

Elementwise semantics in categories with pull-backs

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Introduction

In several places, including [3], II,1-3, is expounded how to work with (generalized) *elements* of the objects in a category \mathcal{E} with finite limits (“Kripke-Joyal semantics”).

In the present note, we describe and extend this “semantics” when the category \mathcal{E} just is assumed to have pull-backs, thus no terminal object 1 is assumed, nor binary products $A \times B$.¹

The basic notions are

- (generalized) element a of an object A of \mathcal{E}
- (generalized) subobject² U of an object A of \mathcal{E}
- (generalized) partial map s from an object A of \mathcal{E} to another object E of \mathcal{E} .

Not included here is the elementwise semantics “object of maps” from A to B , (which will be an object B^A) and related constructs, possible in Cartesian closed categories or related kinds of categories. Elementwise semantics for certain such constructs exist, see e.g. [3] II.4. We do consider such “higher order” objects here, but only with traditional diagram-style

¹The reason for not insisting in the finite products (in particular, a terminal object 1) is the possible application in e.g. the partial toposes, as studied in [1]. Note that there is a qualitative difference between the categorical properties \mathcal{E} itself and the properties of its “slices” \mathcal{E}/X : for \mathcal{E}/X always has a terminal object; this is not assumed for \mathcal{E} .

²Unlike most standard references on Kripke-Joyal semantics like [10] VI.6, we do not assume that \mathcal{E} is a topos. In a topos, *subobjects* of A may be encoded as *elements* of Ω^A ; and the semantics for subobjects and partial maps may be reduced to semantics for elements. The same applies to the (generalized) subobjects and (generalized) partial maps.

tools, see the section on jet *bundles* in 3.4 below. We intend in a later note also to give such things an elementwise semantics.

The maps in \mathcal{E} will be called “actual maps”, or even just “maps” or “arrows”.

I was led to the desire for an explicit semantics for these things, because it is in this way that some synthetic reasoning, e.g. for the geometry of jet bundles, as in [5] 22.7 or in [7] Theorem 11.1, may be fully justified and communicated. We deal with a theory of (section-) jets below; but I believe that the methodology we develop for it in the present note has a more general scope.

Some derived relations

- \in : when is a (generalized) element a of A *contained in* or a *member of* a (generalized) subobject U of A ;
- \subseteq : the partial order of (generalized) subobjects of A
- (generalized) partial map
- *support* of a (generalized) partial map from A to E , as a (generalized) subobject of A
- *value* $s(a)$ of a (generalized) partial map on a (generalized) element a which is a member of the support of s .
- *counterimage* of a (generalized) subobject of A along a map $A' \rightarrow A$.

And finally, we have some results (“principles”), e.g. the extensionality principles:

- a (generalized) subobject is determined by the (generalized) elements contained in it
- a (generalized) partial map s is determined by the values of s on the (generalized) elements contained in its support.

The (generalized) elements, (generalized) subobjects, (generalized) partial maps, are all given at a certain “stage” $X \in \mathcal{E}$, which is subject to variation (“change of state”) as in Kripke’s work, and explained in category theoretic terms in [3], say; a stage X is any object of \mathcal{E} , and the “change of stage” from X to Y takes place along a map $\alpha : Y \rightarrow X$ in \mathcal{E} . (This change of stage can be formulated by saying that the notions etc. described are contravariant, set-valued constructions, i.e. that they take place in the functor category $Set^{\mathcal{E}^{op}}$. More generally, the change of stage can be encoded as passing from the slice category \mathcal{E}/X to \mathcal{E}/Y by “the” pull-back

functor $\alpha^* : \mathcal{E}/X \rightarrow \mathcal{E}/Y$. We avoid here using this useful technique, because it involves the choice of definite pull-backs, - but see e.g. [3] for an exposition of this slice category technique. The uses that we shall make of the notation α^* are *exact*, i.e. do not depend on any choice, or the coherence issues that arise from resulting comparison isomorphisms.

If \mathcal{E} happens to have finite (chosen) products, a (generalized) subobject U of A at stage X can be encoded as an actual subobject of $A \times X$.

Not included in the present note is the consideration of generalized *map* from A to B ; a generalized map at stage X is an actual map $A \times X \rightarrow B$ (assuming binary products in \mathcal{E}).

The decoration of an entity, like a (generalized) subobject U of A at stage X , will be $U \subseteq_X A$; the subscript indicates that we are talking about a *generalized* subobject defined at stage X , not about an actual subobject of $U \subseteq A$, in the standard sense of the category \mathcal{E} . If \mathcal{E} happens to have a terminal object 1 , $U \subseteq_1 A$ will be equivalent to $U \subseteq A$ in the standard sense.

In the sequel, the phrase “generalized” is sometimes omitted.

1 Basic notions

1.1 Elements: $a \in_X A$

If A is an object of \mathcal{E} , and $a : X \rightarrow A$ is an arrow in \mathcal{E} , we say that $a \in_X A$, or that a is a *generalized* element of A *defined at stage* X , or *parametrized* by X . If $f : A \rightarrow B$ is an actual map in \mathcal{E} and $a \in_X A$, we have $f \circ a \in_X B$, and it may also be denoted $f(a)$. This usage will be extended to (generalized) partial maps from A to B . Note that we compose maps from the right to the left.

If $\alpha : Y \rightarrow X$, we define $\alpha^*(a) \in_Y A$ to be $a \circ \alpha$, or “ a considered at the later stage Y ”. The associative law for \circ implies that $\alpha^*(f(a)) = f(\alpha^*(a))$; both equal $f \circ a \circ \alpha$.

Consider a pull-back square obtained from two given maps f and p with

common codomain, as in

$$\begin{array}{ccc}
 M & \xrightarrow{d} & B \\
 \downarrow c & \lrcorner & \downarrow p \\
 A & \xrightarrow{f} & C
 \end{array} \tag{1}$$

Then a (generalized) element at stage X of M is uniquely given by a pair of (generalized) elements: $a \in_X A$ and $b \in_X B$ with the property that $f(a) = p(b) \in_X C$. This is just a reformulation of the universal property of a pull-back; and it does not mention any other data than the two given maps f and p . The unique (generalized) element of M thus given, one denotes $\langle a, b \rangle \in_X M$.

1.2 Subobjects and generalized subobjects

Recall that a *subobject* U of an object A in a category \mathcal{E} is *represented* by a monic map $i : N \rightarrow A$; and $i' : N' \rightarrow A$ represents the same subobject if there is some (necessarily unique) isomorphism $\mu : N \rightarrow N'$ with $i' \circ \mu = i$, cf. [9] V.7; thus U is an equivalence class of monics with codomain A . We want to be pedantic about the distinction between a subobject and a representing monic. We write $U \subseteq A$ when U is a subobject of $A \in \mathcal{E}$. But note that a subobject U of A is not an object of \mathcal{E} . We shall use similar pedantry also for some other of the following notions.

A subobject U of A in this sense, we shall also call an *actual* subobject of A , to distinguish it from a *generalized* subobject, *defined at stage* $X \in \mathcal{E}$, as in Definition 1.2 below.

Given objects A and B in \mathcal{E} .

Definition 1.1 A relation \mathfrak{M} from A to B is an equivalence class of jointly monic spans

$$A \xleftarrow{c} M \xrightarrow{d} B. \tag{2}$$

The equivalence relation is the evident one: (c, d) is equivalent to (c', d') if there is an isomorphism $\mu : M \rightarrow M'$ (necessarily unique) with $c' \circ \mu = c$ and $d' \circ \mu = d$.

$$\begin{array}{ccccc}
& & & d' & \\
& & & \longrightarrow & Y \\
& & \lrcorner & & \downarrow \alpha \\
& c' & \downarrow & & \\
A & \longleftarrow & M & \xrightarrow{d} & X
\end{array}
\tag{5}$$

Since (c, d) is jointly monic, then so is (c', d') , by an easy argument. The passage from (c, d) to (c', d') preserves the equivalence relation considered on the set of jointly monic spans with given ends. So we obtain a (generalized) subobject $\alpha^*(U)$, represented by (c', d') , of A at stage Y .

Let $U \subseteq_X A$. If $Z \xrightarrow{\beta} Y \xrightarrow{\alpha} X$ is a composable pair, then $\beta^*(\alpha^*(U)) = (\alpha \circ \beta)^*(U)$ exactly, (because of the equivalence relation defining the notion of (generalized) subject; recall that a subobject of A is not an object of \mathcal{E}).

There is a similar construction, but performed in the A -end rather than in the X -end: Given given $f : A' \rightarrow A$, and given $U \subseteq_X A$, then pulling back c along f provides a well defined subobject of A' at stage X , which is sensibly denoted $f^{-1}(U) \subseteq_X A'$, the *counter image* of U along f . The processes α^* and f^{-1} commute.

Proposition 1.3 *Let $f : A' \rightarrow A$, and let $U' \subseteq_X A'$ and $U \subseteq_X A$ be (generalized) subobjects represented, respectively, by the upper and the lower spans in*

$$\begin{array}{ccccc}
A' & \longleftarrow & M' & & \\
\downarrow f & & \vdots \mu & \searrow & \\
A & \longleftarrow & M & \longrightarrow & X
\end{array}
\tag{6}$$

Then $U' \subseteq_X f^{-1}(U)$ iff there exists a (necessarily unique) μ making the square and the triangle commute.

If further $f' : A'' \rightarrow A'$, and $U'' \subseteq_X A''$, it is trivial to conclude a “transitivity law” (continuing the notation from the Proposition): If $U' \subseteq_X f^{-1}(U)$ and $U'' \subseteq_X f'^{-1}(U')$, then $U'' \subseteq_X f'^{-1}(f^{-1}(U)) = (f \circ f')^{-1}(U)$.

1.3 Partial maps and partial sections $A \dashrightarrow_U E$

A (generalized) *partial map* at stage X from A to E with *support* $U \subseteq_X A$ is an equivalence class of diagrams of the form (ignoring the p)

$$\begin{array}{ccc}
 & E & \\
 & \swarrow t & \\
 p \downarrow & & \\
 A & \xleftarrow{c} M \xrightarrow{d} & X
 \end{array} \tag{7}$$

with (c, d) jointly monic and U being the (generalized) subobject of A represented by (c, d) . The diagram represents a (generalized) *partial section* of p if $p \circ t = c$.

The equivalence relation mentioned is the evident one, extending the one appearing after Definition 1.2; similarly for the contravariant functoriality in X . In particular, if s is a (generalized) partial map at stage X , as represented by (7), and if $\alpha : Y \rightarrow X$ is a map, then we get a (generalized) partial map $\alpha^*(s)$ at stage Y .

If $q : E \rightarrow F$ is a given actual map, one may evidently post-compose a (generalized) partial map $s : A \dashrightarrow_U E$ (with support $U \subseteq_X A$) with q to obtain a (generalized) partial map $q \circ s : A \dashrightarrow_U F$, with the same support.

If $f : A' \rightarrow A$ is an actual map, we may precompose a (generalized) partial map $A \dashrightarrow_U E$ with f to obtain a (generalized) partial map from A' to E , with support $f^{-1}(U) \subseteq_X A'$.

It is straightforward to see that the pre- and post-composition operations on given (generalized) partial maps commute. In a full-fledged theory of (generalized) partial maps in \mathcal{E} , this will be a special case of the associative law for the category of (generalized) partial maps. We do not intend to present such a full-fledged theory here. It seems to require that \mathcal{E} besides pull-backs has some further finite limits.

Consider spans (c', d') and (c, d) representing (generalized) subobjects $U' \subseteq_X A'$ and $U \subseteq_X A$, respectively, both at stage X , as displayed in the

right hand part of the diagram

$$\begin{array}{ccccc}
 E' & \xrightarrow{p'} & A' & \xleftarrow{c'} & M' \\
 \downarrow e & \lrcorner & \downarrow f & & \downarrow \mu \\
 E & \xrightarrow{p} & A & \xleftarrow{c} & M \xrightarrow{d} X,
 \end{array}$$

and suppose that further a pull-back square is given, as in the left hand part of the diagram. Assume that $U' \subseteq_X f^{-1}(U)$, witnessed by the displayed map μ . Let $t : M \rightarrow E$ represent a (generalized) partial section of p with support U , i.e. we assume that $p \circ t = c$. Because of the commutativities assumed, the universal property of the pull-back square implies that there exists a unique $t' : M' \rightarrow E'$ with $p' \circ t' = c'$ and $e \circ t' = t \circ \mu$, which then represents a (generalized) partial section of p' with support U' . Summarizing, (omitting the phrase “generalized” in a couple of places)

Proposition 1.4 *Let $f : A' \rightarrow A$, and let $U' \subseteq_X A'$ and $U \subseteq_X A$ have $U' \subseteq_X f^{-1}(U)$. Then a partial section t of $p : E \rightarrow A$ with support U restricts canonically to a partial section t' of $p' : E' \rightarrow A'$ with support U' , where p' is obtained by pulling p back along f .*

Namely, $t' : M' \rightarrow E'$ is characterized by $p' \circ t' = c'$ and $e \circ t' = t \circ \mu$.

For $A'' \rightarrow A' \rightarrow A$, there is a rather obvious, and easily provable, strict associativity assertion which supplements this Proposition.

1.4 Elements of a (generalized) subobject

If (c, d) is a span representing a (generalized) subobject U of A at stage X , and a is a (generalized) element of A defined at a later stage $\alpha : Y \rightarrow X$, as depicted with full arrows in

$$\begin{array}{ccc}
 Y & & \\
 \downarrow a & \searrow \alpha & \\
 A & \xleftarrow{c} M \xrightarrow{d} X, &
 \end{array}
 \tag{8}$$

then we say that $a \in_\alpha U$ if there is a map $a_0 : Y \rightarrow M$ such that $c \circ a_0 = a$ and $d \circ a_0 = \alpha$. Such a map is unique because (c, d) is jointly monic. In this case, we say that a_0 is the *witness* or the *proof* (relative to the given representative (c, d) for the (generalized) subobject U) that a belongs to U at the later stage. The assertion that $a \in_\alpha U$ is clearly independent of the choice of the span $A \leftarrow M \rightarrow X$ representing U .

An important special case is when no change of stage takes place, i.e. when $Y = X$ and $\alpha : Y \rightarrow X$ is id_X . In this case, we say $a \in_X U$. But note: $a \in_X U$ is not a special case of $a \in_X A$: U is, unlike A , not an object of \mathcal{E} , and $a \in_X U$ is only meaningful for an a which is already $\in_X A$.

The somewhat heavy \in_α -notation can be dispensed with, because it is easily proved that for U , α and a as above,

$$a \in_\alpha U \text{ iff } a \in_Y \alpha^*(U) \tag{9}$$

A particular case of such $\alpha : Y \rightarrow X$ is $d : M \rightarrow X$. We have $c \in_d U$, as witnessed by id_M , equivalently, we have $c \in_M d^*(U)$. Verbally: if U is represented by (c, d) , then $c \in_d U$.

It is easy to see that the relation \in thus defined is stable under change of stage: if $a \in_\alpha U$, then $\beta^*(a) \in_{\alpha \circ \beta} \beta^*(U)$.

1.5 Value of a partial map on an element in its support

Let $U \subseteq_X A$ be represented by the span (c, d) as in (4). Consider a (generalized) partial map $s : A \dashrightarrow_U E$ with support U , thus it is represented by data as in (7) (ignoring the p). Consider also a (generalized) element, at the same stage X , of A , so $a : X \rightarrow A$. Then if $a \in_X U$, witnessed by a_0 as in (8), then we write $s(a)$ for $s \circ a_0 \in_X E$. More generally, if $\alpha : Y \rightarrow X$ and $a \in_\alpha U$, we shall also write $s(a)$ for the element $\in_Y E$ whose full notation is $\alpha^*(s)(\alpha^*(a))$. Experience shows that the “change of stage” symbols α or α^* often can be omitted from notation, improving readability.

In the category of sets, this is the fundamental relation between *composition* \circ in \mathcal{E} on the one side, and *evaluation* of a function s on an element a on the other: The process leading from “evaluation” to “composition \circ ” is the one through which we learned to compose functions - a relation which we want to exploit in a more general category \mathcal{E} with pull-backs.

2 Principles and constructions

2.1 Extensionality principle for subobjects

Proposition 2.1 *Let $U \subseteq_X A$ and $U' \subseteq_X A$ be (generalized) subobjects of A . Then $U \subseteq_X U'$ iff for every $\alpha : Y \rightarrow X$ and every $a \in_Y A$, $a \in_Y \alpha^*(U)$ implies $a \in_Y \alpha^*(U')$.*

Proof. Let the (generalized) subobject U in question be represented by (c, d) as in (4), and similarly let U' be represented by $(c', d') : A \leftarrow M' \rightarrow X$. Assume that $U \subseteq_X U'$, witnessed by $\mu : M \rightarrow M'$, with the relevant commutativities. Let $\alpha : Y \rightarrow X$. Then if $a \in_Y A$ satisfies $a \in_Y \alpha^*(U)$ witnessed by $a_0 : Y \rightarrow M$, it satisfies $a \in_Y \alpha^*(U')$, witnessed by $\mu \circ a_0$. For the converse conclusion, recall (9). We may take $\alpha : Y \rightarrow X$ to be $d : M \rightarrow X$ and $a \in_M A$ to be c . Then $a \in_M U$ (witnessed by id_M). By assumption, therefore, $a \in_M U'$, and this is witnessed by a map $\mu : M \rightarrow M'$, which then proves the desired inequality between the (generalized) subobjects U and U' .

This is a typical Kripke semantics argument, for interpreting the universal quantifier “for all $a \dots$ ”, using “for all later stages Y and for all elements at this later stage \dots ”. Note that the “Extensionality principle for subobjects” in Proposition II.3.1 in [3] only deals with actual subobjects, not (generalized) subobjects, as here.

2.2 Yoneda principle for partial maps

This says roughly that a (generalized) partial map can be constructed and recognized by what it does to (generalized) elements of its support. The clue is that both construction and recognition are supposed to be available for *all* stages, and are preserved by change of stage.

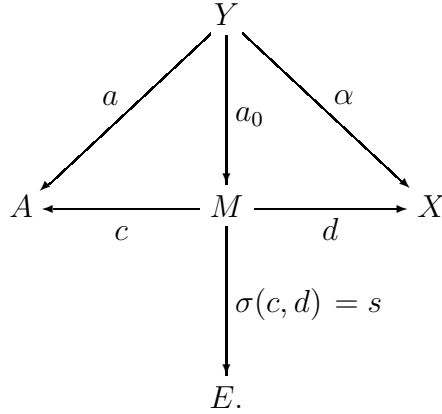
Proposition 2.2 *Let $U \subseteq_X A$ be a (generalized) subobject of A at stage X , represented by the span (c, d) , as in (4). Given a law σ which to each $\alpha : Y \rightarrow X$ and $a \in_Y \alpha^*(U)$ associates $\sigma(a, \alpha) \in_Y E$, in a way which is stable under change of stage, i.e. $\sigma(a, \alpha) \circ \beta = \sigma(a \circ \beta, \alpha \circ \beta)$ for any $\beta : Z \rightarrow Y$. Then there is a unique (generalized) partial map $s : A \dashrightarrow_U E$ such that for $a \in_Y \alpha^*(U)$, we have $\alpha^*(s)(a) = \sigma(a, \alpha)$.*

Proof. Take first $\alpha : Y \rightarrow X$ to be $d : M \rightarrow X$ and take $a \in_M \alpha^*(U)$ to be c . Recall from Subsection 1.4 that $c \in_M d^*(U)$. So $\sigma(c, d) \in_M E$. Then $s := \sigma(c, d)$ is a map $M \rightarrow E$, which represents the desired (generalized) partial map $A \dashrightarrow_U E$. For, consider an arbitrary $\alpha : Y \rightarrow X$, and let $a \in_Y \alpha^*(U)$, witnessed by $a_0 : Y \rightarrow M$. Then we have

$$s(a) = s \circ a_0 = \sigma(c, d) \circ a_0 = \sigma(c \circ a_0, d \circ a_0) = \sigma(a, \alpha)$$

using stability under change of state. Uniqueness of s is clear.

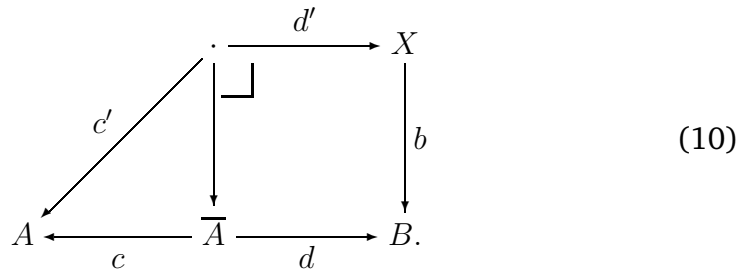
For completeness, here is a diagram for some of the proof:



3 Jets associated with a relation

3.1 Relations and their “monads”

We consider the construction in (5), just with a change of notation



Then the span (c', d') is jointly monic, since (c, d) was assumed so, and hence represents a (generalized) subobject of A at stage X , which we denote $\mathfrak{M}(b)$; it does not depend on the specific choice of span or pull-back

in the construction. It is the “*monad* around b ” in the terminology and notation from [5] 2.1 and [3] I.6 (where it in turn is borrowed from Leibniz, and from there, by non-standard analysis). In the category of sets, if $b \in B$, the subset $\mathfrak{M}(b) \subseteq A$ is $\{a \in A \mid (a, b) \in \mathfrak{M}\}$.

Consider a map $\alpha : Y \rightarrow X$. It is an immediate consequence of the fact of “pulling back $b \circ \alpha$ in stages” that

$$\alpha^*(\mathfrak{M}(b)) = \mathfrak{M}(\alpha^*(b)) (= \mathfrak{M}(b \circ \alpha)). \quad (11)$$

In particular, for composable α and β ,

$$\beta^*(\alpha^*(\mathfrak{M}(b))) = (\alpha \circ \beta)^*(\mathfrak{M}(b))$$

holds exactly.

3.2 Jets and section jets

Let E be any object in \mathcal{E} , and consider a relation \mathfrak{M} from A to B .

Definition 3.1 *Let $b \in_X B$. An E -valued jet j at b (relative to \mathfrak{M}) is a (generalized) partial map $A \dashrightarrow_{\mathfrak{M}(b)} E$.*

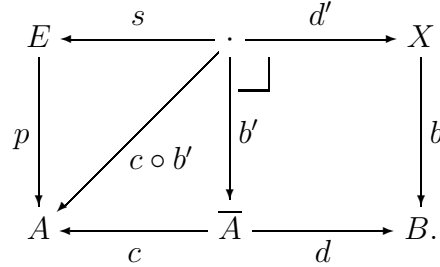
So the support of j is $\mathfrak{M}(b) \subseteq_X A$; the information of X is built into $\mathfrak{M}(b)$. Note that no specific span, like (2), is mentioned in the definition; in this sense, it is a “coordinate free” definition.

Let further $p : E \rightarrow A$ be given.

Definition 3.2 *Let j be an E -valued jet at b . We say that j is a section jet of p at b (relative to \mathfrak{M}) if $p \circ j : A \dashrightarrow_{\mathfrak{M}(b)} A$ is the partial identity map of A with support $\mathfrak{M}(b)$. The set of such section jets, we denote $J_{\mathfrak{M}}(b, p)$ (or $J(b, p)$ when the relation \mathfrak{M} is understood from the context).*

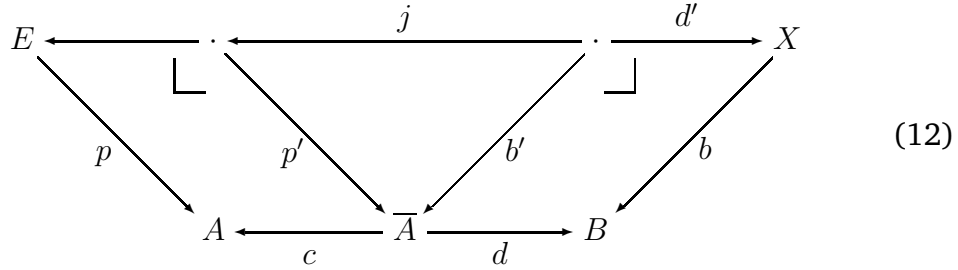
Note that a jet j , or a section jet, is not just given by a (generalized) partial map, but by a pair, consisting of a (generalized) partial map and a (generalized) element b . We shall mainly be concerned with section jets in the following, and it is worthwhile to unravel the definition in terms of representing data:

Consider a diagram, where the span (c, d) represents a relation \mathfrak{M} from A to B , and where the right hand square is a pull-back:



The span $(c \circ b', d') : A \leftarrow \cdot \rightarrow X$ represents the (generalized) subobject $\mathfrak{M}(b) \subseteq_X A$, and the map s represents an E -valued jet at b ; it represents a section jet if the upper triangle commutes, equivalently the left hand square, commutes.

An equivalent way of unravelling the definition diagrammatically is the more symmetric



where j denotes the map into the (un-named) pull back, given by $j := \langle s, b' \rangle$.

There is an obvious equivalence relation on the set of such representatives; it involves a commutativity of a square with the two comparison isomorphisms between the two pull-backs involved.

So for fixed \mathfrak{M} , represented by (c, d) , say, and for $p : E \rightarrow A$, we have the set $J_{\mathfrak{M}}(b, p)$ of section jets j (relative to \mathfrak{M}) of p at b .

The set $J_{\mathfrak{M}}(b, p)$ depends contravariantly on $b \in \mathcal{E}/B$ and covariantly on $p \in \mathcal{E}/A$. For the contravariant dependence on b , one sees from (12) that elements (section jets) in $J_{\mathfrak{M}}(b, p)$ may be represented by factorizations j of the arrow b' across p' . Given a map $\alpha : Y \rightarrow X$, defining a morphism $b \circ \alpha \rightarrow b$ in \mathcal{E}/B , one gets the section jet $\in J_{\mathfrak{M}}(b \circ \alpha)$ represented by the factorization $j \circ \alpha'$ of $b' \circ \alpha'$ over p' , where α' denotes some pull-back of

α along d' . A diagrammatic rendering (which omits the pull-back square defining p') is the following: both the displayed rectangles are pull-backs, and we use standard, incomplete notations, e.g. to denote the object in a pull-back of $b : X \rightarrow B$ along d by $d^*(X)$:

$$\begin{array}{ccc}
 d^*(Y) & \longrightarrow & Y \\
 \downarrow \alpha' & & \downarrow \alpha \\
 d^*(X) & \longrightarrow & X \\
 \downarrow j & & \downarrow b \\
 c^*(E) & & \\
 \downarrow p'=c^*(p) & & \\
 \overline{A} & \xrightarrow{d} & B.
 \end{array} \tag{13}$$

3.3 Morphisms between relations

We shall now consider a morphism between two relations (between two different pairs of objects). Therefore, we make a change in the choice of letters; formerly we considered a relation from A to B ; we now shall write A_0 rather than B . Then we have the letter B free, and can consider a relation \mathfrak{M}_A from A to A_0 , and another one \mathfrak{M}_B from B to B_0 . We are not implying that A and A_0 are otherwise related, nor do we so for B and B_0 , although ultimately, in Section 4 on “classical” jet theory, where we will have $A = A_0$, $B = B_0$, $f = f_0$ (referring to (14) below).

Given a pair of maps $f : A \rightarrow B$ and $f_0 : A_0 \rightarrow B_0$, and given relations \mathfrak{M}_A from A to A_0 and \mathfrak{M}_B from B to B_0 , then there is an obvious notion of when (f, f_0) preserves the given relations: if the relations are represented by monic spans, as in the diagram below, this means that there is a (necessarily unique) map \overline{f} making the two squares commute:

$$\begin{array}{ccccc}
 \mathfrak{M}_A : & A & \longleftarrow \overline{A} & \longrightarrow & A_0 \\
 & \downarrow f & & & \downarrow f_0 \\
 & & \downarrow \overline{f} & & \\
 \mathfrak{M}_B : & B & \longleftarrow \overline{B} & \longrightarrow & B_0.
 \end{array} \tag{14}$$

In case binary products are available and chosen, this is just saying that $f \times f_0 : A \times A_0 \rightarrow B \times B_0$ takes the subobject $\mathfrak{M}_A \subseteq A \times A_0$ into the subobject $\mathfrak{M}_B \subseteq B \times B_0$, in the sense that $\mathfrak{M}_A \subseteq (f \times f_0)^{-1}(\mathfrak{M}_B)$.

Using the Extensionality Principle in Subsection 2.1, it is straightforward to see that this preservation property for (f, f_0) can equally well be formulated in either of the following ways: For any $a_0 \in_X A_0$, we have $\mathfrak{M}_A(a_0) \subseteq f^{-1}(\mathfrak{M}_B(f_0(a_0)))$; or: for any $a_0 \in_X A_0$ and any $a \in_Y \alpha^*(\mathfrak{M}_A(a_0))$, we have $f(a) \in_\alpha \mathfrak{M}_B(f_0(a_0))$.

Consider now a morphism of relations $\mathfrak{M}_A \rightarrow \mathfrak{M}_B$, as represented by the diagram (14). Let J_A and J_B denote the respective section jet constructions. Let $p : E \rightarrow B$ be given, and suppose that we have a pull-back square h like

$$\begin{array}{ccc}
 E' & \xrightarrow{\quad} & E \\
 \downarrow p' & \lrcorner & \downarrow p \\
 & h & \\
 A & \xrightarrow{\quad f \quad} & B
 \end{array} \tag{15}$$

and let $a_0 \in_X A_0$. We shall construct a map of sets

$$\phi(a_0, h) : J_B(f_0(a_0), p) \rightarrow J_A(a_0, p') \tag{16}$$

natural in $a_0 \in \mathcal{E}/A_0$. An element in $J_B(f_0(a_0), p)$ is a section jet j of $p : B \dashrightarrow_{\mathfrak{M}_B(f_0(a_0))} E$. We shall produce an element in $J(a_0, p')$, meaning a section jet j'

$$A \dashrightarrow_{\mathfrak{M}_A(a_0)} E'.$$

We construct this (generalized) partial map by using the Yoneda principle in Proposition 2.2: for any $\alpha : Y \rightarrow X$ and $a \in_Y A$ with $a \in_\alpha \mathfrak{M}_A(a_0)$, we produce an element $\sigma_h(a, \alpha) \in_Y E'$ which maps to a by p' . Because (f, f_0) is a morphism of relations from \mathfrak{M}_A to \mathfrak{M}_B , we have $f(a) \in_\alpha \mathfrak{M}_B(f_0(a_0))$ as we observed, so $f(a)$ belongs to the support of j , so $j(f(a)) \in_Y E$ is defined as an element $\in_Y E$, mapping to $f(a) \in_Y B$ by p . By the commutativities involved, the pair $\langle a, j(f(a)) \rangle$ defines an element $\in_Y E'$. The law $\sigma_h(a, \alpha)$, given by $j \mapsto \langle a, j(f(a)) \rangle$, is stable under change of stage α , and so by the Yoneda principle, it defines the desired (generalized) partial map, $A \dashrightarrow_{\mathfrak{M}_A(a_0)} E'$. So an explicit description of the law σ_h is, for $\alpha : Y \rightarrow X$ and $a \in_Y \alpha^*(\mathfrak{M}_A(a_0))$:

$$\sigma_h(a, \alpha) := \langle a, j(f(a)) \rangle.$$

This explicit description only mentions j and a , so it follows that if we have another pull-back diagram of f and p , say

$$\begin{array}{ccc} E'' & \longrightarrow & E \\ \downarrow p'' & \lrcorner & \downarrow p \\ A & \xrightarrow{f} & B, \end{array}$$

then the resulting comparison isomorphism $\tau : E' \rightarrow E''$ satisfies $\tau(\sigma_h(a, \alpha)) = \sigma_k(a, \alpha)$. Therefore we not only have a map $\phi(a_0, h)$ as in (16), but a compatible family of maps, one for each choice of pull-back $f^*(p)$, justifying the notation

$$\phi(a_0, p) : J_B(f_0(a_0), p) \rightarrow J_A(a_0, f^*(p)). \quad (17)$$

Consider a composable pair of morphisms between relations \mathfrak{M}_A , \mathfrak{M}_B , and \mathfrak{M}_C , represented by the four right hand squares in the following diagram. The two further squares h and k are assumed to be pull-backs; hence their concatenation $k \circ h$ is likewise a pull-back.

$$\begin{array}{ccccccc} E'' & \xrightarrow{p''} & A & \longleftarrow \overline{A} & \longrightarrow & A_0 & \\ \downarrow & \lrcorner & \downarrow f & \downarrow & & \downarrow f_0 & \\ & h & B & \longleftarrow \overline{B} & \longrightarrow & B_0 & \\ E' & \xrightarrow{p'} & B & \longleftarrow \overline{B} & \longrightarrow & B_0 & \\ \downarrow & \lrcorner & \downarrow g & \downarrow & & \downarrow g_0 & \\ & k & C & \longleftarrow \overline{C} & \longrightarrow & C_0 & \\ E & \xrightarrow{p} & C & \longleftarrow \overline{C} & \longrightarrow & C_0 & \end{array} \quad (18)$$

In these circumstances, we have

Proposition 3.3 *For any $a_0 \in_X A_0$, the composite*

$$J_C(g_0 f_0(a_0), p) \xrightarrow{\phi(f_0(a_0), k)} J_B(f_0(a_0), p') \xrightarrow{\phi(a_0, h)} J_A(a_0, p'')$$

equals $\phi(a_0, h \circ h')$.

Proof. Let $j : C \dashrightarrow_U E$ be an element in the common domain of the two maps to be compared, with $U = \mathfrak{M}_C(g_0 f_0(a_0))$. The value of either of the two maps on this j are (generalized) partial maps $A \dashrightarrow_{\mathfrak{M}(a_0)} E''$. To see that they are equal, we use the ‘‘recognition’’ part of the Yoneda principle; it suffices to consider an arbitrary $\alpha : Y \rightarrow X$ and a (generalized) element $a \in_Y \alpha^*(\mathfrak{M}_A(a_0))$, and see that the partial map in either case takes the value $\langle a, j(g(f(a))) \rangle$ on such a . The condition for the partial map j being defined on the argument $g(f(a))$ follows because the relations in question are preserved by (f, f_0) and (g, g_0) , by assumption.

3.4 Jet bundles \mathbb{J}

Note that if we have chosen pull-back functors $c^* : \mathcal{E}/A \rightarrow \mathcal{E}/\bar{A}$ and $d^* : \mathcal{E}/A_0 \rightarrow \mathcal{E}/\bar{A}$, (where (c, d) is a span $A \leftarrow \bar{A} \rightarrow A_0$ representing \mathfrak{M}) we can, using (12), write (for $a_0 \in_X A_0$),

$$J_A(a_0, p) \cong \text{hom}_{\mathcal{E}/\bar{A}}(d^*(a_0), c^*(p)) \quad (19)$$

(where J_A is short for $J_{\mathfrak{M}}$, with \mathfrak{M} is understood from the context). If further d^* admits a right adjoint d_* , we therefore also have

$$J_A(a_0, p) \cong \text{hom}_{\mathcal{E}/A_0}(a_0, d_* c^*(p)). \quad (20)$$

Consider for a fixed $p \in \mathcal{E}/A$ the contravariant set valued functor $J_A(-, p)$ on \mathcal{E}/A_0 , and assume that it is representable (which is the case in (20)). Thus there is a representing object $\mathbb{J}(p) = \mathbb{J}_{\mathfrak{M}}(p) \in \mathcal{E}/A_0$ and a bijection

$$J_A(a_0, p) \cong \text{hom}_{\mathcal{E}/A_0}(a_0, \mathbb{J}(p)).$$

More precisely, we have a generic section $\text{jet } \epsilon$ of p , defined at stage $\mathbb{J}(p) \in \mathcal{E}/A_0$, such that for any $a_0 : X \rightarrow A_0$, and any $j \in J_A(a_0, p)$, we have $\bar{j}^*(\epsilon) = j \in J_A(a_0, p)$ for a unique $\bar{j} \in \text{hom}_{\mathcal{E}/A_0}(a_0, \mathbb{J}_{\mathfrak{M}}(p))$; \bar{j} deserves the name the *classifying map* for j .

A rough sketch (specializing (13) of the items here (with incomplete but conventional notation, as in (13)), is found in the following diagram;

both the displayed rectangles are a pull-backs

$$\begin{array}{ccc}
 d^*(a_0) & \longrightarrow & X \\
 \downarrow & & \downarrow \bar{j} \\
 d^*\mathbb{J}(p) & \longrightarrow & \mathbb{J}(p) \\
 \downarrow \epsilon & & \downarrow \\
 c^*(E) & & \\
 \downarrow c^*(p) & & \downarrow \\
 \bar{A} & \xrightarrow{d} & A_0.
 \end{array}
 \tag{21}$$

The generic section jet is denoted ϵ , because it, as an arrow in \mathcal{E}/\bar{A} , is the back adjunction for $d^* \dashv d_*$ (if we have the functors d^* and d_* available). And \bar{j} is the classifying map for j : $\bar{j}^*(\epsilon) = j$, see (13), with different notation.

The object $\mathbb{J}(p) \rightarrow A_0$ in \mathcal{E}/A_0 deserves the name: the section-jet-bundle of $p : E \rightarrow A$ (relative to \mathfrak{M}). It exists if \mathcal{E} is a locally Cartesian closed category. The existence of $\mathbb{J}(p)$ may depend on \mathfrak{M} , as well as on p , and may, if it exists, be described, without any chosen pull-back functors or right adjoints for them, in terms of what [13] calls *distributivity pull-backs*, here: a distributivity pull-back around $(c^*(p), d)$. The lower rectangle in (21) is a distributivity pull-back. In Section 5, we shall give the following verbal rendering of the property that makes the lower rectangle above a distributivity pull-back, namely: ϵ is terminal in the category of comorphisms over d with domain $c^*(p)$.

If $g : A_0 \rightarrow B_0$ is a map, and $q : F \rightarrow B$, then the functor $a_0 \mapsto J(g(a_0), q)$, for $a_0 \in \mathcal{E}/A_0$ is representable as well: the pull-back $g^*\mathbb{J}(q)$ will do the job. This fact is a version of the Beck-Chevalley condition.

If every $p \in \mathcal{E}/A$ is provided with a (chosen) generic section jet ϵ of it, it follows from standard properties of bifunctors that $\mathbb{J}(p) \in \mathcal{E}/A_0$ depends functorially on $p \in \mathcal{E}/A$; in other words, we have (for the given relation \mathfrak{M} from A to B) a functor

$$\mathbb{J} : \mathcal{E}/A \rightarrow \mathcal{E}/A_0 :$$

A relation \mathfrak{M} from A to A_0 gives rise to a functor which to a bundle p over A associates its \mathfrak{M} -jet-bundle $\mathbb{J}(p)$, which is a bundle over A_0 .

Given a morphism of relations $\mathfrak{M} \rightarrow \mathfrak{M}'$, in analogy with (16), (denoting the corresponding jet bundle functors \mathbb{J} and \mathbb{J}' , respectively). Then the naturality of the map described in (16) implies that it is mediated by a map

$$\Phi(h) : f^*(\mathbb{J}(p)) \rightarrow \mathbb{J}'(p'), \quad (22)$$

or denoting p' by $f^*(p)$, since p' comes from a pull-back of p along f , by a map $f^*(\mathbb{J}(p)) \rightarrow \mathbb{J}'(f^*(p))$ (recall the assumption that \mathfrak{M} maps into \mathfrak{M}'). In Section 4 below, we assume that the \mathfrak{M} s are uniformly given, in a certain sense; we also assume that $A = B$, $A' = B'$, and $f = f_0$, in which case the map constructed is a map $\Phi(f) : f^*(\mathbb{J}(p)) \rightarrow \mathbb{J}(f^*(p))$.

4 Classical jets

Our interest in relations presented by jointly monic spans $A \leftarrow M \rightarrow A_0$ comes from algebraic geometry, where $A = A_0$ is an affine scheme, and M is the “ r th neighborhood of the diagonal”, $A_{(r)} \subseteq A \times A$ (or “prolongation space $A_{(r)}$ ” in [8] for the C^∞ case); it is a reflexive symmetric relation. The role of this for jet-theory was made explicit in [8], Chapter 1, see also [11] IV.2; some of it is expounded synthetically in [5]. All morphisms of affine schemes preserve the r th neighbourhood relation.

So we consider in the following that every object $A \in \mathcal{E}$ comes together with an (endo-) relation $\mathfrak{M} = \mathfrak{M}_A$, represented by a (jointly monic) span

$$A \xleftarrow{c} \overline{A} \xrightarrow{d} A, \quad (23)$$

preserved by all maps in \mathcal{E} ; in particular, for $f : A \rightarrow B$, we have the diagram like (14), but now with $A = A_0$, $B = B_0$, and with $f = f_0$:

$$\begin{array}{ccccc} A & \xleftarrow{c} & \overline{A} & \xrightarrow{d} & A \\ \downarrow f & & \downarrow \overline{f} & & \downarrow f \\ B & \xleftarrow{c'} & \overline{B} & \xrightarrow{d'} & B; \end{array} \quad (24)$$

the map \overline{f} is the witness that f preserves the endo-relations involved.

The jet functors for the given endo-relation \mathfrak{M}_A on an object A are denoted J_A , and (if they exist) similarly for the jet bundles \mathbb{J}_A .

Because of the assumption that any map $f : A \rightarrow B$ preserves the given endo-relations (“takes \overline{A} into \overline{B} ”), we have in particular the ϕ -construction in (16) available: for any map $f : A \rightarrow B$ and for any pull-back square h :

$$\begin{array}{ccc} E' & \xrightarrow{f_1} & E \\ \downarrow p' & \lrcorner & \downarrow p \\ A & \xrightarrow{f} & B \end{array} \quad h$$

we have, for $a_0 \in_X A$, a set mapping

$$\phi(a_0, f) : J_B(f(a_0), p) \rightarrow J_A(a_0, p').$$

Consider two commutative squares on top of each other

$$\begin{array}{ccc} F' & \xrightarrow{f_2} & F \\ \downarrow r' & & \downarrow r \\ E' & \xrightarrow{f_1} & E \\ \downarrow p' & \lrcorner & \downarrow p \\ A & \xrightarrow{f} & B \end{array} \quad \begin{array}{l} q' \\ \\ \\ q \end{array} \quad (25)$$

and assume that the lower square h is a pull-back, and also that the total square, which we denote k , is a pull-back,

$$\begin{array}{ccc} F' & \xrightarrow{f_2} & F \\ q' \downarrow & k & \downarrow q \\ A & \xrightarrow{f} & B \end{array}$$

where $q' = p' \circ r'$ and $q = p \circ r$. (Then the upper square is also a pull-back, but we do not need to name it). Recall by (16) we have, for $a_0 \in_X A$ a set map $\phi(a_0, h) : J_B(f(a_0), p) \rightarrow J_A(a_0, p')$, and similarly we have $\phi(a_0, k) : J_B(f(a_0), q) \rightarrow J_A(a_0, q')$.

Proposition 4.1 *The following diagram of sets commutes*

$$\begin{array}{ccc}
 J_B(f(a_0), q) & \xrightarrow{\phi(a_0, k)} & J_A(a_0, q') \\
 \downarrow J_B(f(a_0), r) & & \downarrow J_A(a_0, r') \\
 J_B(f(a_0), p) & \xrightarrow{\phi(a_0, h)} & J_A(a_0, p').
 \end{array}$$

Proof. Since it is a diagram of sets, it suffices to see that the value of the two composites on an element $j \in J_B(f(a_0), q)$ give the same element in $J_A(a_0, p')$. Here, j is a section jet $j : B \dashrightarrow_{\mathfrak{M}(f(a_0))} F$ of q . So let $a \in_{\alpha} \mathfrak{M}(a_0)$. Then both composites produce the element $\langle a, r(j(a)) \rangle \in_Y E'$ where $\alpha : Y \rightarrow X$. This hinges on a general fact, valid for generalized elements in a concatenation of two pull-back squares: referring to notation as in (25), it is the fact that $r'(\langle a, c \rangle) = \langle a, r(c) \rangle$ whenever $a \in_Y A$, $c \in_Y F$ satisfy $f(a) = q(c)$.

4.1 Reflexivity and symmetry

For a relation \mathfrak{M} from A to itself, it makes sense to ask whether it is reflexive; this is simply that it contains the diagonal relation Δ . It is easily seen to be equivalent to: for every generalized element $a_0 \in_X A$, we have $a_0 \in_X \mathfrak{M}(a_0)$, - justifying the terminology that $\mathfrak{M}(a_0)$ is the monad *around* a_0 .

Consider a reflexive relation \mathfrak{M} , and a section jet j at $a_0 \in_X A$ of $p : E \rightarrow A$, so the support of j is $\mathfrak{M}(a_0)$. Then since $a_0 \in_X \mathfrak{M}(a_0)$, by reflexivity, the value $j(a_0) \in_X E$ makes sense. For a section jet $j : A \dashrightarrow_{\mathfrak{M}(a_0)} E$ of $p : E \rightarrow A$, we therefore have (cf. Subsection 1.5) a particular (generalized) element in E , namely $j(a_0) \in_X E$, with $p(j(a_0)) = a_0$.

A possible symmetry of a relation \mathfrak{M} from A to itself can also be expressed in terms of (generalized) elements: $a \in_X \mathfrak{M}(b)$ iff $b \in_X \mathfrak{M}(a)$.

For \mathfrak{M} reflexive and symmetric, we may think of \mathfrak{M} as encoding a neighbour relation: $a\mathfrak{M}a_0$ meaning “ a is r th order neighbour of a_0 ”; and $\mathfrak{M}(a_0)$ would then mean the r th order (infinitesimal) neighbourhood of a_0 . This is how it is used in [2] and in [5].

5 Formulation in terms of fibered categories \mathcal{E}^2 and $(\mathcal{E}^2)^*$

Recall that if \mathcal{E} is a category with pull-backs, one has the codomain fibration $\mathcal{E}^2 \rightarrow \mathcal{E}$, associating to an object $a : X \rightarrow A$ its codomain A , and the arrows in \mathcal{E}^2 are the commutative squares in \mathcal{E} . The fibre over $A \in \mathcal{E}$ thus is the category \mathcal{E}/A . Note that \mathcal{E} itself is not of the form \mathcal{E}/A , unless we have a terminal object in \mathcal{E} .

The Cartesian arrows in \mathcal{E}^2 are the pull-back squares in \mathcal{E} . Choosing pull-backs amounts to a cleavage of the codomain fibration. Such cleavage amounts to giving pull-back *functors* $f^* : \mathcal{E}/A \rightarrow \mathcal{E}/A'$ for any $f : A' \rightarrow A$. But the present Section is “cleavage free”.

Associated to the codomain fibration $\mathcal{E}^2 \rightarrow \mathcal{E}$, we have its fibrewise dual fibration $(\mathcal{E}^2)^* \rightarrow \mathcal{E}$, as we shall recall below.

Some of the notions and constructions may as well be formulated for an arbitrary fibered category $\mathcal{X} \rightarrow \mathcal{E}$. This in particular applies to the description of the “fibrewise dual” fibration $\mathcal{X}^* \rightarrow \mathcal{E}$:

5.1 The fibrewise dual of a fibration $\mathcal{X} \rightarrow \mathcal{E}$

Let $\pi : \mathcal{X} \rightarrow \mathcal{E}$ be any fibration. We have in mind the codomain fibration $\mathcal{E}^2 \rightarrow \mathcal{E}$. In this case, $\mathcal{X}_A = \mathcal{E}/A$, and therefore, we find it convenient to denote the objects in \mathcal{X}_A by lower case letters like p , since $p : E \rightarrow A$ is the name of a typical object in \mathcal{E}/A . Recall from [6] or [7] that an arrow in \mathcal{X}^* over the arrow $f : A' \rightarrow A$ in \mathcal{E} , from $p' \in \mathcal{X}_{A'}$ to $p \in \mathcal{X}_A$, is represented by a “vh-span” (v, h) where v is vertical over A' and h Cartesian over f

$$\begin{array}{ccc} \cdot & \xrightarrow{h} & p \\ \downarrow v & & \\ p' & & \end{array}$$

Two such spans, say (v, h) and (v', h') are *equivalent* if there is a (necessarily unique) vertical isomorphism i with $v \circ i = v'$ and $h \circ i = h'$. The equivalence classes, we call *comorphisms* from p' to p ; they are the arrows of \mathcal{X}^* . The

composition in \mathcal{X}^* is a standard composition of spans. The functor $\mathcal{X}^* \rightarrow \mathcal{E}$ associates to the comorphism represented by a vh span (v, h) , (as above) the arrow $\pi(h)$ in \mathcal{E} . This functor $\mathcal{X}^* \rightarrow \mathcal{E}$ is a fibration. A Cartesian arrow in \mathcal{X}^* has a unique representative of the form (v, h) with v an identity arrow in \mathcal{X} . So there is an isomorphism between the category $C(\mathcal{X})$ of Cartesian arrows in \mathcal{X} and the category $C(\mathcal{X}^*)$ of Cartesian arrows in \mathcal{X}^* . Also, $(\mathcal{X}^*)_A$ is may be identified with $(\mathcal{X}_A)^{op}$;

Assume that we for each $A \in \mathcal{E}$ have an endofunctor $\mathbb{J}_A : \mathcal{X}_A \rightarrow \mathcal{X}_A$, hence also $\mathbb{J}_A^{op} : \mathcal{X}_A^{op} \rightarrow \mathcal{X}_A^{op}$. Assume also that we have a functor $\Phi : C(\mathcal{X}) \rightarrow \mathcal{X}^*$, which for any $A \in \mathcal{E}$ agrees with \mathbb{J}_A on objects, and such that Φ agrees with \mathbb{J}_A^{op} on the arrows that are simultaneously Cartesian and vertical. Consider a vh square in \mathcal{X} , i.e. a commutative square in \mathcal{X} with the h_i s Cartesian over $f : A' \rightarrow A$, and w and v vertical,

$$\begin{array}{ccc} q_2 & \xrightarrow{h_2} & p_2 \\ w \downarrow & & \downarrow v \\ q_1 & \xrightarrow{h_1} & p_1 \end{array}$$

over $f : A' \rightarrow A$ in \mathcal{E} .

There is a compatibility condition between the \mathbb{J} s and the Φ , namely commutativity of the following square in \mathcal{X}^* ,

$$\begin{array}{ccc} \mathbb{J}_{A'}(q_2) & \xrightarrow{\Phi(h_2)} & \mathbb{J}_A(p_2) \\ \mathbb{J}_{A'}^{op}(w) \uparrow & & \uparrow \mathbb{J}_A^{op}(v) \\ \mathbb{J}_{A'}(q_1) & \xrightarrow{\Phi(h)} & \mathbb{J}_A(p_1). \end{array}$$

Proposition 5.1 *Assume the compatibility for vh squares. Then there is a canonical functor $\mathbb{J} : \mathcal{X}^* \rightarrow \mathcal{X}^*$ over \mathcal{E} agreeing with Φ on Cartesian arrows, and (except for variance) with the \mathbb{J}_A s on vertical arrows.*

This is an easy consequence of the way composition of comorphisms are defined (“standard span composition”). (The special case for $\mathcal{X} = \mathcal{E}^2$ was given in [7], Theorem 11.1, in synthetic terms.)

The following is an explanation in fibrational terms of the notion of distributivity pull-back:

Given a map $d : A \rightarrow B$ in \mathcal{E} , and given $q \in \mathcal{X}_A$, we have the following category: its objects are comorphisms over f with domain q and codomain any object t in \mathcal{X}_B ; the arrows from the comorphism represented by the vh span $(e \circ v, h_1)$ to the one represented by (e, h_2) are represented by diagrams of the following form, where the square is a vh-square over d , and where e is vertical

$$\begin{array}{ccc}
 \cdot & \xrightarrow{h_1} & t_1 \\
 \downarrow v & & \downarrow \\
 \cdot & \xrightarrow{h_2} & t_2 \\
 \downarrow e & & \\
 q & &
 \end{array}$$

In these terms, if $\mathcal{X} \rightarrow \mathcal{E}$ is the codomain fibration, a terminal object in the category of comorphisms thus described is a distributivity pull-back in Weber's sense around q, d .

5.2 Jets as a functor $(\mathcal{E}^2)^* \rightarrow (\mathcal{E}^2)^*$

The present Subsection gives the ultimate aim of the present note, namely to establish the existence of jet-bundle formation as a global functor $(\mathcal{E}^2)^* \rightarrow (\mathcal{E}^2)^*$ (for the "classical" case of uniformly given endo-relations in \mathcal{E} , as in Section 4), agreeing with the jet bundle formation \mathbb{J}_A on the individual fibres \mathcal{E}/A (cf. also [7] Theorem 11.1). There are cleavage choices involved in the description of the \mathbb{J}_A s, and in the choice of right adjoints, like the d_* s used. But the text given provides a choice-free description of the various set-valued functors which the \mathbb{J}_A s represent, where no choice-generated coherence questions arise.

So consider the case where $\mathcal{X} = \mathcal{E}^2$, and where each object A of \mathcal{E} comes with a relation $A \leftarrow \overline{A} \rightarrow A$, as in Section 4. Therefore, we have for each

$A \in \mathcal{E}$ a functor

$$J_A : (\mathcal{E}/A)^{op} \times \mathcal{E}/A \rightarrow Sets,$$

with $J_A(b, p)$ as described in Definition 3.2. We observe that Proposition 4.1 can be formulated in abstract fibrational terms, applied to the codomain fibration $\mathcal{E}^2 \rightarrow \mathcal{E}$: for, the concatenation in (25) can be seen as commutative square in \mathcal{E}^2

$$\begin{array}{ccc} q' & \xrightarrow{k} & p \\ r' \downarrow & & \downarrow r \\ p' & \xrightarrow{h} & p \end{array}$$

with the two horizontally displayed arrows being Cartesian arrows (namely pull-back squares in \mathcal{E}) over f , and the two vertically displayed arrows r' and r being vertical over A and B , respectively (the equality $p \circ r = q$ is the commutativity that qualifies r as a vertical arrow $q \rightarrow p$ over B , and similarly for r'). So it is a “vh square” over f . Conversely, such a vh square over f is given by a concatenation (25) of pull-backs in \mathcal{E} . And in this form, the result in the Proposition can be interpreted as the compatibility condition needed to establish existence of a functor $(\mathcal{E}^2)^* \rightarrow (\mathcal{E}^2)^*$, agreeing with the \mathbb{J}_A s (except for variance) on vertical arrows, and with the Φ on Cartesian arrows. Note that the value of $\Phi(h)$ as in (22) may be seen as a comorphism in \mathcal{E}^2 over f from $\mathbb{J}_A(p')$ to $\mathbb{J}_B(p)$ (which in turn refers to a pull-back diagram (Cartesian arrow) displayed in (15)). The fact that Φ preserves composition of Cartesian arrows follows from Proposition 3.3. Note that the values of Φ are not Cartesian arrows in general.

5.3 Jet functors in fibered categories with internal products

We consider a fibered category $\pi : \mathcal{X} \rightarrow \mathcal{E}$ with internal products. We assume a cleavage given: for each arrow $d : M \rightarrow B$ in \mathcal{E} , we therefore have a functor $d^* : \mathcal{X}_B \rightarrow \mathcal{X}_M$; and we furthermore assume that these functors admit right adjoints (internal products), which likewise are assumed chosen, and denoted d_* . (At this point, we are not assuming Beck-Chevalley

conditions.) Consider a span (not assumed jointly monic)

$$A \xleftarrow{c} M \xrightarrow{d} B.$$

It gives rise to a functor $\mathbb{J} : \mathcal{X}_A \rightarrow \mathcal{X}_B$, namely the composite

$$\mathcal{X}_A \xrightarrow{c^*} \mathcal{X}_M \xrightarrow{d_*} \mathcal{X}_B.$$

We may also consider a similar span (c', d') , as displayed as the upper line in the following commutative diagram

$$\begin{array}{ccccc} A' & \xleftarrow{c'} & M' & \xrightarrow{d'} & B' \\ \downarrow f & & \downarrow \bar{f} & \xrightarrow{h} & \downarrow g \\ A & \xleftarrow{c} & M & \xrightarrow{d} & B \end{array} \quad (26)$$

and we similarly let \mathbb{J}' denote $d'_* \circ c'^* : \mathcal{X}_{A'} \rightarrow \mathcal{X}_{B'}$. Then there is a canonical natural transformation $g^* \circ \mathbb{J} \Rightarrow \mathbb{J}' \circ f^* : \mathcal{X}_A \rightarrow \mathcal{X}_{B'}$; it is the 2-cell obtained by the pasting

$$\begin{array}{ccccc} \mathcal{X}_{A'} & \xrightarrow{c'^*} & \mathcal{X}_{M'} & \xrightarrow{d'_*} & \mathcal{X}_{B'} \\ \uparrow f^* & \cong & \uparrow \bar{f}^* & \uparrow & \uparrow g^* \\ \mathcal{X}_A & \xrightarrow{c^*} & \mathcal{X}_M & \xrightarrow{d_*} & \mathcal{X}_B \end{array}$$

where the left hand 2-cell comes from the commutativity of the left hand square in (26), and the right hand 2-cell is obtained as the mate, under $d^* \dashv d_*$, $d'^* \dashv d'_*$, of the natural isomorphism $\bar{f}^* \circ d^* \Rightarrow d'^* \circ g^*$, which in turn comes from the commutativity of the right hand square in (26). (If this square h is a pull-back, the 2-cell will be an isomorphism, by the Beck-Chevalley condition.)

This gives a construction, in terms of the calculus of polynomial functors, of the 2-cells $\Phi(h)$. There is also in these terms a compatibility with the composite of two morphisms of spans, as in (18); however, then even more canonical isomorphisms, and therefore coherence questions, present

themselves. This is due to the choices involved in a cleavage, and of the choice of internal products. The approach to jet bundles which we have chosen in the present note was motivated by the desire to get rid of such choices, by making constructions more “coordinate free”.

A functor of the form $d_* \circ c^*$ is a special case of a polynomial functor in a locally Cartesian closed category \mathcal{E} . The possibility of a “polynomial functor” approach to jet bundles was observed in [5], Remark 7.3.1.

This locally Cartesian closed category approach to jet bundles does not insist that the spans considered (c, d) are jointly monic; so they include the idea of non-holonomous jets, as studied “synthetically” in [4]. I conjecture that the construction of $(\mathcal{E}^2)^* \rightarrow (\mathcal{E}^2)^*$ also works for this non-holonomous case. However, I have not been able in this generality to circumvent the necessity for using cleavages, i.e. the choice of pull-back functors like c^* and their right adjoints, and the coherence questions arising.

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Most diagrams were made with Paul Taylor's package.