Extensive quantities and monads[1]

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The basis of Functional Analysis is the ability to form function spaces. The categorical expression for this is: Cartesian Closed Categories. To say that $\mathscr{E}$ is Cartesian Closed means that for X and Y "spaces" (objects in $\mathscr{E}$), there is a function "space" in $\mathscr{E}$, denoted Y^X "consisting of" maps (in the category $\mathscr{E}$) from X to Y. More precisely, there is a bijection ("exponential transposition" or "lambda-conversion") between maps in $\mathscr{E}$

$$Z \to Y^X$$

and maps, likewise in $\mathscr{E}$,

$$Z \times X \to Y$$

and this bijection is to be mediated by an "evaluation" map $ev : Y^X \times X \to Y$, or equivalently $X \times Y^X \to X$.

If further X and Y have some algebraic kind of structure, say vector space structure, one may form a subspace $L(X,Y) \subseteq Y^X$ consisting of the those maps $X \to Y$ which are furthermore homomorphisms ("linear"); but the non-linear category is the basic one.

Now the word "space" could mean many other things than "topological space", one has e.g. a category of *bornological spaces*, or of *diffeological* spaces; these categories are already Cartesian Closed, whereas if "space" means "manifold", one needs to extend the category to get function spaces.

I shall not settle on any specific of these categories of "spaces", but treat them in a uniform way, by a category theoretic/axiomatic exposition. But for motivation, let me mention two results by Frölicher and Kriegl, [FK] Theorem 5.1.1.

They construct several interesting cartesian closed categories of spaces, let me mention $\underline{l}^\infty$, a full subcategory of the category of bornological spaces, and Lip^∞, a full subcategory of the category of diffeological spaces. A convenient vector space has canonical structure of both kinds. So there are forgetful functors ("CVS" means "convenient vector space", and also the category of such, with bounded linear maps)

$$CVS \to \underline{l}^\infty \qquad \text{and} \qquad CVS \to Lip^\infty.$$

Both of these forgetful functors have left adjoints, meaning that any l^∞ space X embeds in a universal way into a CVS denoted $\lambda(X)$, (in this lecture: $T(X)$) and similarly for Lip^∞ spaces.

Composing a pair of adjoint functors category $\mathscr{E} \leftrightarrow \mathscr{E}$ gives rise to an endofunctor T on $\mathscr{E}$ (a monad, in fact). Let me for concreteness describe the endofunctor coming from the first mentioned adjoint situation: it is the functor T, which to a bornological space X

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associates the space $T(X)$ of those functions $f : X \to \mathbb{R}$ which have the property that the support of f is countable and bounded (w.r.to the given bornology on X), and such that

$$\sum_{x \in X} |f(x)] < \infty.$$

For $X \in \underline{l}^\infty$, the set $T(X)$ of such functions has a natural bornology, and is in fact an $\underline{l}^\infty$ space. It clearly has vector space structure as well; with its a natural bornology, it is even a CVS. There is a map $\eta_X : X \to T(X)$, sending $x \in X$ to δ_x (value 1 on x, value 0 else). [FK] prove that this map is universal for bornological maps of X into CVSs.

They also construct a universal map from any Lip^∞ space X into a CVS $T(X)$; this $T(X)$ is constructed by a double dulization procedure, which when X is a smooth manifold is vector space $T(X)$ of Schwartz distributions of compact support on X, likewise a convenient vecor space.

In either of the two cases: just because of the universal property of $\eta_{T(X)} : T(X) \to T(T(X))$, there is a (linear) map $\mu_X : T(T(X)) \to T(X)$, namely the unique linear CVS map making the triangle

$$\begin{array}{ccc} T(T(X)) & \xrightarrow{\mu_X} & T(X) \\ \uparrow{\scriptstyle \eta_{T(X)}} & \nearrow{\scriptstyle id} & \\ T(X) & & \end{array} \tag{1}$$

commute. The maps η_X and μ_X are natural in $X \in \mathscr{E}$, so we have natural transformations

$$\eta : I \Rightarrow T \quad \text{and} \quad \mu : T \circ T \Rightarrow T \tag{2}$$

This is the situation T, η, μ, (plus one more data, a "strength", see below) which I shall describe in general category theoretic terms, in particular, the relationship between T and a suitable double-dualization construct. Such T provides, under certain assumptions, a "universal" link between non-linear and linear functional analysis. The endofunctor $T : \mathscr{E} \to \mathscr{E}$, together with the transformations η and μ, constitute a *monad*, meaning that there is an associative law for μ and two unitary laws, "η as a two-sided unit for μ. There is a notion of T-algebra (in the sense of Eilenberg and Moore) for such monad, namely a $B \in \mathscr{E}$ together with an "action" $\beta : T(B) \to B$, satisfying an associative and unitary law. For the case we are interested in, it is better to use the phrase that such β makes B into a *T-linear* space. The map $\mu_X : T(T(X)) \to T(X)$ makes $T(X)$ into such a T-linear, called the *free* T-algebra on X. There is an evident notion of T-homomorphism between T-linear spaces: essentially a map that preserves the action by T; see e.g. Borceux's "Handbook in Cat. Algebra" for these standard monad theoretic notions.

Henceforth, we consider a Cartesian Closed Category $\mathscr{E}$, and a monad $T = (T, \eta, \mu)$ on it. There is one piece more of structure, also present in the quoted Theorem of [FK], namely a *strength* or *$\mathscr{E}$-enrichment* [EK]: not only does T, being a functor, give a map

$\hom_{\mathscr{E}}(X,Y) \to \hom_{\mathscr{E}}(T(X),T(Y))$, but it gives a map between the hom-objects (exponetial objects) in $\mathscr{E}$,

$$Y^X \to T(Y)^{T(X)}, \tag{3}$$

called the *strength* or $\mathscr{E}$-enrichment of T. (In the $\underline{l}^\infty$ case, say, the strength follows from the assertion, likewise in loc. cit., that $\hom_{\mathscr{E}}(X,Y) \to \hom_{\mathscr{E}}(T(X),T(Y))$ is a *bornological* map.)

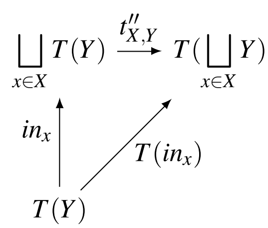
The strength of the functor T is assumed to be compatible with η and μ, in a rather straightforward sense (so (T,η,μ) is a "strong monad"); but for a moment, we consider just T, not η or μ. The strength (3) of an endofunctor $T : \mathscr{E} \to \mathscr{E}$ can equivalently, cf. [K70]...[K71b], be encoded in two other forms, the "tensorial" form

$$t''_{X,Y} : X \times T(Y) \to T(X \times Y),$$

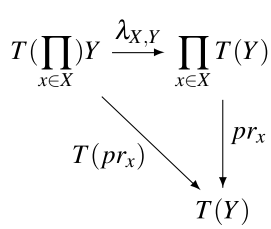
(or its twin sister $t'_{X,Y} : T(X) \times Y \to T(X \times Y)$) natural in X and Y, and the "cotensorial" form

$$\lambda_{X,Y} : T(Y^X) \to T(Y)^X,$$

likewise natural in X and Y. When $\mathscr{E}$ is the category of sets, either of the three manifestations of the strength are automatic; for the basic form (3), this is just because Y^X equals $\hom_{\mathscr{E}}(X,Y)$; for the two other forms, let us be explicit. For the tensorial strength, we rethink $X \times Y$ as a coproduct of X copies of Y, $X \times Y = \bigsqcup_{x \in X} Y$. Let in_x be the inclusion of the xth summand in an X-fold coproduct. Then t'' is the unique map making the diagram



commute, for all $x \in X$. (The twin sister t' comes about similarly by rethinking $X \times Y$ as $\bigsqcup_{y \in Y} X$). The cotensorial strength is "dual"; rethink Y^X as a product of X copies of Y, $Y^X = \prod_{x \in X} Y$:



where pr_x is the projection to the xth factor.

In the category of sets, one knows how to define algebraic structure on a product like $\prod_X B = B^X$ "coordinatewise", given algebraic structure on B. For the case of T-linear structure in the sense of Eilenberg-Moore, say $\beta : T(B) \to B$, the combinator λ implies that this can be generalized to B^X in $\mathscr{E}$ with X a *space*, $X \in \mathscr{E}$, namely B^X is endowed with the structure

$$T(B^X) \xrightarrow{\lambda_{X,B}} T(B)^X \xrightarrow{\beta^X} B^X$$

(recall that $(-)^X$ for fixed X is a covariant functor $\mathscr{E} \to \mathscr{E}$).

This construction gives us a way to ask when a map $f : X \times C \to B$ is a "T-linear in the second variable", provided $C = (C, \gamma)$ and $B = (B, \beta)$ are T-algebras; this is taken to mean that the exponential transpose of f, which is a map $\hat{f} : C \to B^X$, is a T-homomorphism, with the T-algebra structure on B^X described above. Similarly, one can make sense to $C \times X \to B$ is a T-homomorphism in the first variable. (These notions of "partial" T-homomorphisms can also be described in terms of, respectively, $t'_{C,X}$ and $t''_{X,C}$, see [K71].)

It is well known how $T(X)$ is a *free* T-linear space on $\eta_X : X \to T(X)$; given $B = (B, \beta)$ a T-linear space, and a map $\phi : X \to B$, the map $\overline{\phi} : T(X) \to B$ given as $\beta \circ T(\phi)$ is a T-linear map $T(X) \to B$, and the unique one with $\overline{\phi} \circ \eta_X = \phi$.

Similarly, if $B = (B, \beta)$ is a T-linear space, and $\phi : X \times Y \to B$ is any map, there is a unique map $\overline{\phi} : T(X) \times Y \to B$ which is T-linear in the first variable with $\overline{\phi} \circ (\eta_X \times Y) = \phi$. It can be described explicitly using $t'_{X,Y}$, or it can be obtained by passing to the exponential transpose $X \to B^Y$, and using the T-structure on B^Y described in terms of $\lambda_{Y,B}$.

We shall have occasion to use the uniqueness assertion in the universal properties thus described many times.

We can now present a fundamental construction for T-linear spaces $B = (B, \beta)$: it gives, for any $X \in \mathscr{E}$ a "pairing" map

$$T(X) \times B^X \xrightarrow{\langle -,- \rangle} B. \tag{4}$$

The bracket should ideally be decorated by symbols X and (B, β), and it will be natural in both these. Ultimately, the pairing will have as a special case that pairing between distributions and test functions[2] which define the notion of (Schwartz-) distribution; in our context, it is defined as the unique map, T-linear in the first variable, which extends over $\eta_X \times B$ the evaluation map $ev : X \times B^X \to B$, thus the diagram

$$\begin{array}{ccc} T(X) \times B^X & \xrightarrow{\langle -,- \rangle} & B \\ {\scriptstyle \eta_X \times B}\Big\uparrow & \nearrow{\scriptstyle ev} & \\ X \times (X \pitchfork B) & & \end{array} \tag{5}$$

commutes. If T is a commutative monad, see below, the pairing is T-bilinear.

The exponential transpose of the pairing is a map

$$T(X) \xrightarrow{\tau} B^{(B^X)}, \tag{6}$$

[2]the test functios here are not supposed to have bounded support, therefore, the distribution notion will be: distribution of compacr support.

which is T-linear, because the pairing is a T-linear in the first variable. (Again, τ ought to be decorated with symbols X and (B,β).) Alternatively, τ may be described as the unique T-linear map with $\tau \circ \eta_X = \delta$, where δ is the standard embedding into double dual, ($x \in X$ maps to "evaluation at x", in set theoretic terms). – The codomain of τ is a "double dualization" construction. Such will appear quite often in the following, and it is clearly typographically inconvenient with the exponential notation; an on-line notation is to be preferred, and we shall use one such (used e.g. in [HW]), namely

$$X \pitchfork Y := Y^X.$$

Thus, τ (for given X and $B = (B,\beta)$) is a map (T-homomorphism, in fact, by construction)

$$T(X) \xrightarrow{\tau} (X \pitchfork B) \pitchfork B. \tag{7}$$

It is natural in X; note that the codomain is the value on X of the covariant functor $(- \pitchfork B) \pitchfork B$; it is covariant, being the composite of two contravariant ones $(- \pitchfork B)$.

Remark. One way to interpret τ, for $\mathscr{E}$ the category of sets is: $P \in T(X)$ is a *name* for an X-ary operation on T-algebras; $\tau(P) \in (X \pitchfork B) \pitchfork B$ is the X-ary operation on B *named by* P: $\tau(P)$ is an X-ary operation on B, since it is a map $X \pitchfork B \to B$, thus a construction which to X-tuples of elements in B returns single elements of B. Thus if "algebra" means "commutative ring", $T(X)$ is the set of formal polynomials in variables from X, and τ itself returns to such a polynomial P the *polynomial function* $X \pitchfork B \to B$ to which P gives rise, for B is a commutative ring. Here, the word T-linear is misleading; this is because T here is not a *commutative* monad, in the sense to be described.

Thus, τ itself is: "semantics".

Let $A = (A,\alpha)$ and $B = (B,\beta)$ be T-linear spaces. Under a weak completeness condition on $\mathscr{E}$ (existence of equalizers), one can describe a subobject $A \pitchfork_T B$ of $A \pitchfork B$, consisting of those maps $A \to B$ which happen to be T-linear. Recall that a T-linear structure on B gives rise to a T-linear structure on $X \pitchfork B$ for any X. In particular, $A \pitchfork B$ inherits a T-linear structure from that of B. But $A \pitchfork_T B \subseteq A \pitchfork B$ need not be a T-linear subspace. It is a a T-linear subspace, if T is what is called a *commutative* monad, [K70], [K71], [K71b]. In particular, the subobject,

$$(X \pitchfork B) \pitchfork_T B \subseteq (X \pitchfork B) \pitchfork B$$

is a sub-T-linear space. Therefore, since $\delta : X \to (X \pitchfork B) \pitchfork B$ in any case factors through the subobject $(X \pitchfork B) \pitchfork_T B$, it follows that, for T is commutative, τ factors through $(X \pitchfork B) \pitchfork_T B$, so that we have

$$T(B) \xrightarrow{\tau} (X \pitchfork B) \pitchfork_T B, \tag{8}$$

(and it is T-linear since the original τ was so).

Remark continued. Recall that when $\mathscr{E}$ is the category of sets, the values of τ are the "operations" on T-algebras; so commutativity of T thus implies (in fact, is equivalent to) the assertion "the operations of T are themselves T-homomorphisms". This is a classical notion of commutativity in universal algebra.

Example. If $\mathscr{E}$ is the category Lip^∞, and T the free-convenient vector space monad of [FK], then $\mathbb{R}$ is $T(1)$, in particular, it is a T-algebra. Then for a [paracompact] manifold X (which may be considered as an object of $\mathscr{E}$), $X \pitchfork \mathbb{R}$ is the CVS of smooth $\mathbb{R}$-valued functions on X; the space $(X \pitchfork \mathbb{R}) \pitchfork \mathbb{R}$ of smooth maps $X \pitchfork \mathbb{R} \to \mathbb{R}$ is a (quite unwieldy) convenient vector space, but the subspace of the *linear* smooth maps $X \pitchfork \mathbb{R} \to \mathbb{R}$ is the convenient vector space of (compactly supported) Schwartz distributions on X, cf. [FK] (with $(X \pitchfork \mathbb{R}$ the CVS of (unbounded) test functions; and the τ of (8) is in this case an isomorphism, cf. [FK]).

If X is a more general Lip^∞-space, τ may not be an isomorphism, but it does make $T(X)$ a subspace of $(X \pitchfork \mathbb{R}) \pitchfork_T \mathbb{R}$, in fact, [FK] construct $T(X)$, with the requisite universal property, as a subspace of $(X \pitchfork \mathbb{R}) \pitchfork_T \mathbb{R}$.

Tensor products and convolution

The contention is that many aspects of distribution theory live already at the level of the monad T, independent of its relationship to the double-dualization construction, which is a 20th century sophistication (Schwartz distributions), whereas T embodies a more fundamental notion of "extensive quantity", e.g. with $T(X)$ (a mathematical model of) the vector space of distributions of electric charge over the space X.

It is well known that for Schwartz distributions, we have notions of tensor product, and convolution. They exist at the level of the monad T, provided T is commutative (and it is preserved by the canonical comparison τ with "Schwartz distributions").

Recall that we have the tensorial strength, natural in X and Y,

$$t'' : X \times T(Y) \to T(X \times Y).$$

By the universal property quoted above, it extend in a unque way over $\eta_X \times T(Y)$ to a map $T(X) \times T(Y) \to T(X \times Y)$ to a map which is a T-homomorphism in the first variable; the extended map we call $\otimes_{X,Y}$ or just $\otimes$; thus, we have a commutative

$$\begin{array}{ccc} T(X) \times T(Y) & \xrightarrow{\otimes} & T(X \times Y) \\ {\scriptstyle \eta_X \times T(Y)} \uparrow & \nearrow {\scriptstyle t''} & \\ X \times T(Y) & & \end{array}$$

with $\otimes$ a T-homomorphism in the first variable. If T is commutative, one can prove that $\otimes$ is also a T-homomorphism in the second variable. With some simple properties on T to be quoted in a moment, it will follow that $R := T(1)^3$ carries structure of commutative ring, that T-algebras in particular are R-modules, and T-linear maps R-linear.

One might instead use t' and construct a $\overline{\otimes} : T(X) \times T(Y) \to T(X \times Y)$, which is T-linear in the second variable. We proved in [K70]... that $\otimes = \overline{\otimes}$ is equivalent to commutativity of the monad. (The equation that $\otimes = \overline{\otimes}$ agree as maps $T(X) \times T(Y) \to T(X \times Y)$ is essentially

[3] here, 1 denotes the terminal object of $\mathscr{E}$; the "one point space"

Fubini's Theorem, for the case of compact Schwartz distribution on a manifold.) Also we proved in loc. cit. that commutativity of T is equivalent to bilinearity of the map $\otimes$, for all X and Y.

We shall henceforth assume henceforth that T is a commutative monad, so that in particular we have the T-bilinear map $\otimes : T(X) \times T(Y) \to T(X \times Y)$ (natural in X and Y). Then if M is a space with a monoid structure, $m : M \times M \to M$, we get a multiplication "convolution along m" on $T(M)$; it is the composite map

$$T(M) \times T(M) \xrightarrow{\otimes} T(M \times M) \xrightarrow{T(m)} T(M). \tag{9}$$

It will actually be a monoid structure again, and will be commutative if m is so.

Similarly, if the monoid M acts on a space X by $a : M \times X \to X$, we will get an action of $T(M)$ on $T(X)$, which is unitary and associative if a is so.

We have in particular a unique (and trivial) monoid structure on the space 1. Convolution along this unique $1 \times 1 \to 1$ yields a monoid structure in $T(1)$, written $\cdot$; this monoid will play the role of (the multiplicative monoid of the ring of) *scalars*, and we denote it R. In both the specific examples quoted from [FK], it will be $\mathbb{R}$.

Since the trivial monoid 1 acts uniquely (and trivially) on any space X, we get an (unitary and associative) action of the monoid $R = T(1)$ on the space $T(X)$. Any T-linear map $f : T(X) \to T(Y)$ will be equivariant for this action; this is clear for f of the form $T(\phi)$ where $\phi : X \to Y$; for general T-linear $f : T(X) \to T(Y)$, an argument is needed, see [MEQ] Proposition 11 for an even more general result; or observe that the two maps $T(X) \times T(1) \to T(Y)$ to be compared are both T-bilinear, so that it suffices to check that their precomposition with $\eta_X \times \eta_1$ agree, and this is easy.

If $M \to M'$ is a monoid homomorphism, it follows from the naturality of $\otimes$ that the induced map $T(M) \to T(M')$ is a homomorphism with respect to the respective convolution structures. In particular, the unique map $! : M \to 1$ (=the terminal object of $\mathscr{E}$) is trivially a monoid homomorphism, and it induces a monoid homomorphism $T(M) \to T(1) = R$.

For any $P \in T(X)$, we have a scalar $\in R$ associated, the *total* of P, namely: apply $T(!) : T(X) \to T(1) = R$ to P. From naturality of $\otimes$ follows that for $P \in T(X)$ and $Q \in T(Y)$, the total of $P \otimes Q \in T(X \times Y)$ is the product in the monoid R of the totals of P and Q.

Extensive quantities

According to Lawvere, a mathematical model, which makes aspects of the physical and philosophical notion of extensive quantity explicit, is: it is a covariant functor from a cartesian closed category into an additive category, with certain properties.

We show here that if a commutative monad T on a cartesian closed category $\mathscr{E}$ has a certain property, then the Kleisli category $Kl(T)$ for T, i.e. category of free T-algebras, with its T-linear maps, is such a category, and the functor T (viewed as a functor $\mathscr{E} \to Kl(T)$), satisfies the properties stated by Lawvere. The category $Kl(T)$ is in fact an additive subcategory of the category of modules over the rig R (rig = commutative semiring). (I don't know in general when $Kl(T)$ is in fact a *full* subcategory of the, likewise additive, category of T-algebras.)

According to Kant, a quantity is extensive, provided the concept of its parts is condition for the concept of the whole quantity. We read this here in over-simplified form: if $X = X_1 + X_2$ (making X a disjoint union of two parts), then a quantity P distributed over X, i.e. a $P \in T(X)$, is conditioned (= given) by its parts, i.e. by a pair (P_1, P_2) with P_i distributed over X_i, i.e. with $P_1 \in T(X_1)$ and $P_2 \in T(X_{2)}$; so there is a bijection

$$T(X_1 + X_2) \cong T(X_1) \times T(X_2);$$

To make this precise, one has to describe *how* the isomorphism here is obtained, in terms of the functorality of $T : \mathscr{E} \to \mathscr{E}$.

This is rather straightforward, and it is probably known since early days of category theory. Most recently, it was (re-) discovered by Coumans and Jacobs 2010 [CJ], and by myself [K11]. The short story is that one has to assume, first that $T(\emptyset) = 1$ (where $\emptyset$ is the initial object of $\mathscr{E}$, "the empty space"); this makes 1 into a zero object in $Kl(T)$. Secondly, using the zero object, one can construct a natural map $T(X+Y) \to T(X) \times T(Y)$, and the property is then the assumption that this map is invertible. When this is the case, $Kl(T)$ has biproducts $T(X) \oplus T(Y) = T(X+Y)$, and therefore becomes an additive category; so all objects in $Kl(T)$ are abelian monoid objects (in $Kl(T)$ as well as in $\mathscr{E}$); and T-linear maps $T(X) \to T(Y)$ preserve the additive structure

In particular, $R := T(1)$ carries such structure, denoted $+$, and from T-bilinearity of the multiplication $\cdot : R \times R \to R$ already described, it follows that this multiplication is bi-additive, so R with $+$ and $\cdot$ is a commutative rig[4]. Likewise, the multiplicative action of R on objects of the form $T(X)$ is T-bilinear, hence bi-additive, so any $T(X)$ carries R-module structure, and T-linear maps between such are R-linear. (I don't know under which conditions the converse holds.)

Extensive quantities on R

For any space X, we have the space $T(X)$ of extensive quantities on X. The case where X is R itself (or, more generally, R^n), the extensive quantities on it have special significances, and a rich structure. One significance is that random variables (on an arbitrary outcome space) have probability distributions which may be construed as elements of $T(R)$ with total 1. Among the important structures on $T(R)$ is the convolution $*$ along the addition map $+ : R \times R \to R$ (which in the particular case of random variables gives the probability distribution of a sum of two independent random variables).

There are also certain scalars $\in R$ associated to a $P \in T(R)$. The most important is $\langle P, id_R \rangle$, which for the case of probability distributions deserves the name *expectation* of P; let us denote it $E(P) \in R$. So E is a map $T(R) \to R$, and it is T-linear since the pairing $T(X) \times (X \pitchfork R) \to R$ is T-linear in the first variable. If $X = R$, then it is easy to see that

$$\langle \eta_R(r), id_x \rangle = r$$

for any $r \in R$, so $E(\eta_R(r)) = r$, or $E \circ \eta_R = id_R$. (You may want to view $\eta_R(r)$ as the probability distribution of a random variable whose value is r with certainty; in this verbal garbe, the equation says that the expectation of such a random variable is r.)

[4]"rig" means commutative semiring", i.e. like commutative ring, but not necessarily with "minus" (negatives), hence the missing "n"

Now, the very construction of pairing (4) depends on a T-algebra structure $\beta : T(B) \to B$ on the codomain B; in the case at hand, $B = R = T(1)$, and $\beta : T(R) \to R$ is $\mu_1 : T^2(1) \to T(1)$. Recall that μ_X in general was constructed as the unique T-linear extension over $\eta_{T(X)}$ of the identity map on $T(X)$. In particular, $\mu_1 \circ \eta_R$ is the identity map on R. By the uniqueness of T-linear extensions over η, we conclude that $E = \mu_1$ as maps $T(R) \to R$.

If $P \in T(R)$ has that its total λ is an invertible element in the multiplicative monoid of R, it may be viewed as (a mathematical model of) a distribution of mass on the line R, and in this case λ^{-1} times the total of P may be viewed as the *center of gravity* of this distributed mass. It is easy to prove (see Prop. 24 in [MEQ] in this generality) the physically obvious fact that affine maps $R \to R$ preserve the formation of center of gravity.

Physical quantities as torsors

It is a reasonable idea that some particular kind of physical quantity, like mass, is a covariant functor M from the category of spaces to the category of modules over the rig $\mathbb{R}_{\geq 0}$ of non-negative reals; $M(X)$ is the module of possible distributions of mass over the space X. If $P \in M(X)$ is such a mass distribution, we can form its *total* $\in M(1)$, by applying the covariant functor M to the unique map $! : X \to 1$. We note that $M(1)$ is not a *number*: $M(1)$ is not $= T(1) = \mathbb{R}_{\geq 0}$; $M(1)$ is *isomorphic* to $T(1)$, but not canonically. An isomorphism amounts to *choosing* a unit of mass; then $M(1) \cong \mathbb{R}_{\geq 0} = T(1)$ by an isomorphism which is linear.

We shall make an explicit theory of why it is that choosing a unit of mass provides an ismorphism of functors $T \cong M$.

Let T be a commutative monad on $\mathscr{E}$. Consider another strong endofunctor M on $\mathscr{E}$, equipped with an action ν by T,

$$\nu : T(M(X)) \to M(X)$$

strongly natural in X, and with ν satisfying a unitary and associative law. Then every $M(X)$ is a T-linear space by virtue of $\nu_X : T(M(X)) \to M(X)$, and morphisms of the form $M(f)$ are T-linear. Let M and M' be strong endofunctors equipped with such T-actions. There is an evident notion of when a strong natural transformation $\lambda : M \Rightarrow M'$ is compatible with the T-actions, so we have a category of T-actions. The endofunctor T itself is an object in this category, by virtue of μ. We say that M is a *T-torsor* if it is isomorphic to T in the category of T-actions. Note that no particular such isomorphism is chosen.

Our contention is that the category of T-torsors is a mathematical model of (not necessarily pure) quantities M of type T (which is the corresponding pure quantity). Thus if T is the free $\mathbb{R}$-vector space monad, the functor M which to a space $X \in \mathscr{E}$ associates the space of distributions of electric charges[5] over X, is a T-torsor.

The following Proposition expresses that isomorphisms of actions $\lambda : T \cong M$ are determined by $\lambda_1 : T(1) \to M(1)$; in the example, the latter data means: choosing a *unit* of electric charge.

[5]I take *electric charge* rather than *mass* as the kind of physical quantity for this discussion, because then we have negatives: vector spaces over the ring $\mathbb{R}$ rather than over the rig $\mathbb{R}_{\geq 0}$

Proposition *If g and $h : T \Rightarrow M$ are isomorphisms of T-actions, and if $g_1 = h_1 : T(1) \to M(1)$, then $g = h$.*

Proof. By replacing h by its inverse $M \to T$, it is clear that it suffices to prove that if $\rho : T \to T$ is an isomorphism of T-actions, and $\rho_1 = id_{T(1)}$, then ρ is the identity transformation. As a morphism of T-actions, ρ is in particular a *strong* natural transformation, which implies that right hand square in the following diagram commutes for any $X \in \mathscr{E}$; the left hand square commutes by assumption on ρ_1:

$$\begin{array}{ccccc} X\times 1 & \xrightarrow{X\times\eta_1} & X\times T(1) & \xrightarrow{t''} & T(X\times 1)\\ {\scriptstyle =}\Big\downarrow & & \Big\downarrow{\scriptstyle X\times\rho_1} & & \Big\downarrow{\scriptstyle \rho_{X\times 1}} \\ X\times 1 & \xrightarrow[X\times\eta_1]{} & X\times T(1) & \xrightarrow[t'']{} & T(X\times 1)\end{array}$$

Now both the horizontal composites are $\eta_{X\times 1}$, by general theory of tensorial strengths. Also $\rho_{X\times 1}$ is T-linear. Then uniqueness of T-linear extensions over $\eta_{X\times 1}$ implies that the right hand vertical map is the identity map. Using the natural identification of $X \times 1$ with X, we then also get that ρ_X is the identity map of $T(X)$.

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