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FIBRE BUNDLES IN GENERAL CATEGORIES

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A notion of fibre bundle is described, which makes sense in any category with finite inverse limits. An assumption is that some of the structural maps that occur are descent maps: this is the categorical aspect of the notion of glueing objects together out of local data. Categories of fibre bundles are proved to be equivalent to certain categories of groupoid actions.

Some applications in locale theory are indicated.

Introduction

We describe a notion of fibre bundle which makes sense in any category $\mathscr{E}$ with finite inverse limits. To get a sufficiently rich theory, one needs to assume that certain of the maps entering into the definition are descent maps, in a well-known sense which we shall recapitulate.

The theory applies to the case where $\mathscr{E}$ is the category of locales, cf. e.g. [4] (or [5], whose authors call them 'spaces'). Here one has a good class of descent maps, namely the open surjections. This category contains the category of (sober) topological spaces as a full subcategory, and our theory specializes here to the classical one.

In any topos, the epimorphisms are descent maps, and for this special case we have elsewhere [7, 8] developed some of the theory of principal fibre bundles from the viewpoint of what we called 'pregroupoids'. We recapitulate this notion in the present more general setting and adjoin the general fibre bundle notion. Only under descent assumptions will a pregroupoid itself be a fibre bundle, and in that case it deserves the name 'principal fibre bundle' which thus here is a derived concept. This is why we insist on keeping the homemade word pregroupoid.

The last section contains some observations on pregroupoids in the category of locales.

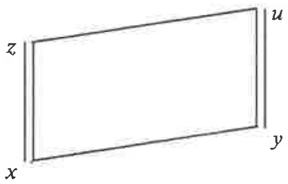
1. Pregroupoids and fibre bundles

We consider a category $\mathscr{E}$ with finite inverse limits. We shall talk about its objects as if they were sets.

To motivate the pregroupoid notion, consider a groupoid in $\mathscr{E}$, i.e. a small category where all arrows are invertible. Let the 'set' of objects be B, and let $* \in B$ be a given base point. The set of all arrows x of Φ with domain $*$ comes equipped with a map $\pi : X \to B$, namely 'take codomain', and it carries a 4-ary relation Λ given by

$$\Lambda(x, y, z, u) \quad \Leftrightarrow \quad y \cdot x^{-1} = u \cdot z^{-1} \quad \Leftrightarrow \quad x^{-1} \cdot z = y^{-1} \cdot u. \tag{1.1}$$

It is convenient to represent the statement $\Lambda(x, y, z, u)$ by a diagram



(1.2)

The diagram suggests the symmetries of the situation

$$\Lambda(x, y, z, u) \quad \Leftrightarrow \quad \Lambda(z, u, x, y) \quad \Leftrightarrow \quad \Lambda(y, x, u, z)$$

as well as the 'book-keeping' condition

$$\Lambda(x, y, z, u) \quad \Rightarrow \quad \pi(x) = \pi(z) \quad \text{and} \quad \pi(y) = \pi(u). \tag{1.3}$$

We now abstract the situation and give the formal definition of the notion of pregroupoid.

Let $\pi : X \to B$ be a given map. A *pregroupoid structure* on it is a 4-ary relation Λ on X (i.e. a subobject of X^4) such that the following four conditions hold:

(1) the book-keeping condition (1.3);

(2) given any three elements satisfying the relevant book-keeping condition, there is a unique fourth element making up a Λ-quadruple (e.g. if x, y, z are given with $\pi(x) = \pi(z)$, there is a unique u with $\Lambda(x, y, z, u)$;

(3) the relation $\sim_v$ between pairs of elements of X, given by

$$(x, z) \sim_v (y, u) \quad \text{iff} \quad \Lambda(x, y, z, u)$$

is an equivalence relation;

(4) the relation $\sim_h$ between pairs of elements of X, given by

$$(x, y) \sim_h (z, u) \quad \text{iff} \quad \Lambda(x, y, z, u)$$

is an equivalence relation.

If $X \to B$ is equipped with a Λ in this way, we shall feel free to say 'X is a pregroupoid over B'.

Pregroupoids arise from groupoids, and, as a special case, any kind of 'frame bundle' carries a pregroupoid structure. Also, the universal covering space of a space is a pregroupoid over it, cf. Section 5 below.

Let $\varepsilon : E \to B$ be given, and let X be a pregroupoid over B. Let F be an object in

$\mathscr{E}$. To *equip $E \to B$ with the structure of a fibre bundle for X with fibre F* means to give a map

$$\alpha : X \times F \to E \tag{1.4}$$

over B (meaning $\varepsilon(\alpha(x,f)) = \pi(x)$) such that the induced map

$$\bar{\alpha} : X \times F \to X \times_B E \tag{1.5}$$

(given by $\bar{\alpha}(x,f) = (x, \alpha(x,f))$) is invertible, and such that

$$\Lambda(x,y,z,u) \wedge (\alpha(x,f_1) = \alpha(z,f_2)) \quad \Rightarrow \quad \alpha(y,f_1) = \alpha(u,f_2). \tag{1.6}$$

Equivalent, more equational, formulations of (1.6) appear as (4.3) and (4.4) below.

A *fibre bundle for X* is a triple $(E \to B, F, \alpha)$, where α equips $E \to B$ with the structure of a fibre bundle for X with fibre F; a morphism of fibre bundles for X,

$$(E \to B, F, \alpha) \to (E' \to B, F', \alpha')$$

is a pair of maps, $E \to E'$ (over B), and $F \to F'$, compatible with the α's. For a given pregroupoid X we thus get a category **Fib**(X) of fibre bundles for X.

The intention is that an $x \in X$ with $\pi(x) = b$ is a 'frame' at the point b of the space B; that $\Lambda(x,y,z,u)$ means 'the coordinate change from x to z is the same as from y to u'; that F is the coordinatizing space; and that $\alpha(x,f)$ is the point in E which in the frame x has coordinates f.

More concretely, if $\mathscr{E}$ is the category of sets, or a topos, then we may form the 'concrete' pregroupoid

$$H = H\left(F, \begin{matrix} E \\ \downarrow \\ B \end{matrix}\right)$$

where an element of H over $b \in B$ is a bijective map $F \to E_b$ from F to the b-fibre of E; the Λ for H is given by (1.1). To equip $E \to B$ with a fibre bundle structure for X with fibre F is then equivalent to giving a pregroupoid homomorphism over B,

$$\hat{\alpha} : X \to H\left(F, \begin{matrix} E \\ \downarrow \\ B \end{matrix}\right),$$

$\hat{\alpha}$ being essentially the exponential adjoint of the α of (1.4).

2. Descent

We consider again a category $\mathscr{E}$ with finite inverse limits. A morphism is called *regular epi* if it is a coequalizer of its kernel pair, and *stable regular epi* if pulling it back along an arbitrary map yields a regular epi.

Given a map $\gamma : X \to C$ and a map $\eta : Y \to X$. Consider the diagram (full arrows)

$$\begin{tikzcd}
Y \times_C X \arrow[loop left, dashed, "\tau"] \arrow[r, "p_0"] \arrow[d, "\eta \times X"'] & Y \arrow[r, dashed, "q"] \arrow[d, "\eta"] \arrow[dr, phantom, "*"] & Q \arrow[d, dashed] \\
X \times_C X \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[r, "\gamma"'] & C
\end{tikzcd} \tag{2.1}$$

By *descent data* for descent of Y along γ we mean a map $\tau : Y \times_C X \to Y \times_C X$ 'living over the twist map $\mathrm{tw} : X \times_C X \to X \times_C X$ via $\eta \times X$' (meaning

$$\mathrm{tw} \circ (\eta \times X) = (\eta \times X) \circ \tau,$$

where $\mathrm{tw}(x_1, x_2) = (x_2, x_1)$), and such that the pair of maps $(p_0, p_0 \circ \tau) : Y \times_C X \to Y$ is an equivalence relation on Y, with τ as its symmetry.

By a *solution* of such descent data we mean a pullback square as $*$ in (2.1), such that $(p_0, p_0 \circ \tau)$ is the kernel pair of q. We refer to the whole diagram (2.1) as a 'descent situation defining Q'.

By a *descent map* γ we mean a stable regular epi $\gamma : X \to C$ with the property that any descent data for descent along it has a solution.

Since γ is now assumed to be a stable regular epi, it follows that q is a (stable) regular epi, and so in particular is a coequalizer of its kernel pair $(p_0, p_0 \circ \tau)$ – from which in turn it follows that a solution for the descent data is uniquely determined up to a unique isomorphism.

3. Group and groupoid associated to a regular pregroupoid

By a *regular* pregroupoid we understand a pregroupoid $X \to B$ such that $X \to B$ and $X \to 1$ are descent maps. For such a pregroupoid we have the following two descent situations, defining objects X_* and X^*, respectively. The first one is

$$\begin{tikzcd}
(X \times_B X) \times X \arrow[loop left, "\tau_\upsilon"] \arrow[r, "p_0"] \arrow[d, "p_0 \times X"'] & X \times_B X \arrow[r, "q_\upsilon"] \arrow[d, "p_0"] & X_* \arrow[d] \\
X \times X \arrow[r, shift left, "p_0"] \arrow[r, shift right] & X \arrow[r] & 1
\end{tikzcd} \tag{3.1}$$

with $\tau_\upsilon(x, z, y) = (y, u, x)$, where u is the unique element satisfying $\Lambda(x, y, z, u)$. (The book-keeping condition is here $\pi(x) = \pi(z)$; then $\pi(y) = \pi(u)$ follows.) The second one is

$$\begin{tikzcd} (X\times X)\times_B X \arrow[loop left, "\tau_h"] \arrow[r, "p_0"] \arrow[d, "p_0\times_B X"'] & X\times X \arrow[r, "q_h"] \arrow[d, "p_0"] & X^* \arrow[d, "\partial_0"] \\ X\times_B X \arrow[r, shift left, "p_0"] \arrow[r, shift right] & X \arrow[r, "\pi"'] & B \end{tikzcd} \tag{3.2}$$

with $\tau_h(x,y,z)=(z,u,x)$, where u is again the unique element satisfying $\Lambda(x,y,z,u)$. (The book-keeping condition being again $\pi(x)=\pi(z)$.)

It is immediately clear that τ_v and τ_h do satisfy the conditions for descent data. In the case of τ_v, the relation $(p_0, p_0\circ\tau_v)$ on $X\times_B X$ is exactly the subobject of $(X\times_B X)\times(X\times_B X)$ described by $\{(x,z,y,u)\,|\,\Lambda(x,y,z,u)\}$, which is an equivalence relation by assumption 3 on pregroupoids. Similarly, assumption 4 takes care of (3.2).

Since the maps q_v and q_h are stable regular epis, it follows that one may represent 'elements' in X_* (respectively X^*) by 'elements' of $X\times_B X$ (respectively $X\times X$); denoting $q_v(x,z)$ by $\underline{xz}$ and $q_h(x,y)$ by $\overline{xy}$, we are back to the situation of [8] as far as the construction of algebraic structure on X_* and X^* is concerned (note that $q_v\times q_v$ and $q_h\times q_h$ are also stable regular epis, for general categorical reasons, cf. e.g. [6, Proposition 5.11]). Thus X_* carries a group structure given by

$$\underline{xz}\cdot\underline{zw}:=\underline{xw},$$

and X^* carries the structure of groupoid over B, with

$$\partial_0(\overline{xy})=\pi(x),\qquad \partial_1(\overline{xy})=\pi(y)$$

and

$$\overline{yv}\circ\overline{xy}:=\overline{xv}.$$

The group X_* acts fibrewise on $X\to B$ from the right, and the groupoid X^* acts on $X\to B$ from the left, the actions being given by

$$x\cdot\underline{xz}=z$$

and

$$\overline{xy}\cdot x=y,$$

respectively.

Alternatively, since the right-hand square of (3.1) is a pullback, we have an isomorphism

$$X\times X_*\xrightarrow{\simeq}X\times_B X, \tag{3.3}$$

and composing with the projection to the second factor defines a map $X\times X_*\to X$ over B, which is the action. We may further observe that this action provides X with the structure of a fibre bundle for X with fibre X_*; the isomorphism required in

(1.5) is now just (3.3), and the proof of (1.6) is quite straightforward, but we shall not give it since it is a special case of a calculation in Section 4 below. When $X \to B$ is made a fibre bundle for X (with fibre X_*) in this way, it deserves the name *principal* fibre bundle (for X).

Similarly, since the right-hand square in (3.2) is a pullback, we have an isomorphism

$$X^* \times_B X \xrightarrow{\simeq} X \times X, \tag{3.4}$$

and composing it with the projection to the second factor defines the action of the groupoid X^* on X.

The two actions commute with each other, and both are principal homogeneous in a certain sense; at least for the X_*-action, this is in the standard sense that "to any two elements in the same fibre, there is a unique group element which takes the one element to the other" (or, "X is torsor over B"). For, making the phrase in quotation marks into a diagrammatic statement yields exactly the statement that (3.3) is an isomorphism.

For the X^*-action, the statement that it is principal homogeneous can be formulated in an analogous way, whose diagrammatic expression is the statement that (3.4) is an isomorphism. It can also be formulated by saying that the category $\hat{X}$ of elements of the action X^* on X is the codiscrete category $X \times X$ on X.

We may remark that the category of regular pregroupoids over B may be described in an alternative way:

Call a group action $\alpha : X \times G \to X$ *principal homogeneous* if the pair consisting of $p_0 : X \times G \to X$ and α has a coequalizer $X \to X/G$ with kernel pair (p_0, α). And call the action *regular* principal homogeneous if furthermore $X \to X/G$ and $X \to 1$ are descent maps.

For any principal homogeneous action $\alpha : X \times G \to X$, X can be made into a pregroupoid over X/G by postulating $\Lambda(x, y, xg, yg)$ for all $x, y \in X$ and $g \in G$. If furthermore the action is regular, the pregroupoid is regular, and has $X_* \cong G$ canonically, and compatibly with the action, as is seen by comparing the coequalizer $X \times X \times G \rightrightarrows X \times G \to G$ with the top row of (3.1).

On the other hand, if X is a regular pregroupoid over B, the isomorphism (3.3) shows that the X_* action is principal homogeneous and with $B = X/G$, so that $X \to X/G$ is a descent map, and so the action is also regular. This gives the equivalence of the category of regular pregroupoids over B with the category of regular principal homogeneous actions (X, G, α) with $X/G \cong B$.

4. Fibre bundles as actions

We shall prove that for a regular pregroupoid X, the category **Fib**(X) may be described in two other ways, namely as the category of left X_*-, respectively left

X^*-actions. The viewpoint of fibre bundles as groupoid actions is due to C. Ehresmann [2], see also [1].

Let $\alpha: X \times F \to E$ equip $\varepsilon: E \to B$ with the structure of a fibre bundle for X. We then have an inverse $\bar{\beta}$ for $\bar{\alpha} = \langle \mathrm{id}_X, \alpha \rangle$,

$$X \times F \underset{\bar{\beta}}{\overset{\bar{\alpha}}{\rightleftarrows}} X \times_B E,$$

and $\bar{\beta}$ may be written $\bar{\beta} = \langle \mathrm{id}_X, \beta \rangle$ for a unique $\beta: X \times_B E \to F$. The fact that $\bar{\alpha}$ and $\bar{\beta}$ are mutually inverse implies

$$\alpha(x, f) = e \quad \Leftrightarrow \quad \beta(x, e) = f \tag{4.1}$$

for $x \in X$, $f \in F$, $e \in E$ and $\pi(x) = \varepsilon(e)$. Also clearly

$$\alpha(\beta(x, e)) = e; \qquad \beta(\alpha(x, f)) = f. \tag{4.2}$$

Finally, condition (1.6) is equivalent to $\Lambda(x, y, z, u)$ implying either of

$$\beta(x, \alpha(z, f)) = \beta(y, \alpha(u, f)), \tag{4.3}$$

$$\alpha(y, \beta(x, e)) = \alpha(u, \beta(z, e)) \tag{4.4}$$

(where $\pi(x) = \varepsilon(e)$.

From these, we immediately deduce the well-definedness of the left action by the group X_* on F given by

$$\underline{xz} \cdot f = \beta(x, \alpha(z, f)), \tag{4.5}$$

as well as the well-definedness of the left action by the groupoid X^* on $E \to B$ given by

$$\overline{xy} \cdot e = \alpha(y, \beta(x, e)) \tag{4.6}$$

(in both cases with $\pi(x) = \varepsilon(e)$).

We can now analyze the kernel pairs of α and β: they are given by the actions of X_* and X^*, respectively. More precisely, we claim that the diagrams (4.7) and (4.8) below are exact (= kernel pair/coequalizer diagrams):

$$(X \times F) \times X_* \underset{t}{\overset{p_0}{\rightrightarrows}} X \times F \xrightarrow{\alpha} E \tag{4.7}$$

with $t(x, f, g) = (x \cdot g, g^{-1} \cdot f)$ for $g \in X_*$;

$$(X \times_B E) \times X^* \underset{t}{\overset{p_0}{\rightrightarrows}} X \times_B E \xrightarrow{\beta} F \tag{4.8}$$

with $t(x, e, h) = (h \cdot x, h \cdot e)$, for $h \in X^*$ with $\partial_0(h) = \pi(x) = \varepsilon(e)$.

To see that (4.7) is a kernel pair diagram, first observe that if $g = \underline{xz}$,

$$\alpha(x \cdot g, g^{-1} \cdot f) = \alpha(z, \underline{zx} \cdot f) = \alpha(z, \beta(z, \alpha(x, f))) = \alpha(x, f)$$

by (4.5) and (4.2), so (4.7) commutes. And if $\alpha(x, f_1) = \alpha(z, f_2)$, then $\pi(x) = \pi(z)$, and then by (4.1) and (4.5), $\underline{zx}\cdot f_1 = f_2$; therefore the element $(x, f_1, \underline{xz}) \in (X\times F)\times X_*$ goes by p_0 and t to (x, f_1) and (z, f_2), respectively.

The proof that (4.8) is a kernel pair diagram is similar. The fact that α and β are coequalizers of their kernel pairs follows because by $X\times F = X\times_B E$ they sit in pullback squares over $X\to B$ and $X\to 1$, respectively, and these maps were assumed stable regular epis.

Letting $\mathbf{Act}(X_*)$ denote the category of objects F with a left X_*-action, and similarly letting $\mathbf{Act}(X^*)$ denote the category of objects $E\to B$ with a left action by the groupoid $X^* \rightrightarrows B$, we thus have functors

$$\mathbf{Fib}(X)\to\mathbf{Act}(X_*);\qquad \mathbf{Fib}(X)\to\mathbf{Act}(X^*) \tag{4.9}$$

sending the fibre bundle $(E\to B, F, \alpha)$ into, respectively, F with the action (4.5), and $E\to B$ with the action (4.6).

Theorem 4.1. *The functors* (4.9) *are equivalences of categories.*

Proof. We construct inverses for them. Let $F\in\mathbf{Act}(X_*)$. Then we have a descent situation, defining α and $X\wedge F$,

$$\begin{tikzcd}
(X\times F)\times X_* \arrow[loop left, "\tau"] \arrow[r, "p_0"] \arrow[d, "r"'] & X\times F \arrow[r, "\alpha"] \arrow[d] & X\wedge F \arrow[d] \\
X\times_B X \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[r] & B
\end{tikzcd} \tag{4.10}$$

with $\tau(x,f,g) := (x\cdot g, g^{-1}\cdot f, g^{-1})$ and $r(x,f,g) := (x, x\cdot g)$. The fact that the left-hand square is a pullback follows because

$$(X\times F)\times X_* \cong (X\times X_*)\times F \cong (X\times_B X)\times F,$$

by (3.3), and under this identification, r corresponds just to the projection. The fact that $p_0, p_0\cdot\tau$ is an equivalence relation with τ as symmetry is immediate from the fact that it is derived out of the (diagonal) group action of X_* on $X\times F$.

We shall see that the $X\wedge F$ thus constructed is made into a fibre bundle for X with fibre F by means of α. First, because the right-hand square in (4.10) is a pullback (by the construction of $X\wedge F$ by descent), we immediately get that $\bar\alpha$ (cf. (1.5)) is invertible. Let $\beta : X\times_B (X\wedge F)\to F$ be derived out of $\bar\alpha^{-1}$ as above, so that (4.1) and (4.2) hold. Then we can prove (4.5), or equivalently (by (4.1))

$$\alpha(x, \underline{xz}\cdot f) = \alpha(z, f),$$

which however is clear, since α coequalizes the group action.

From validity of (4.5), however, immediately follows (4.3) and thus (1.6), so that

α indeed does make $X \wedge F$ into a fibre bundle with fibre F. Also validity of (4.5) tells us that the X_*-action on F derived from this fibre bundle structure agrees with the given one on F. This implies that the composite $\mathbf{Act}(X_*) \to \mathbf{Fib}(X) \to \mathbf{Act}(X_*)$ is the identity functor.

Also the composite $\mathbf{Fib}(X) \to \mathbf{Act}(X_*) \to \mathbf{Fib}(X)$ is isomorphic to the identity functor, for E and $X \wedge F$ both appear as coequalizer of the same parallel pair, namely (4.7), respectively top row of (4.10). This proves that the first functor in (4.9) is an equivalence.

The proof that the second functor is an equivalence is quite similar, so we only indicate the construction of the inverse. Let $(E \to B) \in \mathbf{Act}(X^*)$. Then we have a descent situation defining β and $X \vee E$,

$$\begin{tikzcd}
(X \times_B E) \times_B X^* \arrow[loop left, "\tau"] \arrow[r, "p_0"] \arrow[d, "r"'] & X \times_B E \arrow[r, "\beta"] \arrow[d] & X \vee E \arrow[d] \\
X \times X \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r] & 1
\end{tikzcd} \tag{4.11}$$

with $\tau(x, e, h) := (h \cdot x, h \cdot e, h^{-1})$ and $r(x, e, h) := (x, h \cdot x)$. First, because the right-hand square is a pullback (by the construction of $X \vee E$ by descent), we get an isomorphism $\bar{\alpha}$

$$X \times (X \vee E) \cong X \times_B E$$

which, when followed by the projection to E, provides the latter with a structure α of fibre bundle for X with fibre $X \vee E$.

We leave the rest of the details of proving $\mathbf{Fib}(X) \cong \mathbf{Act}(X^*)$ to the reader. The arguments are 'symmetric' to those carried out for $\mathbf{Fib}(X) \cong \mathbf{Act}(X_*)$. In fact, this symmetry suggests that one might generalize to the case where X is structured with *two* maps $X \to B_1$ and $X \to B_2$, with X_* and X^* becoming groupoids over B_1 and B_2, respectively.

For the convenience of possible future reference, we shall indicate also directly the composite equivalence $\mathbf{Act}(X_*) \cong \mathbf{Act}(X^*)$: to $F \in \mathbf{Act}(X_*)$, associate $X \wedge F \to B$ with left action by X^* given by

$$\overline{xy} \cdot \alpha(z, f) = \alpha(u, f), \tag{4.12}$$

where $\Lambda(x, y, z, u)$; and to $E \to B \in \mathbf{Act}(X^*)$, associate $X \vee E$ with left action by X_* given by

$$\underline{xz} \cdot \beta(u, e) = \beta(y, e), \tag{4.13}$$

where again $\Lambda(x, y, z, u)$. $\square$

5. Applications in locale theory

In the category **Loc** of locales (cf. e.g. [4]; 'spaces' in the terminology of [5]), open surjections are known to be descent maps. We consider in what follows *open* pregroupoids $\pi : X \to B$ in **Loc**, meaning that π and $X \to 1$ are open surjections. Open pregroupoids are therefore in particular regular in the sense of Sections 3 and 4, so the theory developed there applies.

Although we shall not need it here, we may remark that the category of open pregroupoids over B is equivalent to the category of principal homogeneous actions (X, G, α) with $X/G \cong B$ and with $G \to 1$ and $X \to 1$ open surjections. For, if $\alpha : X \times G \to X$ is an action, the map $\bar{\alpha} : X \times G \to X \times G$ given by $(x, g) \mapsto (xg, g)$ will be invertible, and satisfy $p_0 \cdot \bar{\alpha} = \alpha$, and since p_0 is an open surjection, so is α. But then, by [9, 1.3], say, the coequalizer $\pi : X \to X/G$ will also be an open surjection. The rest of the construction of the equivalence is then as in Section 3.

We shall develop only one issue concerning fibre bundles in locale theory, related to the notion of étale map (= local homeomorphism, in topological terms). Recall that 'F discrete' means '$F \to 1$ is étale', (cf. [5]).

Proposition 5.1. *Let $\varepsilon : E \to B$ be structured as a fibre bundle for X with fibre F. Then ε is étale iff F is discrete.*

Proof. The two squares in

$$\begin{tikzcd} F \arrow[d] & X \times F \cong X \times_B E \arrow[l] \arrow[r] \arrow[d] & E \arrow[d, "\varepsilon"] \\ 1 & X \arrow[l] \arrow[r, "\pi"'] & B \end{tikzcd}$$

are pullback diagrams, and the two bottom maps are open surjections. But pulling back along an open surjection preserves and reflects the notion of étaleness. □

Proposition 5.2. *The following conditions on the pregroupoid X are equivalent:*

(i) *$\pi : X \to B$ is étale.*

(ii) *X_* is discrete.*

(iii) *$(\partial_0, \partial_1) : X^* \to B \times B$ is étale.*

Proof. The equivalence (i) ⇔ (ii) follows from the previous proposition since X 'is' a fibre bundle for X with fibre X_* as observed in Section 3. Next, from the right-hand pullback square in (3.2) which defines X^*, it follows by a pure diagrammatic argument that the right-hand square in

$$\begin{tikzcd} X \arrow[d, "\pi"'] & X\times X \arrow[l, "p_1"'] \arrow[r] \arrow[d, "X\times\pi"] & X^* \arrow[d, "(\partial_0,\partial_1)"] \\ B & X\times B \arrow[l, "p_1"] \arrow[r, "\pi\times X"'] & B\times B \end{tikzcd} \tag{5.1}$$

is also a pullback, and the two bottom maps are open surjections. But pulling back along an open surjection preserves and reflects the notion of étaleness, whence π is étale iff (∂_0, ∂_1) is. □

Example. If a path-connected space B has a universal covering space $\pi : X \to B$, then X carries a canonical pregroupoid structure over B (and is étale); namely, if $x, y, z \in X$ have $\pi(x) = \pi(z) = a$ and $\pi(y) = b$, we define u with $\pi(u) = b$ as follows: choose a path γ from x to y. Then $\pi \circ \gamma$ is a path from a to b, and since $\pi(z) = a$, there is a unique lifting δ of $\pi \circ \gamma$ starting in z. The other end point of δ is then u with $\Lambda(x, y, z, u)$. It does not depend on the choice of path γ since another path γ' from x to y is homotopic to γ (X being simply connected), and then the resulting homotopy from $\pi \circ \gamma$ to $\pi \circ \gamma'$ lifts to a homotopy from δ to δ'.

In this example, X^* is canonically isomorphic to the fundamental groupoid of B, whereas X_* is isomorphic to 'the' fundamental group (which is only defined up to non-canonical isomorphism; of course, 'the' universal covering space has the same defect).

Pregroupoids satisfying the conditions of Proposition 5.2 deserve the name *étale pregroupoids*. Note that if X is an étale pregroupoid, the groupoid X^* will usually not be étale over B in the sense of $\partial_0 : X^* \to B$ being étale. The proper word for property (iii) of the groupoid X^* is that it is *locally codiscrete*. For recall that a groupoid Φ over B is called codiscrete if $(\partial_0, \partial_1) : \Phi \to B \times B$ is bijective. Since an étale map is 'locally bijective' (in a sense which can be made precise), we arrive at the word '*locally* codiscrete' if (∂_0, ∂_1) is étale.

Note that for any regular pregroupoid X, the map $(\partial_0, \partial_1) : X^* \to B \times B$ is stably regular epic; this can be seen by contemplating the right-hand pullback square in (5.1) and using standard properties of stable regular epis (e.g. [6, Proposition 5.12]).

We finish by presenting some observations on classifying toposes for localic groupoids (cf. [9]). Recall that if Φ is a groupoid over B in **Loc**, with $\partial_0, \partial_1 : \Phi \to B$ and $B \to 1$ open surjections, then the classifying topos $\mathbb{B}(\Phi)$ may be described as the category of étale locales over B equipped with a left Φ-action (thus it is a full subcategory of $[\Phi, \mathbf{Loc}]$).

Theorem 5.3. *Let $X \to B$ be a pregroupoid in* **Loc**, *with $X \to B$ and $X \to 1$ open surjections. Then $\mathbb{B}(X^*) \simeq \mathbb{B}(X_*)$.*

Proof. The equivalence $\mathbf{Act}(X_*) \simeq \mathbf{Act}(X^*)$ of Theorem 4.1 takes discrete X_*-actions to X^*-actions which are étale over B, by Proposition 5.1. □

Corollary 5.4. *Consider a groupoid Φ over B in* **Loc**. *Assume that Φ is locally co-discrete (i.e. $(\partial_0, \partial_1) : \Phi \to B \times B$ is an étale surjection) and assume that B has a global point $* : 1 \to B$. Then $\mathbb{B}(\Phi)$ is equivalent to $\mathbf{Set}^G$ for a (discrete) group G. (For applications to base toposes other than* **Set**, *one should explicitly require B to be an open locale,* [5].)

Proof. We construct a pregroupoid X over B, namely by the pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\pi"'] & \Phi \arrow[d, "(\partial_0, \partial_1)"] \\ B \cong 1 \times B \arrow[r, "* \times B"'] & B \times B \end{tikzcd}$$

cf. the motivating remarks at the beginning of Section 1. It is almost immediate that $X^* \cong \Phi$ as a groupoid over B. Also π is an étale surjection since (∂_0, ∂_1) is. By Theorem 5.3, we have the middle equivalence in the string

$$\mathbb{B}(\Phi) \simeq \mathbb{B}(X^*) \simeq \mathbb{B}(X_*) \simeq \mathbf{Set}^{X_*};$$

and the last equivalence follows because X_* is discrete, by Proposition 5.2. □

Remark 5.5. All these results relativize to base toposes other than **Set**. In particular, even if B does not have a global point, we may adjoin one generically by taking sh(B) as base topos. Then X_* becomes a sheaf Φ_* of groups on B (morally the sheaf of all vertex groups), and by the stability theorem of [9], Theorem 6.7, $\mathbb{B}(\Phi_B)$ is the pullback topos of $\mathbb{B}(\Phi)$ along sh(B) $\to$ **Set**. We deduce that

$$\mathrm{sh}(B) \times_{\mathbf{Set}} \mathbb{B}(\mathbf{Set}, \Phi) \cong \mathrm{sh}(B)^{\Phi_*}.$$

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