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Every étendue comes from a local equivalence relation

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Abstract

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We first prove that, under suitable connectedness assumptions, the equivariant sheaves for a local equivalence relation on a space (or a locale) form an étendue topos. Our main result is that conversely, every étendue can be obtained in this way.

Introduction

An étendue is a topos $\mathcal{T}$ for which an object $U \in \mathcal{T}$ exists such that $U \to 1$ is epi and the slice topos $\mathcal{T}/U$ is localic, that is, $\mathcal{T}/U$ is equivalent to the category of sheaves on a locale. These étendue topoi were introduced by Grothendieck and Verdier [1, p. 478 ff.] in the context of foliations and local equivalence relations. It was suggested that for a suitable local equivalence relation r on a topological space, the category of r-invariant sheaves form an étendue topos. In this paper, we will consider the notion of a local equivalence relation r on a locale M. We will show that if r is locally simply connected (in an appropriate sense), then the category of r-invariant sheaves on M is a topos, and in fact an étendue. (We will also explain, in Example 2.3, how this result relates to a similar statement for local equivalence relations on topological spaces in [16].)

Our main result is that every étendue can be obtained this way. Indeed, in Theorem 7.1 we will show that for any étendue $\mathcal{T}$, there exists a local equivalence

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relation r on some locale M for which there is an equivalence of topoi

$$(*) \qquad \mathrm{Sh}(M, r) \approx \mathcal{T} .$$

Moreover, this local equivalence relation is locally simply connected in the sense referred to above. The construction of M and r is based on the observation that étendue topoi in some sense 'classify' local equivalence relations: every locally connected geometric morphism from a topos of sheaves on a locale M into an étendue $\mathcal{T}$ gives rise to a canonical local equivalence relation on M. Furthermore, essential use is made of the construction from [6] of a localic cover with contractible fibres of any given topos.

For an étendue $\mathcal{T}$ with enough points, there exists a *topological space* with a local equivalence relation r for which there is an equivalence of form (*), but this space has to be obtained by a completely different construction, cf. [10, 11]; here the reader will also find a discussion of the relation between étendues, foliations, holonomy and monodromy.

Our main result is a presentation theorem for étendues; we wish to point out that this result bears no relation to the type of presentation considered in [9].

1. Equivalence relations on locales

Let X be a locale. By an equivalence relation on X we shall always mean a sublocale $R \subseteq X \times X$ satisfying the usual conditions of reflectivity, symmetry, and transitivity, and in addition having the property that the two projection maps

$$d_0, d_1 : R \rightrightarrows X \tag{1.1}$$

are open maps. This implies that the coequalizer $\pi_R : X \to X/R$ of d_0 and d_1 is also an open map [14]. Note that, unlike the case of topological spaces, R need not coincide with the kernel pair of π_B, cf. [8]. Since R is reflexive and transitive, there is a (truncated) simplicial complex of locales

$$R \times_X R \substack{\rightarrow\\\rightarrow\\\rightarrow} R \rightrightarrows X . \tag{1.2}$$

By applying the functor sh(–), we obtain a similar diagram of topoi and geometric morphisms. We write sh(X; R) for the associated descent topos. So the objects of sh(X; R) are sheaves E on X equipped with 'descent data' $\theta_E : d_0^* E \to d_1^* E$ satisfying a unit and cocycle condition. (This construction will be discussed in greater generality in Section 3.) Equivalently, θ can be given in the form of an action by R on E, or of a transport on E along R, i.e. a map $R \times_X E \to E$ satisfying usual associativity and unit laws. The localic reflection of this topos sh(X; R) is (the topos of sheaves on) the quotient locale X/R.

In the context of topological spaces, it is well known that sh(X; R) coincides with sh(X/R) in case the map $d_0 : R \to X$ has 'enough local sections' (cf. [1, p.

480]). In fact, it is enough to require d_0 to be an open map; the argument also works for locales:

Proposition 1.1. *For any equivalence relation R on a locale X (with d_0 and d_1 open), the map* $\mathrm{sh}(X; R) \to \mathrm{sh}(X/R)$ *is an equivalence of topoi.*

Proof. It is enough to show that the topos $\mathrm{sh}(X; R)$ is generated by subobjects of the terminal object 1. Such subobjects are R-saturated open sublocales of X, i.e. sublocales of the form $d_1 d_0^{-1}(U) \subseteq X$, where $U \subseteq X$ is any open sublocale. Such R-saturated sublocales carry a unique action by R, hence are objects of $\mathrm{sh}(X; R)$. Now consider an arbitrary object E of $\mathrm{sh}(X; R)$, given as an étale map $p : E \to X$ with an action $\theta : R \times_X E \to E$. Let $s : W \to E$ be any section of $E \to X$ over an open $W \subseteq X$. We wish to show that W is covered by opens $W_i \subseteq W$ with the property that $s|_{W_i} : W_i \to E$ can be extended to an R-equivariant section

$$\tilde{s}_i : d_1 d_0^{-1}(W_i) \to E . \tag{1.3}$$

Thus each such section $\tilde{s}_i$ is a map in the topos $\mathrm{sh}(X; R)$ from a subobject of 1 into E. All these sections $\tilde{s}_i$, for all possible sections $s : W \to E$, cover E since $p : E \to X$ is a local homeomorphism. So this indeed shows that $\mathrm{sh}(X; R)$ is generated by subobjects of 1.

To construct these local extensions $\tilde{s}_i$ from the given section $s : W \to E$, consider first the pullback $d_1^*(E|_W) = E|_W \times_W R$ of $E|_W \to W$ along d_1: $R|_W = R \cap (W \times W) \to W$, as in

$$\begin{array}{ccc} d_1^*(E|_W) & \longrightarrow & E|_W \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ R|_W = R \cap (W \times W) & \xrightarrow[d_1]{} & W \end{array} \tag{1.4}$$

The map p' in this diagram has two sections induced by $s : W \to E$, namely

$$s_1 = \theta(\mathrm{id}, sd_0) : R|_W \to R \times_W E \to E , \qquad s_2 = sd_1 .$$

(In point-set notion, $d_1^*(E|_W) = \{(x, y, e) \mid (x, y) \in R,\ x, y \in W,\ e \in p^{-1}(y)\}$, and $s_1(x, y) = \theta((x, y), s(x))$, $s_2(x, y) = s(y)$.) These two sections agree on the diagonal $\Delta : W \to R|_W$. Since p' is étale, it follows that they must agree on a neighbourhood N of the diagonal. We may assume that this neighbourhood is of the form

$$N = \bigcup_i R|_{W_i} = \bigcup_i R \cap (W_i \times W_i) ,$$

for some open cover $W = \bigcup_i W_i$. By definition of s_1 and s_2, this means that each restriction $s|_{W_i} : W_i \to E$ is $(R|_{W_i})$-equivariant. It follows that $s|_{W_i}$ can be extended to the R-saturation $d_1 d_0^{-1}(W_i)$ of W_i. Indeed, let P be the kernel pair of

$d_1 : d_0^{-1}(W_i) \to d_1 d_0^{-1}(W_i)$, as in the diagram

$$P \overset{\pi_0}{\underset{\pi_1}{\rightrightarrows}} d_0^{-1}(W_i) \to d_1 d_0^{-1}(W_i)\,. \tag{1.5}$$

Since $d_1 : R \to X$ is an open surjection by assumption, this diagram is a coequalizer, cf. [7]; furthermore, since $s|_{W_i}$ is $(R|_{W_i})$-equivariant, the cocycle condition for the action θ by R on E implies that the map $s_1 : d_0^{-1}(W_i) \to E$, given in point-set terms by $s_1(x, y) = \theta((x, y), s(x))$, satisfies the identity $s_1\pi_1 = s_1\pi_2$. Thus s_1 factors through the coequalizer (1.5), to give the desired section $\tilde{s}_i : d_1 d_0^{-1}(W_i) \to E$. □

An equivalence relation R on a locale X, as above, is said to be *connected* (respectively *locally connected*) if d_0 and d_1 are connected (respectively locally connected) maps of locales, i.e. if the corresponding geometric morphisms $d_0, d_1 : \mathrm{sh}(R) \to \mathrm{sh}(X)$ are connected, respectively locally connected.

Proposition 1.2. *If d_0, d_1 are connected (respectively locally connected) maps, the quotient map $X \to X/R$ is a connected (respectively locally connected) map.*

Proof. For the locally connected case, if $d_0, d_1 : R \rightrightarrows X$ are locally connected, then by [15] so is the geometric morphism $\mathrm{sh}(X) \to \mathrm{sh}(X; R)$, and hence by Proposition 1.1, $X \to X/R$ is a locally connected map. For the connected case, assume that $d_0, d_1 : R \rightrightarrows X$ are connected. Again by Proposition 1.1, it suffices to see that $\mathrm{sh}(X) \to \mathrm{sh}(X; R)$ is a connected geometric morphism. Consider two R-equivariant sheaves (E, θ) and (F, μ), and a map $\phi : E \to F$ in $\mathrm{sh}(X)$. We must prove that ϕ is R-equivariant, i.e. a map in $\mathrm{sh}(X; R)$. Consider the two maps $\alpha, \beta : d_0^* E \to d_0^* F$ described, in point-set notation, for $(x, y) \in R$ and $e \in E_x$, by

$$\alpha((x, y), e) = ((x, y), \mu((y, x), \phi(\theta((x, y), e))))\,,$$

$$\beta((x, y), e) = ((x, y), \phi(e))\,.$$

Thus $\beta = d_0^*(\phi)$, and since d_0^* is full and faithful, $\alpha = d_0^*(\alpha')$ for a unique map $\alpha' : E \to F$ in $\mathrm{sh}(X)$. By the unit-condition for the actions θ and μ, we have $\triangle^*(\alpha) = \triangle^*(\beta)$, where $\triangle : X \to R$ is the diagonal. Hence $\alpha' = \triangle^* d_0^* \alpha' = \triangle^* \alpha = \triangle^* \beta = \phi$, and thus, applying d_0^*, $\alpha = \beta$. This identity expresses that ϕ is an R-equivariant map. This proves the proposition. □

As a consequence, we obtain the following:

Proposition 1.3. *Let R be a connected equivalence relation on a locale X. Then for any sheaf E on X, there is at most one action by R on E.*

(If there is such an action, we call E an *R-invariant* sheaf.)

Proof. Let θ and θ' be two R-actions on E. Since the forgetful functor $\mathrm{sh}(X; R) \to \mathrm{sh}(X)$ is full and faithful, the identity map on E in $\mathrm{sh}(X)$ must also be an R-equivariant map $(E, \theta) \to (E, \theta')$, thus $\theta = \theta'$. □

2. Local equivalence relations and sheaves

For a locale M, consider for each open $U \subseteq M$ the set $E_M(U)$ of equivalence relations R on U, as defined in Section 1. For open sublocales $V \subseteq U \subseteq M$ there is an evident restriction map $E_M(U) \to E_M(V)$, making E_M into a presheaf on M. By definition [1, p. 485], a local equivalence relation on M is a global section of the associated sheaf $\tilde{E}_M$. An equivalence relation R on any locale U gives rise to a local equivalence relation $L(R)$ on U. Let r be a local equivalence relation on M. An equivalence relation R on an open $U \subseteq M$ will be called a *chart* for r if $L(R)$ agrees with the restriction of r to U; if $V \subseteq U$, then $(V, R|_V)$ is also a chart for r; we call it a *subchart* of (U, R). An *atlas* for r is a family $\{(U_i, R_i)\}$ of charts for r such that the U_i's cover M. A family $\{(U_i, R_i)\}$ will be an atlas for some local equivalence relation iff for any two indices i and j, $U_i \cap U_j$ is covered by open W such that $R_i|_W = R_j|_W$. An atlas is a *refinement* of another if each chart of the former is a subchart of some chart of the latter.

By our conventions in Section 1, it follows that any local equivalence relation r has an atlas consisting of charts (U, R) for which $R \rightrightarrows U$ are open maps. Furthermore, r is said to be *locally connected* if any atlas for r can be refined by an atlas consisting of connected and locally connected charts, i.e. charts (U, R) for which $R \rightrightarrows U$ are connected and locally connected maps. Such an atlas will be called a *connected atlas* for r.

Following [1], we now define, for a local equivalence relation r on a locale M and a sheaf F on M, the notion of an r-transport on F. Consider for an open $U \subseteq M$ the set $T_F(U)$ of pairs (R, θ), where R is an equivalence relation on U and $\theta : R \times_U (F|_U) \to (F|_U)$ is an action by R on $F|_U$ (as in Section 1). With the obvious restrictions maps $T_F(U) \to T_F(V)$ for opens $V \subseteq U \subseteq M$, this gives a presheaf T_F on M, with a projection map $\pi : T_F \to E_M$. Passing to the associated sheaves, we obtain a map $\tilde{\pi} : \tilde{T}_F \to \tilde{E}_M$. An *r-transport* on the sheaf F is by definition a global section t of $\tilde{T}_F$ such that $\tilde{\pi}(t) = r$. A sheaf equipped with an r-transport is called an *r-invariant sheaf*, or an *r-sheaf*. Such an r-transport is thus given by an open cover $\bigcup U_i = M$, equivalence relations R_i on U_i, and actions θ_i of R_i on $F|_{U_i}$, all locally compatible on intersections $U_i \cap U_j$. As before, we call $\bar{\mathcal{U}} = \{(U_i, R_i, \theta_i)\}$ an *atlas* for t, and each of its members a *chart* for t.

Any atlas or chart for t has an evident underlying atlas or chart for r. We note that if $\bar{\mathcal{U}}$ is an atlas for t with underlying atlas $\mathcal{U}$ for r, and $\mathcal{V}$ is another atlas for r which refines $\mathcal{U}$, then $\bar{\mathcal{U}}$ can be refined by an atlas $\bar{\mathcal{V}}$ for t which has the given $\mathcal{V}$ as underlying atlas for r. It follows that for two sheaves with r-transport (F, t) and (F', t'), there exists atlases for t and t' with identical underlying atlas for r. It also follows that if r is locally connected, any atlas for t can be refined by an atlas whose underlying atlas for r is connected.

In this paper, we shall only consider local equivalence relations r which are locally connected. For such an r, it follows readily from Proposition 1.2 that for a given sheaf F on M, there is at most one r-transport t on F. Thus (the existence of) an r-transport on F is a property, rather than an additional structure. For a locally connected r, we therefore define the category $\mathrm{sh}(M, r)$ to be the full subcategory of $\mathrm{sh}(M)$ consisting of sheaves on M which admit an r-transport (necessarily unique).

Remark 2.1. The property of being an r-sheaf on X is a local property. More explicitly, if $q : Y \to X$ is an étale map (a local homeomorphism), then any local equivalence relation r on X induces, in an evident way, a local equivalence relation on Y, which we denote $q^{\#}r$; if r is locally connected, then so is $q^{\#}r$; and conversely, provided q is surjective. In this case, it is clear that if $E \in \mathrm{sh}(X)$, then E is an r-sheaf iff q^*E is an $q^{\#}r$-sheaf.

Remark 2.2. More generally, for an arbitrary local equivalence relation r on a locale M and two sheaves with r-transport (F, t) and (F', t'), there is a straightforward definition of transport-preserving map $F \to F'$, so that one obtains a category $\mathrm{sh}(M, r)$. Using the remarks in Section 1, one can easily show that in case r is locally connected, any sheaf map $F \to F'$ is transport-preserving, so that for such r, the fact that the forgetful functor $\mathrm{sh}(M, r) \to \mathrm{sh}(M)$ is full and faithful is a result, rather than a definition.

Example 2.3. For any locale M, there is a 'maximal' local equivalence relation $r_{\max}$ on M, given by the single chart (U, R), where $U = M$ and $R = M \times M$. If M is a locally connected locale, then $r_{\max}$ is also locally connected. The category $\mathrm{sh}(M, r_{\max})$ is exactly the category of locally constant sheaves on M. This category is not in general a Grothendieck topos. For example [3, p. 314] when M is the Hawaiian earring, $\mathrm{sh}(M, r_{\max})$ is not closed under infinite sums; on the other hand, if $\mathrm{sh}(M, r_{\max})$ is a Grothendieck topos, it must have infinite sums, and these sums must be preserved by the forgetful functor, cf. loc. cit., Theorem 6; cf. also [10]. The fact that $\mathrm{sh}(M, r_{\max})$ is not a Grothendieck topos disproves Theorem 4.14 in [16].

Example 2.4. Let r be a (locally connected) local equivalence relation on a locale M. For any locale T, there is a sheaf $T^{(r)}$ on M of germs of r-invariant maps $M \to T$. A typical section of $T^{(r)}$ over an open $U \subseteq M$ is a map $s : U \to T$ which has the property that U is covered by r-charts (U_i, R_i) such that each restriction $s|_{U_i} : U_i \to T$ factors through the quotient map $U_i \to R_i/R_i$. This sheaf $T^{(r)}$ has r-transport, hence is an object of $\mathrm{sh}(M, r)$. When T is the Sierpinski space, $T^{(r)}$ is a subobject classifier for $\mathrm{sh}(M, r)$, and $\mathrm{sh}(M, r)$ is an elementary topos. This is discussed more fully in [10].

3. Simplicial topoi and descent

Recall that a simplicial topos is a simplicial object $\mathscr{E}.$ in the category of (Grothendieck) topoi, except that the simplicial identities are required to hold only up to coherent isomorphisms. Thus a simplicial topos consists of a sequence of topoi $\mathscr{E}_n$ $(n \geq 0)$, and for each nondecreasing function $\alpha : [n] \to [m]$ (where $[n] = \{0, 1, \ldots, n\}$) a geometric morphism

$$\mathscr{E}(\alpha) : \mathscr{E}_m \to \mathscr{E}_n ;$$

furthermore, for each such $\alpha : [n] \to [m]$ and $\beta : [m] \to [k]$, there is given an isomorphism $\theta_{\alpha,\beta} : \mathscr{E}(\alpha) \circ \mathscr{E}(\beta) \to \mathscr{E}(\beta\alpha)$, and these θ's are required to satisfy suitable coherence conditions. (Thus, a simplicial topos is a homomorphism of bicategories from the category Δ^{op} into the bicategory of Grothendieck topoi.)

We adopt the standard notation from simplicial sets; for example, we write $d_j : \mathscr{E}_n \to \mathscr{E}_{n-1}$ for $\mathscr{E}(\partial_j)$, where $\partial_j : [n-1] \to [n]$ is the strictly increasing function which omits j (for $0 \leq j \leq n$).

For each simplicial topos $\mathscr{E}.$ one can construct a universal augmentation $\mathscr{D}(\mathscr{E}.)$, as in

$$\mathscr{E}_2 \overrightarrow{\underrightarrow{\rightarrow}} \mathscr{E}_1 \rightrightarrows \mathscr{E}_0 \to \mathscr{D}(\mathscr{E}.) . \tag{3.1}$$

The category $\mathscr{D}(\mathscr{E}.)$ can be explicitly described in various equivalent ways; e.g. as the category of *descent objects*: thus an object of $\mathscr{D}(\mathscr{E}.)$ is a pair $(\mathscr{E}, \mu)$ where $\mathscr{E}$ is an object of $\mathscr{E}_0$ and $\mu : d_0^* E \to d_1^* E$ is an isomorphism satisfying the appropriate unit and cocycle conditions (cf. [15, section 3]); the arrows in $\mathscr{D}(\mathscr{E}.)$ between two such objects (E, μ) and (E', μ') are arrows $E \to E'$ in $\mathscr{E}_0$ which are compatible with the 'descent data' μ and μ'. It follows from the general existence theorem for colimits of Grothendieck topoi ([15, Section 2] and [12]) that $\mathscr{D}(\mathscr{E}.)$ is a Grothendieck topos, and is the colimit of the diagram $\mathscr{E}.$. The augmentation geometric morphism $a : \mathscr{E}_0 \to \mathscr{D}(\mathscr{E}.)$ has as its inverse image the forgetful functor $a^* : \mathscr{D}(\mathscr{E}.) \to \mathscr{E}_0$, so that $a^*(E, \mu) = E$.

The following is part of [15, Theorem 3.6]:

Lemma 3.1. *For a simplicial topos $\mathscr{E}.$, if all the face maps $d_j : \mathscr{E}_n \to \mathscr{E}_{n-1}$ are open (respectively locally connected, or atomic), then so is the augmentation $a : \mathscr{E}_0 \to \mathscr{D}(\mathscr{E}.)$.* □

In particular, if $X.$ is a simplicial locale, we obtain a simplicial topos $\mathrm{sh}(X.)$ by constructing the topos of sheaves $\mathrm{sh}(X_n)$ on each locale X_n, and hence a descent topos $\mathscr{D}(\mathrm{sh}(X.))$, and Lemma 3.1 gives the following:

Lemma 3.2. *For a simplicial locale $X.$ in which all the face maps $d_j : X_n \to X_{n-1}$*

are étale, the augmentation $\mathrm{sh}(X_0) \to \mathscr{D}(\mathrm{sh}(X.))$ *is an atomic geometric morphism, and* $\mathscr{D}(\mathrm{sh}(X.))$ *is an étendue.*

Proof. Since the d_j are étale, the induced geometric morphisms $d_j : \mathrm{sh}(X_n) \to \mathrm{sh}(X_{n-1})$ are atomic. By Lemma 3.1, the augmentation $\mathrm{sh}(X_0) \to \mathscr{D}(\mathrm{sh}(X.))$, which is evidently surjective, must also be atomic. Since this augmentation is also clearly a localic geometric morphism, it must be a slice, and thus $\mathscr{D}(\mathrm{sh}(X.))$ is an étendue. □

A map of simplicial topoi $f : \mathscr{F}. \to \mathscr{E}.$ is given by geometric morphisms $f_n : \mathscr{F}_n \to \mathscr{E}_n$ for each $n \geq 0$, together with, for each $\alpha : [n] \to [m]$, an isomorphism

$$f_\alpha : f_n \circ \mathscr{F}(\alpha) \to \mathscr{E}(\alpha) \circ f_m ,$$

and these isomorphisms are required to be compatible with the isomorphisms $\theta_{\alpha,\beta}$ for $\mathscr{E}.$ and $\mathscr{F}.$. Such a map $f : \mathscr{F}. \to \mathscr{E}.$ induces a geometric morphism $\mathscr{D}(f) : \mathscr{D}(\mathscr{F}.) \to \mathscr{D}(\mathscr{E}.)$ between descent topoi, which is compatible with the augmentations in the sense that the square

$$\begin{array}{ccc} \mathscr{F}_0 & \xrightarrow{f_0} & \mathscr{E}_0 \\ {\scriptstyle a_{\mathscr{F}}}\downarrow & & \downarrow{\scriptstyle a_{\mathscr{E}}} \\ \mathscr{D}(\mathscr{F}.) & \xrightarrow[\mathscr{D}(f)]{} & \mathscr{D}(\mathscr{E}.) \end{array} \tag{3.2}$$

commutes up to canonical isomorphism. Later we will use the following lemma concerning connected geometric morphisms (these are morphisms whose inverse image functor is full and faithful).

Lemma 3.3. *Let* $f : \mathscr{F}. \to \mathscr{E}.$ *be a map of simplicial topoi. If* f_0 *is connected and* f_1 *is surjective, then the induced geometric morphism* $\mathscr{D}(f) : \mathscr{D}(\mathscr{F}.) \to \mathscr{D}(\mathscr{E}.)$ *is again connected.*

Proof. Consider two objects (E, μ) and (E', μ') in $\mathscr{D}(\mathscr{E}.)$. We wish to show that arrows $(E, \mu) \to (E', \mu')$ in $\mathscr{D}(\mathscr{E}.)$ correspond bijectively to arrows $\mathscr{D}(f)^*(E, \mu) \to \mathscr{D}(f)^*(E, \mu')$ in $\mathscr{D}(\mathscr{F}.)$. Since f_0^* is full and faithful by assumption, it evidently suffices to show that for an arrow $\alpha : E \to E'$ in $\mathscr{E}_0$, α is compatible with descent data $\mu : d_0^* E \to d_1^* E$ and $\mu' : d_0^* E' \to d_1^* E'$ (in $\mathscr{E}_1$) iff $f_0^*(\alpha)$ is compatible with the induced descent data (in $\mathscr{F}_1$)

$$f_1^*(\mu) : d_0^* f_0^*(E) \cong f_1^* d_0^*(E) \to f_1^* d_1^*(E) \cong d_1^* f_0^*(E)$$

on $f_0^*(E)$ and (similarly) $f_1^*(\mu')$ on $f_0^*(E')$. But this readily follows by the assumption that $f_1^* : \mathscr{E}_1 \to \mathscr{F}_1$ is a faithful functor. □

Recall that by the existence theorem for colimits [12, 15], already used in the construction of descent topoi $\mathscr{D}(\mathscr{E}.)$, the pushout topos $\mathscr{B} \cup_{\mathscr{A}} \mathscr{C}$ of any two geometric morphisms $f : \mathscr{A} \to \mathscr{B}$ and $g : \mathscr{A} \to \mathscr{C}$ between Grothendieck topoi exists,

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{g} & \mathscr{C} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle v} \\ \mathscr{B} & \xrightarrow[u]{} & \mathscr{B} \cup_{\mathscr{A}} \mathscr{C} \end{array} \tag{3.3}$$

and can be constructed simply as follows: the objects of $\mathscr{B} \cup_{\mathscr{A}} \mathscr{C}$ are triples (B, C, v), where B is an object of the topos $\mathscr{B}$ and C one of $\mathscr{C}$, while $v : f^*(B) \to g^*(C)$ is an isomorphism in the topos $\mathscr{A}$. An arrow $(B, C, v) \to (B', C', v')$ in the pushout topos $\mathscr{B} \cup_{\mathscr{A}} \mathscr{C}$ is given by a pair of arrows $\beta : B \to B'$ in $\mathscr{B}$ and $\gamma : C \to C'$ in $\mathscr{C}$ such that $v' \circ f^*(\beta) = g^*(\gamma) \circ v$ in $\mathscr{A}$. In the square (3.3), the inverse images u^* and v^* of the indicated geometric morphisms are the evident forgetful functors.

One can easily verify that for a pushout square, u^* is full and faithful whenever g^* is; in other words, we have the following:

Lemma 3.4. *The pushout of a connected geometric morphism along any other geometric morphism is again connected* ('*connectedness is preserved under co-base-change*'). □

Slightly more involved is the following lemma:

Lemma 3.5. *Let $f : \mathscr{F}. \to \mathscr{E}.$ be a map of simplicial topoi, with induced geometric morphism $\mathscr{D}(f) : \mathscr{D}(\mathscr{F}.) \to \mathscr{D}(\mathscr{E}.)$. If $f_1 : \mathscr{F}_1 \to \mathscr{E}_1$ is connected and $f_2 : \mathscr{F}_2 \to \mathscr{E}_2$ is surjective, then the square* (3.2) *is a pushout.*

Proof. Let us write $\mathscr{P}$ for the pushout topos. Then the objects of $\mathscr{P}$ are of the form

$$(F, \mu, E, v)\,, \tag{3.4}$$

where F is an object of $\mathscr{F}_0$ with descent data $\mu : d_0^* F \to d_1^* F$, while E is an object of $\mathscr{E}_0$ and $v : F \to f_0^* E$ is an isomorphism. This gives an arrow $f_1^* d_0^* E \to f_1^* d_1^* E$ in $\mathscr{F}_1$: the broken arrow in the following diagram

$$\begin{array}{ccc} f_1^* d_0^* E \cong d_0^* f_0^* E \cong d_0^* F \\ \downarrow \qquad\qquad\qquad\qquad \downarrow{\scriptstyle \mu} \\ f_1^* d_1^* E \cong d_1^* f_0^* E \cong d_1^* F \end{array}$$

Since f_1^* is full and faithful by assumption, this arrow comes from a unique arrow $\sigma : d_0^* E \to d_1^* E$. This arrow σ satisfies the cocycle condition in $\mathscr{E}_2$ because in $\mathscr{F}_2$, the map μ, and hence also $f_1^*(\sigma)$, does, while $f_2^* : \mathscr{E}_2 \to \mathscr{F}_2$ is faithful by assumption. The arrow σ also satisfies the unit condition in $\mathscr{E}_0$ for a similar reason, since $f_0^* : \mathscr{E}_0 \to \mathscr{F}_0$ is again faithful (in fact $f_0 : \mathscr{E}_0 \to \mathscr{F}_0$ is a retract of $f_1 : \mathscr{E}_1 \to \mathscr{F}_1$, so f_0 is connected since f_1 is). This shows that from an object (3.4) in the pushout $\mathscr{P}$, one can construct an object in $\mathscr{D}(\mathscr{E}.)$.

Conversely, any object (E, σ) in $\mathscr{D}(\mathscr{E}.)$ gives an object (F, μ, E, v) in the pushout, where $F = f_0^* E$ and μ is defined as

$$\mu : d_0^* F = d_0^* f_0^* E \cong f_1^* d_0^* E \xrightarrow{f_1^* \sigma} f_1^* d_1^* E \cong d_1^* f_0^* E \cong d_1^* F ,$$

and $v : F \to f_0^* E$ is defined to be the identity.

These constructions establish a suitable equivalence of categories $\mathscr{D}(\mathscr{E}.) \cong \mathscr{P}$, proving the lemma. □

4. The topos defined by an atlas

This section is of auxiliary character. It defines a topos sh$(M, \mathscr{U})$ out of an atlas $\mathscr{U}$ for a local equivalence relation r on the locale M, and sh$(M, \mathscr{U})$ in general will depend on the choice of $\mathscr{U}$ (and even, in the most general case, on some further choice of a 'hypercovering').

For any atlas $\mathscr{U} = \{(U_j, R_j)\}_{j \in I}$ for a local equivalence relation, we construct a simplicial locale $U.$ (a hypercovering of M, in fact): the locale U_0 of vertices is the disjoint sum

$$U_0 = \coprod_{j \in I} U_j , \tag{4.1}$$

while the space U_1 of 1-simplices is defined as

$$U_1 = \coprod_{i,j \in I} \coprod_{k \in K_{i,j}} U_{ijk} , \tag{4.2}$$

where $K_{i,j}$ is an index set for some open covering U_{ijk} of $U_i \cap U_j$ by sublocales on which R_i and R_j agree. The simplicial operators

$$U_1 \rightrightarrows U_0 \tag{4.3}$$

are defined in the obvious way (if we assume, as we may, that $K_{ii} = \{*\}$, a one-point set, and that $U_{j,j,*} = U_j$). We now define $U.$ as the coskeleton of the truncated simplicial locale (4.3).

$$U. = \mathrm{Cosk}(U_1 \rightrightarrows U_0) . \tag{4.4}$$

Thus, U_2 is a coproduct with an index set whose typical element is given by data $((i_0, i_1, i_2), (k_0, k_1, k_2))$ with the i's in I, and $k_2 \in K_{i_0 i_1}$, etc., and the summand corresponding to this index is

$$U_{i_0 i_1 k_2} \cap U_{i_1 i_2 k_0} \cap U_{i_0 i_2 k_1} \, .$$

The simplicial locale $U.$ has an evident augmentation a to M given by the inclusions $U_i \to M$ $(i \in I)$. All maps in the diagram

$$\cdots U_2 \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} U_1 \rightrightarrows U_0 \to M \tag{4.5}$$

are étale, so $U.$ is a simplicial sheaf on M.

Lemma 4.1. *The descent topos* $\mathcal{D}(\mathrm{sh}(U.))$ *is equivalent to the topos* $\mathrm{sh}(M)$ *of sheaves on* M, *by an equivalence compatible with the augmentations* (3.1) *and* (4.5).

Proof. We view $U.$ as a simplicial sheaf on M. Since $U_1 \to U_0 \times U_0$ is surjective and $U.$ is defined as a coskeleton, $U.$ is clearly a hypercover of M (i.e. an internal contractible simplicial set inside $\mathrm{sh}(M)$). By standard theory of simplicial covering spaces [5, Appendix] applied in $\mathrm{sh}(M)$, an object of $\mathcal{D}(\mathrm{sh}(U.))$ can be identified with a covering projection into $U.$. But by contractibility of $U.$, each such is a trivial covering projection, i.e. it corresponds to a sheaf of M. This proves the lemma. □

The sum of the equivalence relations R_j defines an equivalence relation R_0 on the sum U_0 (cf. (4.1)); similarly, the sum of the equivalence relations

$$R_i|_{U_{ijk}} = R_j|_{U_{ijk}}$$

defines an equivalence relation R_1 on the sum U_1 (cf. (4.2)), and on U_2, etc. By the evident compatibilities, we get a morphism of simplicial locales

$$q_n : U_n \to U_n / R_n \quad (n = 0, 1, 2, \ldots)$$

and hence a morphism of their respective descent topoi; we denote the descent topos for the simplicial topos $(\mathrm{sh}(U_n/R_n))_n$ by $\mathrm{sh}(M, U.)$. All this is depicted in the following diagram (utilizing Lemma 4.1 for the descent of the left-hand column):

$$\begin{array}{ccc}
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
\mathrm{sh}(U_1) & \longrightarrow & \mathrm{sh}(U_1/R_1) \\
\downarrow\downarrow & & \downarrow\downarrow \\
\mathrm{sh}(U_0) & \longrightarrow & \mathrm{sh}(U_0/R_0) \\
\downarrow & & \downarrow \\
\mathrm{sh}(M) & \longrightarrow & \mathrm{sh}(M, U.)
\end{array} \tag{4.6}$$

Lemma 4.2. *For any open atlas $\mathcal{U}$ and choice of hypercovering $U_\bullet$ the topos* $\mathrm{sh}(M, U_\bullet)$ *is an étendue.*

Proof. If (W, R) is an open chart, and $V \subseteq W$ is an open sublocale, then one obtains an inclusion of an open sublocale $V/(R|_V) \to W/R$. In the right-hand column of (4.6) each map $U_n/R_n \to U_{n-1}/R_{n-1}$ is a sum of such inclusions, hence is étale. By Lemma 3.2, the descent topos $\mathrm{sh}(M, U_\bullet)$ is an étendue. □

The situation simplifies considerably for the case where r is an (open and) locally connected local equivalence relation. If $\mathcal{U}$ is a connected atlas for r, we may choose the U_{ijk} so small that the charts (U_{ijk}, R_i) are all connected (and open, locally connected, of course). If this is the case, we say that the hypercovering $U_\bullet$ is connected; then the geometric morphisms

$$\mathrm{sh}(U_n) \to \mathrm{sh}(U_n/R_n) \quad (n = 0, 1)$$

are connected geometric morphisms. Consequently, we have by Lemmas 3.5 and 3.4, the following lemma:

Lemma 4.3. *For a connected atlas $\mathcal{U}$ and any connected hypercovering $U_\bullet$ associated to it,*

$$\begin{array}{ccc} \mathrm{sh}(U_0) & \rightarrowtail & \mathrm{sh}(U_0/R_0) \\ \downarrow & & \downarrow \\ \mathrm{sh}(M) & \xrightarrow[\pi]{} & \mathrm{sh}(M, U_\bullet) \end{array} \tag{4.7}$$

is a push-out, and the geometric morphism $\pi : \mathrm{sh}(M) \to \mathrm{sh}(M, U_\bullet)$ is again connected. □

(The geometric morphism π is also locally connected.)

Thus, $\mathrm{sh}(M, U_\bullet)$ may be identified, via π^*, with a full subcategory of $\mathrm{sh}(M)$, and since the remaining parts of the diagram (4.7) do not depend on the choice of $U_\bullet$, it follows that $\mathrm{sh}(M, U_\bullet)$ only depends on the atlas $\mathcal{U}$ itself, not on the choice of hypercovering $U_\bullet$, as long as $U_\bullet$ is taken to be connected. Therefore, we may write $\mathrm{sh}(M, \mathcal{U})$ for $\mathrm{sh}(M, U_\bullet)$. It is an étendue, by Lemma 4.2. The objects of $\mathrm{sh}(M, \mathcal{U})$ we call *$\mathcal{U}$-sheaves.*

We already observed that for r locally connected, $\mathrm{sh}(M, r)$ is a full subcategory of $\mathrm{sh}(M)$, so we may compare it with the $\mathrm{sh}(M, \mathcal{U})$'s. It is clear from Proposition 1.1 that if the structure of r-sheaf on a sheaf E is given by an atlas $\mathcal{U}$, then $E \in \mathrm{sh}(M, \mathcal{U})$; and conversely, every $\mathcal{U}$-sheaf is an r-sheaf, so that $\mathrm{sh}(M, r)$ is the union of all the subcategories $\mathrm{sh}(M, \mathcal{U})$ as $\mathcal{U}$ ranges over the connected atlases for r. This union is actually a filtered one; for, any two connected atlases for r

have a common refinement, and it is easy to see that if $\mathcal{U}'$ refines $\mathcal{U}$, then $\mathrm{sh}(M, \mathcal{U}) \subseteq \mathrm{sh}(M, \mathcal{U}')$.

5. Simply connected maps and étendues

Let $f : Y \to X$ be a map of locales. We shall call f *simply connected* if

(i) f is connected (i.e. $f^* : \mathrm{sh}(X) \to \mathrm{sh}(Y)$ is full and faithful),

(ii) for every sheaf E on Y, if there exists an open cover $\bigcup U_i = Y$ of Y and sheaves D_i on X such that $E|_{U_i} \cong f^*(D_i)|_{U_i}$, then there exists a sheaf D on X such that $E \cong f^*(D)$.

Condition (ii) expresses that if a sheaf E on Y is locally in the image of f^*, then it is in the image of f^* (up to isomorphism). (The conditions together express the intuitive idea that f is a map with simply connected fibers, in a very weak way, but sufficient for our purposes in this paper. Surely for a general theory of simply connected maps, one should use a stronger notion, which is stable under pullback.)

Examples. (a) The unique map $Y \to 1$ is simply connected iff every locally constant sheaf on the locale Y is constant. In particular, if a path-connected topological space T is simply connected in the usual sense (defined in terms of paths), then the unique map $T \to 1$ is simply connected.

(b) If $T \to B$ is a locally connected map of topological spaces with connected and simply connected fibers (in the usual topological sense), then as a map of locales, f is simply connected in the sense just defined. (This is not trivial; a detailed proof is given in [10, Lemma 3.2] and [11].)

(c) The standard argument that a locally constant sheaf on the (localic) unit interval I is constant will (when applied internally in $\mathrm{sh}(X)$) show that the projection $X \times I \to X$ is simply connected, for every locale X.

(d) Let Y be a connected and locally connected locale, and suppose that the restriction map $Y^{\triangle} \to Y^{\partial\triangle}$ is a stable surjection (here $\triangle$ is the standard 2-simplex, and $\partial\triangle$ is its boundary). Then example (c) and [6, Lemma 3.4] show that the map $Y \to 1$ is simply connected.

(e) The previous example can be relativized: A connected and locally connected map of locales $f : Y \to X$ is simply connected, in the sense defined above, whenever $(Y^{\triangle})_X \to (Y^{\partial\triangle})_X$ is a stable surjection. (Here, for any locale A, $(Y^A)_X$ denotes the relative exponential 'of maps $A \to Y$ which become constant when composed with $f : Y \to X$', i.e. the locale defined by the pull-back diagram

$$\begin{array}{ccc} (Y^A)_X & \longrightarrow & Y^A \\ \downarrow & & \downarrow{\scriptstyle f^A} \\ X & \longrightarrow & X^A \end{array}$$

where the map $X \to X^A$ is the exponential adjoint of the projection map $X \times A \to A$.)

The notion of simply connected map given here is related to local equivalence relations in the following way. For any open map $f : Y \to X$, its kernel pair $\mathrm{Ker}(f) = Y \times_X Y \subseteq Y \times Y$ defines an equivalence relation on Y. The induced local equivalence relation on Y, given by the atlas consisting of the single chart $(Y, \mathrm{Ker}(f))$, is called the *local kernel pair* of f, and denoted $\mathrm{Lker}(f)$. If the map f is locally connected, then so is this local equivalence relation on Y, and we have a category $\mathrm{sh}(Y; \mathrm{Lker}(f))$, together with an evident factorization of $f^* : \mathrm{sh}(X) \to \mathrm{sh}(Y)$ through the forgetful functor $\mathrm{sh}(Y; \mathrm{Lker}(f)) \to \mathrm{sh}(Y)$.

The following is now obvious from the definition, and from Proposition 1.1.

Lemma 5.1. *A locally connected map $f : Y \to X$ is simply connected iff f^* induces an equivalence of categories* $\mathrm{sh}(X) \approx \mathrm{sh}(Y; \mathrm{Lker}(f))$. □

An equivalence relation R on a locale X is said to be *simply connected* if the quotient map $X \to X/R$ is a simply connected map. (If $d_0, d_1 : R \to X$ are locally connected, it can be shown that $X \to X/R$ is simply connected whenever d_0, d_1 are; but we will neither use nor prove this here.) Moreover, an atlas for a local equivalence relation is called simply connected if all its charts are; and a local equivalence relation r is called *locally simply connected* if every atlas for r can be refined by a simply connected atlas (this implies that r is locally connected).

Lemma 5.2. *Let r be a locally connected local equivalence relation on the locale M, and let $\mathcal{U}$ be a simply connected atlas for r. Then the inclusion functor* $\mathrm{sh}(M, \mathcal{U}) \to \mathrm{sh}(M, r)$ *is an equivalence of categories.*

Proof. The inclusion functor is a functor between full subcategories of $\mathrm{sh}(M)$, hence is full and faithful. To see that it is essentially surjective, consider a sheaf E on M with r-transport. We have to show that there exists an atlas for this r-transport with underlying r-atlas the given atlas $\mathcal{U}$. By the uniqueness of transport, this means that we have to show that for any chart (U, R) of $\mathcal{U}$, the restricted sheaf $E|_U$ is isomorphic to $\pi^*(D)$ for some sheaf D on U/R (where π is the quotient map $U \to U/R$). Since E has r-transport, there exists an atlas $\mathcal{V}$ for r, whose charts (V_j, R_j) act on $E|_{V_j}$, so for the given U, there exists a covering $\bigcup V_i$ of U such that for each index i there exist a sheaf D_i on V_i/R_i with $E|_{V_i} \cong \pi_1^*(D_i)$, where $\pi_i : V_i \to V_i/R_i$ is the quotient map. Let $\mu_i : V_i \to U$ and $\nu_i : V_i/R_i \to U/R$ be the inclusions, so that $\nu_i \pi_i = \pi\mu_i$. Then $D_i \cong \nu_i^* \nu_{i*}(D_i)$, so $E|_{V_i} \cong \pi_i^* \nu_i^* (\nu_{i*} D_i) \cong \mu_i^* \pi^*(\nu_{i*} D_i) \cong \pi^*(\nu_{i*} D_i)|_{V_i}$. Thus $E|_{V_i}$ is in the image of π^*, up to isomorphism. Since by assumption the quotient map $\pi : U \to U/R$ is simply connected, it follows that $E|_U$ is isomorphic to $\pi^*(D)$ for some sheaf D on U/R, as required. □

This lemma, together with Lemma 4.2, yields the following theorem:

Theorem 5.3. *Let r be a local equivalence relation on a locale M. If r is locally simply connected, then* sh(M, r) *is an étendue topos.* □

6. Maps from locales into étendues

Let $\mathcal{T}$ be a fixed étendue topos. In this section, we will show how for any locale M, a locally connected geometric morphism $a : \mathrm{sh}(M) \to \mathcal{T}$ gives rise to a local equivalence relation on M. Recall that for locally connected a, the inverse image functor a^* has a left adjoint $a_! : \mathrm{sh}(M) \to \mathcal{T}$.

Lemma 6.1. *The locale M has a basis of open sublocales $U \subseteq M$ with the property that $\mathcal{T}/a_!U$ is a localic topos.*

Proof. Let G be an object of $\mathcal{T}$ for which $\mathcal{T}/G$ is a localic topos, and construct the locale B by the pull-back

$$\begin{array}{ccc} \mathrm{sh}(B) & \longrightarrow & \mathcal{T}/G \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{sh}(M) & \xrightarrow[a]{} & \mathcal{T} \end{array}$$

Then, by construction of B, the topos $\mathrm{sh}(B)$ is equivalent over $\mathrm{sh}(M)$ to $\mathrm{sh}(M)/a^*(G)$. And q is induced by an étale map (a local homeomorphism) of locales, also denoted $q : B \to M$. The required basis for M consists of those open $U \subseteq M$ over which q has a section. Indeed, let $s : U \to B$ be a section of q. This section can be viewed as a map $s : U \to a^*(G)$ in $\mathrm{sh}(M)$, and hence corresponds by adjunction to a map $\hat{s} : a_!(U) \to G$ in $\mathcal{T}$. But then the topos $\mathcal{T}/a_!(U) = (\mathcal{T}/G)/\hat{s}$ is localic since $\mathcal{T}/G$ is. □

By the lemma, any open $U \subseteq M$ in this basis for M gives rise to a locale $a_\#(U)$ and a map $\varepsilon_U : U \to a_\#(U)$, for which there is an equivalence of topoi under $\mathrm{sh}(U)$, as in

$$\begin{array}{ccc} \mathrm{sh}(U) & \xrightarrow{\varepsilon_U} & \mathrm{sh}(a_\# U) \approx \mathcal{T}/a_!(U) \\ \downarrow & & \downarrow \\ \mathrm{sh}(M) & \longrightarrow & \mathcal{T} \end{array}$$

Notice that by construction, ε_U is a connected and locally connected map of locales. Thus, since connected locally connected maps are stable under pullback,

$$R_U := \mathrm{Ker}(\varepsilon_U) \subseteq U \times U$$

is a connected and locally connected equivalence relation on U. We shall prove the following:

Lemma 6.2. *The charts* (U, R_U), *for all open* $U \subseteq M$ *for which* $\mathcal{T}/a_!(U)$ *is localic, form an atlas for a local equivalence relation on* M.

We will call this local equivalence relation the *local kernel of* a, and denote it by $\mathrm{Lker}(a)$. (This is compatible with the similar notation used in Section 5.) Clearly $\mathrm{Lker}(a)$ is locally connected.

Proof. For two such open $V \subseteq U \subseteq M$, it is enough to show that $\mathrm{Lker}(\varepsilon_U)|_V = \mathrm{Lker}(\varepsilon_V)$. Consider the diagram

$$\begin{array}{ccc} V & \stackrel{i}{\hookrightarrow} & U \\ {\scriptstyle \varepsilon_V}\downarrow & & \downarrow{\scriptstyle \varepsilon_U} \\ a_\# V & \xrightarrow[a_\#(i)]{} & a_\# U \end{array}$$

obtained from the inclusion $i : V \subseteq U$. Since $\mathcal{T}/a_!V \to \mathcal{T}/a_!U$ is a map of slice topoi over $\mathcal{T}$, the corresponding map of locales $a_\# V \to a_\# U$ is étale. Thus

$$\begin{aligned} \mathrm{Lker}(\varepsilon_U)|_V &= \mathrm{Lker}(\varepsilon_U i) \qquad \text{(since } i \text{ is an open inclusion)} \\ &= \mathrm{Lker}(a_\#(i)\varepsilon_V) \\ &= \mathrm{Lker}(\varepsilon_V)\,, \end{aligned}$$

where the latter equality holds by the following lemma.

Lemma 6.3. *Let* $f : Y \to X$ *and* $e : X \to B$ *be maps of locales, where* e *is étale. Then* $\mathrm{Lker}(f) = \mathrm{Lker}(ef)$.

Proof. Consider an open $U \subseteq X$ such that $e|_U$ is a homeomorphism $U \cong e(U)$. Then

$$\begin{aligned} \mathrm{Ker}(f)|_{f^{-1}U} &= \mathrm{Ker}(f|_{f^{-1}U}) \\ &= \mathrm{Ker}(ef|_{f^{-1}U}) \quad \text{(since } e|_U \text{ is an embedding)} \\ &= \mathrm{Ker}(ef)|_{f^{-1}U}\,. \end{aligned}$$

Since this holds for all such U, $\mathrm{Ker}(f)$ and $\mathrm{Ker}(ef)$ agree on an open cover of Y. Hence $\mathrm{Lker}(f) = \mathrm{Lker}(ef)$. $\square$

This construction of the local equivalence relation $\mathrm{Lker}(a)$ on M from the locally connected geometric morphism $a : \mathrm{sh}(M) \to \mathcal{T}$ enjoys various naturality properties. We single out the following. Recall the #-construction of Section 2 for lifting a local equivalence back along an étale map. Then the following holds:

Lemma 6.4. *For a pull-back square*

$$\begin{array}{ccc} \mathrm{sh}(N) & \xrightarrow{b} & \mathcal{R} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{sh}(M) & \xrightarrow[a]{} & \mathcal{T} \end{array}$$

where a (and hence b) are locally connected geometric morphisms and p is an étale map between étendues, we have $q^{\#}(\mathrm{Lker}(a)) = \mathrm{Lker}(b)$.

Proof. Let $V \subseteq N$ be an open sublocale of N, so small that $q|_V$ is a homeomorphism $V \cong q(V)$, and moreover so small that both $\mathcal{R}/b_!V$ and $\mathcal{T}/a_!(q(V))$ are localic topoi. (Note that this property is inherited by smaller open sublocales.) Then there is an induced map $b_{\#}(V) \to a_{\#}(q(V))$ which is étale (in fact a homeomorphism) since p is. The result follows from Lemma 6.3. □

Remark 6.5 (which we shall not use). If r is a local equivalence relation on a locale M, and $\mathcal{U}$ is a connected atlas for r, then there is an induced geometric morphism $a : \mathrm{sh}(M) \to \mathrm{sh}(M, \mathcal{U})$, as in Section 4. The local equivalence relation $\mathrm{Lker}(a)$ is in general larger than r. It coincides with r if r has an atlas consisting of charts (U, R) with the property that R is the kernel pair of $U \to U/R$.

7. The main theorem

We now prove the result announced in the title of the paper:

Theorem 7.1. *For every étendue $\mathcal{T}$, there exists a locale M and a locally simply connected local equivalence relation r on M for which there exists an equivalence of topoi* $\mathrm{sh}(M, r) \approx \mathcal{T}$.

In the proof, we shall use the following construction from [6]: for any topos $\mathcal{E}$, there exists a locale $X = X_{\mathcal{E}}$ in $\mathcal{E}$ such that X is (internally) contractible and locally contractible, and moreover such that the topos $\mathcal{E}[X]$ of $\mathcal{E}$-internal sheaves on X is (externally) localic. In particular, this locale X has (internally in $\mathcal{E}$) a basis, containing X itself, and consisting of open $U \subseteq X$ which are connected and locally connected, and 'simply connected' in the sense that $U^{\triangle} \to U^{\partial\triangle}$ is a stable surjection of locales in $\mathcal{E}$ (cf. Example (d) in Section 5 for notation). Moreover, these properties of the internal locale X are stable under pull-back along an arbitrary geometric morphism $f : \mathscr{F} \to \mathscr{E}$. In the special case where $f : \mathscr{F} \to \mathscr{E}$ is such that both $\mathscr{F}$ and $\mathscr{F}[f^{\#}(X_{\mathscr{E}})]$ are localic, then $\mathscr{F}[f^{\#}(X_{\mathscr{E}})] \to \mathscr{F}$ corresponds to an (external) map of locales $b : B \to A$, and the stable internal properties of X just listed can be rephrased as follows: b is an open surjection, and B has a basis, containing B itself, which consists of open sublocales $U \subseteq B$ with the property that each restriction $b|_U : U \to b(U)$ satisfies the conditions for the map $f : T \to X$ in Example (e) in Section 5; in particular, each such restriction is a simply connected map.

For the proof of the theorem, consider for the given étendue $\mathscr{T}$ such a locale $X = X_{\mathscr{T}}$ in $\mathscr{T}$. Since $\mathscr{T}[X_{\mathscr{T}}]$ is localic, there is a locale M and a geometric morphism $a : \mathrm{sh}(M) \to \mathscr{T}$ for which $\mathrm{sh}(M) \approx \mathscr{T}[X_{\mathscr{T}}]$, over $\mathscr{T}$. Moreover, a is a connected and locally connected geometric morphism, since $X_{\mathscr{T}}$ is a connected and locally connected locale in $\mathscr{T}$.

Now let G be an object with full support in $\mathscr{T}$ for which $\mathscr{T}/G$ is a localic topos (such a G exists since $\mathscr{T}$ is an étendue). Thus there is a locale A and an étale surjection $p : \mathrm{sh}(A) \to \mathscr{T}$ such that $\mathscr{T}/G \approx \mathrm{sh}(A)$, over $\mathscr{T}$. It follows that the pull-back of p along $a : \mathrm{sh}(M) \to \mathscr{T}$ is again étale. Hence this pull-back is a localic topos, say $\mathrm{sh}(B)$ as in

$$\begin{array}{ccc} \mathrm{sh}(B) & \xrightarrow{b} & \mathrm{sh}(A) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{sh}(M) & \xrightarrow[a]{} & \mathscr{T} \end{array} \tag{7.1}$$

The local equivalence relation r on M in the statement of the theorem will be $\mathrm{Lker}(a)$, as constructed in Section 6. A comparison of the pushout squares in the following two lemmas will now prove the equivalence of topoi $\mathrm{sh}(M, r) \approx \mathscr{T}$ asserted in the theorem.

Lemma 7.2. *The pull-back square* (7.1) *of topoi is also a pushout square.*

Proof. The maps p and q are étale surjections, hence descent maps [7, 13]. In other words, $\mathrm{sh}(M)$ is obtained by descent from the simplicial topos of sheaves on the simplicial locale

$$B. = (\cdots B \times_M B \times_M B \rightrightarrows B \times_M B \rightrightarrows B),$$

and $\mathscr{T}$ is similarly obtained from the simplicial topos

$$\mathrm{sh}(A.) = (\cdots \mathrm{sh}(A) \times_{\mathscr{T}} \mathrm{sh}(A) \rightrightarrows \mathrm{sh}(A)).$$

Moreover, all the components of the simplicial map $\mathrm{sh}(B.) \to \mathrm{sh}(A.)$ are pullbacks of the map $a : \mathrm{sh}(M) \to \mathscr{T}$, hence are connected and locally connected. The lemma thus follows from Lemma 3.5. □

Lemma 7.3. *With A, B, etc. as above, there is a pushout square of topoi*

$$\begin{array}{ccc} \mathrm{sh}(B) & \xrightarrow{b} & \mathrm{sh}(A) \\ {\scriptstyle q}\downarrow & & \downarrow \\ \mathrm{sh}(M) & \longrightarrow & \mathrm{sh}(M, \mathrm{Lker}(a)) \end{array}$$

Proof. Note first that by the properties of the [6]-construction listed above, the map $b : B \to A$ is connected, locally connected, and simply connected. Hence by Lemma 5.1, the map $b : B \to A$ induces an equivalence of categories $\mathrm{sh}(A) \approx \mathrm{sh}(B, \mathrm{Lker}(b))$. In particular, the latter is a topos. Write $\mathscr{P}$ for the pushout of $\mathrm{sh}(B) \to \mathrm{sh}(B, \mathrm{Lker}(b))$ and $q : \mathrm{sh}(B) \to \mathrm{sh}(M)$. By the explicit description of pushouts of topoi given in Section 3, $\mathscr{P}$ is the category of triples (F, F', σ), where F' is a $\mathrm{Lker}(b)$-sheaf on B, F is a sheaf on M, and σ is an isomorphism $q^*(F) \cong F'$ of sheaves on B. In other words, $\mathscr{P}$ is (equivalent to) the category of sheaves F on M such that $q^*(F)$ is an $\mathrm{Lker}(b)$-sheaf. But, by Lemma 6.4, $\mathrm{Lker}(b) = q^{\#}\mathrm{Lker}(a)$, so by Remark 2.1, we conclude that the pushout $\mathscr{P}$ is equivalent to the category $\mathrm{sh}(M, \mathrm{Lker}(a))$. This proves the lemma. □

Note that we have proved that $\mathrm{sh}(M, \mathrm{Lker}(a))$ is a topos without invoking Theorem 5.3 and the (as yet unproved) fact that $\mathrm{Lker}(a)$ is locally simply connected, as we asserted in the theorem. To prove this fact, it suffices to show that $\mathrm{Lker}(b)$ is locally simply connected, since $q : B \to M$ is étale and $\mathrm{Lker}(b) = q^{\#}\mathrm{Lker}(a)$. But as explained above, the properties of the [6]-construction imply that B has a basis of open sublocales $U \subseteq B$ for which $U \to b(U)$ is a connected, locally connected, and simply connected map, so that $\mathrm{Ker}(b|_U)$ is a simply connected (and locally connected) equivalence relation on U. Since the collection of these $U \subseteq B$ form a basis for B, the local equivalence relation $\mathrm{Lker}(b)$ must be locally simply connected.

This completes the proof of the theorem. □

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