & Appl. Algebra* (to appear).
13. A. KOCK & G.E. REYES, Manifolds in formal differential geometry, *Lecture Notes in Math.* 753, Springer (1977), 514-533.
14. A. KUMPERA & D. SPENCER, Lie equations, *Annals of Math. Studies* 73, Princeton (1972).
15. P. LIBERMANN, Sur les prolongements des fibrés principaux et des groupoïdes différentiables banachiques, in *Analyse Globale*, Presses Univ. Montréal 1971, 7-108.
16. B. MALGRANGE, Equations de Lie I-II, *J. Diff. Geom.* 6 (1972), 503-522; 7 (1972), 117-141.

17. NGO VAN QUE, Du prolongement des espaces fibrés et des structures infinitésimales, *Ann. Institut Fourier* 17, Grenoble (1967), 157-223.

18. G.E. REYES & G.C. WRAITH, A note on tangent bundles in a category with a ring object, *Math. Scand.* 42 (1978), 53-63.

19. J. VERDIER, Fonctorialité des catégories de faisceaux, SGA 4, Exposé III *Lecture Notes in Math.* 269, Springer.

20. P. VER EECKE, *Introduccion a las conexiones de orden superior*, Fac. Cienc. de Zaragoza, 1968.

21. Topos theoretic methods in Geometry, A collection of articles ed. A. KOCK, *Aarhus Var. Publ. Series* 30, 1979.

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