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ON 1-FORM CLASSIFIERS*

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We describe a notion of Fermat theory as a unifying concept for the algebraic theory of polynomials, and the algebraic theory of smooth functions, and we develop some of the commutative algebra for such, in particular concerning modules and derivations. The analysis of derivations, and construction of universal such is of course tied in with the notion of differential form. We describe two synthetic notions of differential 1-form, a "non-linear" and a "linear" one. The former is defined as a map defined on "the first neighbourhood of the diagonal" and vanishing on the diagonal. For "affine" objects, the linear 1-forms have a related description, using a "first neighbourhood inside the tangent bundle", which turns out to be isomorphic to the first neighbourhood of the diagonal. So the two 1-form notions agree for affine objects, and, in some sense, the first neighbourhoods classify 1-forms.

The results and constructions are then used to describe explicitly, in certain toposes, one object Ω which classifies

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(linear) 1-forms in another sense: 1-forms on any object M correspond to maps $M \to \Omega$.

§1. Fermat theories and their algebras

We remind the reader about some standard notation in connection with finitary algebraic theories $\mathbb{T}$ in the sense of [7]. The set of n-ary operations may be identified with $F(n)$, the free $\mathbb{T}$-algebra in n generators, these n generators being the n projections

$$\mathrm{proj}_i : n \to 1$$

making n into an n-fold product of 1. We may also denote the free $\mathbb{T}$-algebra in n generators by $K\{X_1,\ldots,X_n\}$ (or $K\{Y_1,\ldots,Y_n\}$, or $K\{Z_1,\ldots,Z_n\}$, say); in particular, $K = F(0) = \mathbb{T}(0,1)$. Thus $K\{X_1,\ldots,X_n\} = F(n) = \mathbb{T}(n,1)$, and $X_i = \mathrm{proj}_i$.

If $f \in \mathbb{T}(n,1)$ is an n-ary operation, the identification of $\mathbb{T}(n,1)$ with $K\{X_1,\ldots,X_n\}$ is given by $f \leftrightarrow f(X_1,\ldots,X_n)$ (or $f \leftrightarrow f(\underline{X})$, $\underline{X}$ denoting the n-tuple $X_1,\ldots,X_n$).

There is a bijective correspondence between arrows $f: n \to m$ in $\mathbb{T}$ and $\mathbb{T}$-algebra homomorphisms $f: K\{X_1,\ldots,X_m\} \to K\{X_1,\ldots,X_n\}$; to an arrow $n \to m$, i.e. to an m-tuple $f_1,\ldots,f_m$ of n-ary operations, we associate the unique $\mathbb{T}$-algebra-homomorphism $K\{X_1,\ldots,X_m\} \to K\{X_1,\ldots,X_n\}$ which sends $X_j \in K\{X_1,\ldots,X_m\}$ into $f_j(X_1,\ldots,X_n)$.

We denote the binary coproduct in the category $\mathbb{T}$-Alg of $\mathbb{T}$-algebras by the sign $\otimes$. For any $\mathbb{T}$-algebra B, the $\mathbb{T}$-algebra $B \otimes K\{X_1,\ldots,X_n\}$ solves the universal problem of adjoining n elements to B. It is consistent to use the notation

$$B\{X_1,\ldots,X_n\} := B \otimes K\{X_1,\ldots,X_n\},$$

with standard "renaming" of the X_i's if B itself is of the form $C\{X_1,\ldots,X_m\}$. In particular

$$K\{X_1,\dots,X_n\}\otimes K\{X_1,\dots,X_m\} = K\{X_1,\dots,X_{n+m}\}$$
$$= K\{X_1,\dots,X_n,Y_1,\dots,Y_m\}.$$

The algebraic theories we have in mind are given by, respectively,

$$\mathbb{T}(n,1) = \text{polynomials i n variables}$$

(over some fixed base ring), and

$$\mathbb{T}(n,1) = C^{\infty}(\mathbb{R}^n,\mathbb{R});$$

they are the algebraic theories of commutative algebras (over the base ring), and the algebraic theory $\mathbb{T}_\infty$ utilized for building models of synthetic differential geometry (cf. [2], [3], or [6] III). Since every polynomial with integral coefficients defines a smooth function, $\mathbb{T}_\infty$ contains the theory of commutative rings. It is, in fact, a Fermat-theory in the sense of the following definition. Likewise the theory of commutative rings itself is a Fermat theory.

Definition. Let $\mathbb{T}$ be an algebraic theory containing the theory of commutative rings. We say that $\mathbb{T}$ is a Fermat theory if for any $(n+1)$-ary operation

$$f = f(X,\underline{Z}) \in K\{X,Z_1,\dots,Z_n\},$$

there exists a unique $(n+2)$-ary operation

$$g = g(X,Y,\underline{Z}) \in K\{X,Y,Z_1,\dots,Z_n\}$$

such that

$$f(X+Y,\underline{Z}) = f(X,\underline{Z}) + Y\cdot g(X,Y,\underline{Z}) \in K\{X,Y,\underline{Z}\}. \tag{1.1}$$

For the case $\mathbb{T}_\infty$, the (unique) existence of such g (for the case $n = 0$) was known already by Fermat, who observed that, if $f: \mathbb{R} \to \mathbb{R}$ is a (smooth) function, the difference quotient g,

$$\frac{f(x+y) - f(x)}{y} =: g(x,y),$$

which à priori is only defined for $y \neq 0$, may be extended uniquely to a (smooth) function defined for all $(x,y) \in \mathbb{R}^2$. In rigourous form, and with parameters $\underline{z} \in \mathbb{R}^n$, this was observed by Hadamard.

An immediate consequence of the uniqueness assertion in the definition is that Y is <u>regular</u> (= multiplicatively cancellable) in $K\{X,Y,\underline{Z}\} = 0$:

$$(1,2) \qquad Y \cdot g(X,Y,\underline{Z}) = 0 \Rightarrow g(X,Y,\underline{Z}) = 0.$$

From the fact that $K\{Y\} \to K\{X,Y\}$ is injective then also follows that $Y \in K\{Y\}$ is regular. Hence any generator X_i in any free algebra $K\{X_1,\ldots,X_n\}$ $(n \geq 1)$ is regular.

<u>Proposition 1.1</u>. If $f = f(X_1,\ldots,X_n) \in K\{X_1,\ldots,X_n\}$, there exist $g_1,\ldots,g_n \in K\{X_1,\ldots,X_n,Y_1,\ldots,Y_n\}$ such that

$$(1.3) \qquad f(\underline{X}+\underline{Y}) = f(\underline{X}) + \sum_{i=1}^{n} Y_i \cdot g_i(\underline{X},\underline{Y}).$$

<u>Proof</u>. This is a simple n-fold iteration of (1.1):

$$\begin{aligned} f(\underline{X}+\underline{Y}) &= f(X_1+Y_1,\ldots,X_n+Y_n) \\ &= f(X_1,X_2+Y_2,\ldots,X_n+Y_n) + Y_1 \cdot g_1(\underline{X},\underline{Y}) \\ &= f(X_1,X_2,\ldots,X_n+Y_n) + Y_2 \cdot g_2(\underline{X},\underline{Y}) + Y_1 \cdot g_1(\underline{X},\underline{Y}) \\ &= \ldots . \end{aligned}$$

Note that the g_i's are not asserted to be unique. In fact, for $0 \in K\{X_1,X_2\}$, we have in $K\{X_1,X_2,Y_1,Y_2\}$:

$$0 = Y_1 \cdot Y_2 + Y_2 \cdot (-Y_1) = Y_1 \cdot 0 + Y_2 \cdot 0.$$

Since $\mathbb{T}$ is assumed to contain the algebraic theory of commutative rings, any $\mathbb{T}$-algebra B has â fortiori a commutative ring structure ("the underlying commutative ring of B").

Proposition 1.2. Let B be a $\mathbb{T}$-algebra. Let $I \subseteq B$ be an ideal in the usual ring-theoretic sense. Then the relation $b \sim c$ given by I ($b \sim c$ iff $b-c \in I$) is a $\mathbb{T}$-congruence.

Proof. Let $a_1 \sim b_1, \ldots, a_n \sim b_n$, and let $f \in \mathbb{T}(n,1)$. We should prove $f(a_1,\ldots,a_n) \sim f(b_1,\ldots,b_n)$. Let $h_i = b_i - a_i$, so by assumption, $h_i \in I$ $(i = 1,\ldots,n)$. We have

$$\begin{aligned} f(b_1,\ldots,b_n) &= f(a_1 + h_1,\ldots,a_n + h_n) \\ &= f(a_1,\ldots,a_n) + \Sigma h_i \cdot g_i(\underline{a},\underline{h}), \end{aligned}$$

where f and the g_i's are related as in Proposition 1.1. From this immediately follows $f(\underline{b}) - f(\underline{a}) \in I$.

It follows from the Proposition that $B/\sim$ has a unique $\mathbb{T}$-algebra structure such that $B \to B/\sim$ is a $\mathbb{T}$-homomorphism. But $B/\sim$ as a ring is just B/I, the usual residue class ring mod I. So whenever B is a $\mathbb{T}$-algebra, and I an ideal, we get a $\mathbb{T}$-algebra which we of course denote B/I since this is anyway its underlying commutative ring.

Note that any $\mathbb{T}$-algebra B comes equipped with a $\mathbb{T}$-homomorphism $K \to B$ (where $K = F(0)$); being in particular a ring-homomorphism, we see that any $\mathbb{T}$-algebra is a K-algebra in the usual sense of commutative algebra.

Recall that $\otimes$ denotes coproduct in $\mathbb{T}$-Alg. If $b \in B$, its image under the inclusion $B \to B \otimes C$ is denoted $b \otimes 1$. Similarly for $1 \otimes c$ where $c \in C$, and $b \otimes c$ denotes $(b\otimes 1) \cdot (1\otimes c)$. The usual tensor product $B \otimes_K C$ of B and C viewed as K-algebras maps into $B \otimes C$ by a canonical comparison map, and its image consists of finite sums $b_i \otimes c_i$ and will also be denoted $B \otimes_K C$. A general element of $B \otimes C$ is of form

$$\varphi(b_1 \otimes 1, \ldots, b_k \otimes 1,\ 1 \otimes c_{k+1}, \ldots, 1 \otimes c_n)$$

for suitable n, $b_i \in B$, $c_j \in C$, and $\varphi \in \mathbb{T}(n,1)$.

If $I \subseteq B$ is an ideal, $I \otimes C$ does not apriori make sense, since I is not a $\mathbb{T}$-algebra. However, if $I \otimes_K C$ denotes the usual tensor product of K-modules, we can let $I \otimes C$ denote the ideal in $B \otimes C$ generated by $I \otimes_K C$ under the canonical $B \otimes_K C \to B \otimes C$. Elements in $I \otimes C$ are finite sums $\Sigma(u_i \otimes 1) \cdot t_i$ with $u_i \in I$, $t_i \in B \otimes C$.

Proposition 1.3. Let $I \subseteq B$ be an ideal, and C some other $\mathbb{T}$-algebra. 1) The kernel of $B \otimes C \to (B/I) \otimes C$ is $I \otimes C$. 2) If $\beta : B \to C$ is a $\mathbb{T}$-homomorphism, then

$$\binom{\beta}{1}(I \otimes_K C) = \binom{\beta}{1}(I \otimes C).$$

(Here $\binom{\beta}{1} : B \otimes C \to C$ is the map from the coproduct which is induced by β and the identity map of C.)

Proof. 1) is a straightforward consequence of the relevant universal properties. To prove 2), let $t = \Sigma(u_i \otimes 1) \cdot t_i \in I \otimes C$, as above, with $u_i \in I$. Then $\binom{\beta}{1}(t) = \Sigma\beta(u_i) \cdot v_i$ where $v_i = \binom{\beta}{1}(t_i)$. But this in turn equals $\binom{\beta}{1}(\Sigma u \otimes v_i)$ and the sum here is in $I \otimes_K C$.

Proposition 1.4. Let A be any $\mathbb{T}$-algebra. Then the element X in $A \otimes K\{X\} = A\{X\}$ is regular.

Proof. We first consider the case where A is finitely generated, $q: K\{Y_1, \ldots, Y_m\} \twoheadrightarrow A$ (a surjective $\mathbb{T}$-homomorphism). Let $I \subseteq K\{Y_1, \ldots, Y_m\}$ be the kernel of q in the usual ring-theoretic sense, so $A = K\{Y_1, \ldots, Y_m\}/I$ as in the remarks following Proposition 1.2. The map

$$q \otimes 1 : K\{Y_1, \ldots, Y_m\} \otimes K\{X\} \twoheadrightarrow A \otimes K\{X\} = A\{X\}$$

has, by Proposition 1.3, its kernel equal to the ideal $\overline{I} = I \otimes K\{X\}$. So assume that $a \in A\{X\}$ has $X \cdot a = 0$. Let $a = (q \otimes 1)(f)$, where $f \in K\{\underline{Y}\} \otimes K\{X\} = K\{Y_1,\ldots,Y_m,X\}$. The assumption that $X \cdot a = 0$ then means

$$X \cdot f(Y_1,\ldots,Y_m,X) \in \overline{I} = I \otimes K\{X\},$$

so there exists elements $h_j = h_j(\underline{Y}) \in I$ and $k_j(\underline{Y},X) \in K\{\underline{Y},X\}$ (for $j = 1,\ldots,s$, say) so that

$$X \cdot f(\underline{Y},X) = \sum^{s} h_j(\underline{Y}) \cdot k_j(\underline{Y},X). \tag{1.4}$$

By the Fermat property, we may write

$$k_j(\underline{Y},X) = k_j(\underline{Y},0) + X \cdot \ell_j(\underline{Y},X),$$

so that, when substituting into (1.4), we have

$$X \cdot f(\underline{Y},X) = \sum^{s} h_j(\underline{Y}) \cdot k_j(\underline{Y},0) + X \cdot \sum^{s} h_j(\underline{Y}) \cdot \ell_j(\underline{Y},X).$$

Putting $X = 0$, we see from this that the first $\sum^{s}$ is 0, so that

$$X \cdot f(\underline{Y},X) = X \cdot \sum^{s} h_j(\underline{Y}) \cdot \ell_j(\underline{Y},X).$$

But X is regular in $K\{\underline{Y},X\}$, as we saw in connection with (1.2). So this equation implies

$$f(\underline{Y},X) = \sum h_j(\underline{Y}) \cdot \ell_j(\underline{Y},X).$$

Since $h_j \in I$, this proves $f \in I \otimes K\{X\} = \overline{I}$, so $q(f)$ $(= a)$ is zero. This proves the Proposition for finitely generated A. For arbitrary A, use that $A\{X\}$ is a filtered union of algebras $A_0\{X\}$ with A_0 finitely generated.

The addition operation $+ \in \mathbb{T}(2,1)$ corresponds, as described in generality in the beginning, to a map $\tilde{+}\colon F(1) \to F(2)$, i.e. $\tilde{+}\colon K\{X\} \to K\{X,Y\}$. For any $\mathbb{T}$-algebra A, $A \otimes \tilde{+}\colon A \otimes K\{X\} \to A \otimes K\{X,Y\}$

is a map

$$A\{X\} \xrightarrow{\tilde{+}} A\{X,Y\}.$$

For $a \in A\{X\}$, we write $a(X+Y)$ for $\tilde{+}(a)$. Similarly, we might write $a(X)$, respectively $a(Y)$, for the image of a under the inclusion $A\{X\} \to A\{X,Y\}$ given by $X \mapsto X$, respectively by $X \mapsto Y$; and $a \in A\{X\}$ may be denoted $a(X)$. This is consistent with the usage for the case $A = K$. We have

Proposition 1.5. For any $a \in A\{X\}$, there exists a unique $b \in A\{X,Y\}$ such that, in $A\{X,Y\}$

$$a(X+Y) - a(X) = Y \cdot b.$$

Proof. The uniqueness of b is immediate from the regularity of Y (Proposition 1.4). The existence follows, for the case where A is finitely generated, by considering $q: K\{Z_1,\ldots,Z_m\} \twoheadrightarrow A$, and considering the "pairwise" commutative diagram

$$\begin{tikzcd} K\{\underline{Z},X\} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "q\otimes 1"'] & K\{\underline{Z},X,Y\} \arrow[d, "q\otimes 1\otimes 1"] \\ A\{X\} \arrow[r, shift left] \arrow[r, shift right] & A\{X,Y\} \end{tikzcd}$$

where the two top horizontal maps are given by $X \mapsto X+Y$ and $X \mapsto X$, respectively, and similarly for the two bottom horizontal maps. Assume that $f(\underline{Z},X)$ goes to $a \in A\{X\}$ by $q \otimes 1$. Then

$$f(\underline{Z},X+Y) - f(\underline{Z},X) = Y \cdot g(\underline{Z},X,Y)$$

for a (unique) $g \in K\{\underline{Z},X,Y\}$, like in (1.1). Then clearly $(q \otimes 1 \otimes 1)(g)$ can be used for b in the desired conclusion. - If A is not finitely generated, write $A\{X\}$ as a filtered union of algebras $A_0\{X\}$ with A_0 finitely generated.

There are various ways in which Proposition 1.5 can be reformulated. One such way, involving categorical logic, is given and utilized in §4, expressing that "the generic $\mathbb{T}$-algebra" is a "Fermat ring". Another formulation is

Proposition 1.5'. If $\mathbb{T}$ is a Fermat theory, then so is $\mathbb{T}_A$, for an arbitrary $\mathbb{T}$-algebra A ($\mathbb{T}_A$ being the algebraic theory of the category $A \downarrow \mathbb{T}\text{-Alg}$; or, equivalently, $\mathbb{T}_A(n,1) = A\{X_1,\dots,X_n\}$).

We finish this § by deriving directly from the Fermat property of $\mathbb{T}$ some further "Taylor" or "Fermat" developments of the operations of $\mathbb{T}$.

Proposition 1.6. Let $f(\underline{X},\underline{Y}) \in K\{X_1,\dots,X_k,Y_1,\dots,Y_n\}$. Then there exist $h_1,\dots,h_n \in K\{\underline{X},\underline{Y}\}$ so that

$$f(\underline{X},\underline{Y}) = f(\underline{X},\underline{0}) + \sum^{n} Y_i \cdot h_i(\underline{X},\underline{Y}).$$

Proof. For $k=0$, this is a consequence of Proposition 1.1 (substitute 0 for $\underline{X}$ in there). For general k, it follows from the $k=0$ - case, applied to the algebraic theory $\mathbb{T}_{F(k)}$, which is a Fermat theory by Proposition 1.5' (note $\mathbb{T}_{F(k)}(n,1) = \mathbb{T}(k+n,1)$).

Proposition/Definition 1.7. To any $f \in K\{X_1,\dots,X_n\}$ there exist unique $k_1,\dots,k_n \in K\{X_1,\dots,X_n\}$ so that there exist $h_{ij} \in K\{X_1,\dots,X_n,Y_1,\dots,Y_n\}$ with

$$f(\underline{X}+\underline{Y}) = f(\underline{X}) + \sum_{i=1}^{n} Y_i \cdot k_i(\underline{X}) + \sum_{i,j=1}^{n} Y_i \cdot Y_j \cdot h_{ij}(\underline{X},\underline{Y}). \tag{1.5}$$

These $k_1(\underline{X}),\ldots,k_n(\underline{X})$ will be denoted $\frac{\partial f}{\partial x_1}(\underline{X}),\ldots,\frac{\partial f}{\partial x_n}(\underline{X})$.

Proof. By Proposition 1.1, we may find $g_1,\ldots,g_n \in K\{X_1,\ldots,X_n,Y_1,\ldots,Y_n\}$ so that

$$f(\underline{X}+\underline{Y}) = f(\underline{X}) + \sum Y_i \cdot g_i(\underline{X},\underline{Y}).$$

For each $i=1,\ldots,n$ we use Proposition 1.6 to write

$$g_i(\underline{X},\underline{Y}) = g_i(\underline{X},0) + \sum Y_j \cdot h_{ij}(\underline{X},\underline{Y}),$$

and substituting this in the expression for $f(\underline{X}+\underline{Y})$, we get the existence of the expansion (1.5) (with $k_i(\underline{X}) = g_i(\underline{X},\underline{0})$). To prove the uniqueness, assume we have (1.5)' like (1.5) but with k_i' and h_{ij}' instead of k_i and h_{ij}. To prove $k_1 = k_1'$, say, substitute $Y_2 = \ldots = Y_n = 0$ in (1.5) and (1.5)', and subtract, to get

$$0 = Y_1 \cdot (k_1(\underline{X}) - k_1'(\underline{X})) + Y_1^2 \cdot (h_{11}(\underline{X},Y_1,0,\ldots) - h_{11}'(\underline{X},Y_1,0,\ldots))$$

in $K\{X_1,\ldots,X_n,Y_1\}$. Since Y_1 is cancellable here by Proposition 1.4, we derive

$$0 = k_1(\underline{X}) - k_1'(\underline{X}) + Y_1 \cdot (\ldots);$$

now put $Y_1 = 0$ to deduce $k_1(\underline{X}) = k_1'(\underline{X})$. Similarly, $k_j(\underline{X}) = k_j'(\underline{X})$ for $j=2,\ldots,n$. This proves the Proposition. - By the notation introduced there, we may write

$$f(\underline{X},\underline{Y}) = f(\underline{X}) + \Sigma Y_i \cdot \frac{\partial f}{\partial x_i}(\underline{X}) + \sum Y_i \cdot Y_j \cdot h_{ij}(\underline{X},\underline{Y}).$$

Proposition 1.7 may be generalized; we remind the reader about usual conventions about multi-indices $\alpha,\beta,\ldots$ (see e.g. [6] I.5):

Proposition 1.7'. For any $f \in K\{X_1,\ldots,X_n\}$ and any integer $s \geq 0$, there exist unique $k_\alpha \in K\{X_1,\ldots,X_n\}$ $(|\alpha| \leq s)$ so that there exist $h_\beta \in K\{X_1,\ldots,X_n,Y_1,\ldots,Y_n\}$ $(|\beta| = s+1)$ with

$$f(\underline{X}+\underline{Y}) = \sum_{|\alpha| \leq s} \underline{Y}^\alpha \cdot k_\alpha(\underline{X}) + \sum_{|\beta| = s+1} \underline{Y}^\beta \cdot h_\beta(\underline{X},\underline{Y}).$$

Proof. The existence is a straightforward iteration of the existence assertion of Proposition 1.7. The uniqueness can be seen by a double induction in s and n, using the cancellation principle of Proposition 1.4.

The h_β's in this Proposition are not in general uniquely determined. The statements in [2] p.240 and in [6] p.218 about their uniqueness are incorrect.

From the uniqueness of the k_i (i.e. of the $\frac{\partial f}{\partial x_i}$) in Proposition 1.7, we easily conclude that if

$$\begin{tikzcd} & \mathbb{T}_0 \arrow[dl] \arrow[dr] & \\ \mathbb{T}_1 \arrow[rr, "G"'] & & \mathbb{T}_2 \end{tikzcd}$$

is a commutative triangle of homomorphisms of algebraic theories, where $\mathbb{T}_0$ is the theory of commutative rings, and if $\mathbb{T}_1$ and $\mathbb{T}_2$ are Fermat theories, then G preserves the formation of $\frac{\partial f}{\partial x_i}$.

Also, we note that the uniqueness assertions imply that for the theory $\mathbb{T}_0$ of commutative rings (whose arrows are polynomials with integral coefficients), the $\frac{\partial \varphi}{\partial x_i}$ is the usual formal derivative of the polynomial φ. In particular, if $\varphi(X_1,X_2) = X_1 + X_2$,

(1.7) $$\frac{\partial \varphi}{\partial x_1}(X_1,X_2) = 1, \qquad \frac{\partial \varphi}{\partial x_2}(X_1,X_2) = 1,$$

and if $\varphi(X_1,X_2) = X_1 \cdot X_2$,

$$(1.8) \qquad \frac{\partial\varphi}{\partial x_1}(x_1,x_2) = x_2, \qquad \frac{\partial\varphi}{\partial x_2}(x_1,x_2) = x_1.$$

For any commutative ring K, a Weil-algebra over K is a K-algebra whose underlying K-module is of form

$$K^s = \bigoplus_{i=1}^{s} K \qquad (s \text{ an integer } \geq 1),$$

and whose unit is $(1,0,\ldots,0)$ and where $K^{s-1} \subsetneq K^s$ (embedded on the last $s-1$ factors) in an ideal M with $M^s = 0$. (If K is a field, M is a maximal ideal, and the algebra is local, with K as residue field.)

The most important Weil-algebra over K is $K[\varepsilon] = K^2$, the "ring of dual numbers", where the multiplication is given by letting $(1,0)$ be the multiplicative unit, and letting $(0,1) = \varepsilon$ satisfy $\varepsilon^2 = 0$. Alternatively, $K[\varepsilon] = K[X]/(X^2)$.

If K, as above, is $\mathbb{T}(0,1)$ for a Fermat theory $\mathbb{T}$, Weil-algebras over K turn out to have a canonical structure of $\mathbb{T}$-algebras. More generally we have

Proposition 1.8. 1) Let W be a Weil-algebra over K, and let B be a $\mathbb{T}$-algebra. Then there exists a unique $\mathbb{T}$-algebra structure on $B \otimes_K W$ extending its K-algebra structure, and such that the canonical $j: B \to B \otimes_K W$ (= inclusion) becomes a $\mathbb{T}$-homomorphism. In particular, any Weil-algebra over K carries a canonical $\mathbb{T}$-algebra structure. With this,

$$B \otimes_K W = B \otimes W.$$

2) For any $\mathbb{T}$-algebra B and Weil-algebra W,

$$\hom_{\mathbb{T}\text{-Alg}}(B,W) = \hom_K(B,W); \qquad \hom_{\mathbb{T}\text{-Alg}}(W,B) = \hom_K(W,B).$$

3) If $K[X_1,\dots,X_n]/I$ is a Weil algebra, then the canonical K-algebra map

$$K[X_1,\dots,X_n]/I \to K\{X_1,\dots,X_n\}/\bar{I}$$

(where $\bar{I}$ is the ideal generated by the image of I under $K[X_1,\dots,X_n] \to K\{X_1,\dots,X_n\}$) is an isomorphism.

Proof. The proofs of these facts are given in [6], say, for the special Fermat theory $\mathbb{T}_\infty$ (where $K = \mathbb{R}$), cf. loc.cit. Theorem III.5.3, Exercise III.5.1, and Proposition III.5.11. The proofs for a general Fermat theory $\mathbb{T}$ are exactly the same, utilizing Proposition 1.7' instead of Proposition III.5.2 of [6].

We should also remark that if C is a $\mathbb{T}$-algebra, and W a Weil algebra, there exists a $\mathbb{T}$-algebra C^W such that

$$\hom_{\mathbb{T}\text{-Alg}}(C^W,B) = \hom_{\mathbb{T}\text{-Alg}}(C,B\otimes W),$$

for any $\mathbb{T}$-algebra B. The construction of A^* to be given in §2 is in fact a special case of this (with $W = K[\varepsilon]$).

§2. Modules and $\mathbb{T}$-derivations

Let $\mathbb{T}$ be a Fermat theory, and $K = \mathbb{T}(0,1)$, as in §1. Let A be a $\mathbb{T}$-algebra. By an A-module, we understand a module for the underlying ring of A.

Definition. If M is an A-module, we say that a map $d: A \to M$ is a $\mathbb{T}$-derivation if, for any $\varphi \in \mathbb{T}(n,1)$ and any $(a_1,\dots,a_n) \in A^n$

$$d(\varphi(a_1,\dots,a_n)) = \sum_{i=1}^{n} \frac{\partial\varphi}{\partial x_i}(a_1,\dots,a_n)\cdot d(a_i). \tag{2.1}$$

(Recall that $\frac{\partial\varphi}{\partial x_i}$ is a certain well-defined element of $\mathbb{T}(n,1) = K\{X_1,\ldots,X_n\}$, cf. Proposition/Definition 1.7.)

Proposition 2.1. A $\mathbb{T}$-derivation is in particular a derivation in the usual sense of ring/module theory; and it is K-linear.

Proof. To see that $d: A \to M$ is additive, apply (2.1) to the $\varphi \in \mathbb{T}(2,1)$ of (1.7); to prove the Leibniz rule, apply (2.1) to the $\varphi \in \mathbb{T}(2,1)$ of (1.8) to get

$$d(a_1 \cdot a_2) = a_2 \cdot d(a_1) + a_1 \cdot d(a_2). \tag{2.2}$$

Finally, to get K-linearity, let $k \in K = \mathbb{T}(0,1)$, and let us apply (2.1) to the nullary operation $\varphi = k$ to get

$$d(\varphi(\;)) = \sum_{i=1}^{0} = 0,$$

so that d takes value 0 on the image of $K \to A$. Then (2.2) implies K-linearity.

If A is a $\mathbb{T}$-algebra, and M an A-module, we may make $A \oplus M$ into a $\mathbb{T}$-algebra as follows (generalizing the construction of $A[\varepsilon]$). Denote $(1,m) \in A \oplus M$ by $a + \varepsilon m$; for $\varphi \in \mathbb{T}(n,1)$ put

$$\varphi(a_1 + m_1,\ldots,a_n + m_n) = \varphi(a_1,\ldots,a_n) + \varepsilon \sum \frac{\partial\varphi}{\partial x_i}(a_1,\ldots,a_n)\cdot m_i ; \tag{2.3}$$

the $\mathbb{T}$-algebra thus constructed we denote $A[\varepsilon M]$. (The verification that (2,3) actually does give a $\mathbb{T}$-algebra structure is as for $A[\varepsilon] = A[\varepsilon A]$, which in turn is an easy special case of the proof of Proposition 1.8.)

Proposition 2.2. Let A and B be $\mathbb{T}$-algebras, and let

M be an A-module. Consider a map $(f,g): B \to A \oplus M = A \times M$. Then this map is a $\mathbb{T}$-homomorphism $B \to A[\varepsilon M]$ if and only if $f: B \to A$ is a $\mathbb{T}$-homomorphism and $g: B \to M$ is a $\mathbb{T}$-derivation (where the B-module structure on M is defined via the A-module structure on M, and via the map $f: B \to A$).

Proof. This is a straightforward calculation, which we omit. - Thus, the notion of $\mathbb{T}$-derivation could have been derived from the notion "$A[\varepsilon M]$".

We have evident $\mathbb{T}$-homomorphisms

$$\alpha_0: A \to A[\varepsilon M] \qquad\qquad p: A[\varepsilon M] \to A$$
$$a \mapsto (a,0) \qquad\qquad (a,m) \mapsto a\,;$$

more generally, if $g: M \to N$ is an A-module homomorphism, we get a $\mathbb{T}$-homomorphism

$$\tilde{g}: A[\varepsilon M] \to A[\varepsilon N]$$
$$(a,m) \mapsto (a,g(m)),$$

and $\mathbb{T}$-homomorphisms $A[\varepsilon M] \to A[\varepsilon N]$ of this form are characterized by $\tilde{g} \circ \alpha_0 = \alpha_0$, $p \circ \tilde{g} = p$; namely g is reconstructed as $g(m) :=$ second component of $\tilde{g}(0,m)$.

Since the condition of being a $\mathbb{T}$-derivation $\delta: A \to M$ is purely equational, it is possible to deduce from general considerations that there exists to any $\mathbb{T}$-algebra A a universal $\mathbb{T}$-derivation $d: A \to \Omega_A$, i.e. any $\mathbb{T}$-derivation $\delta: A \to M$ factors through d via a unique A-module map $\Omega_A \to M$. We shall describe Ω_A explicitly, by a description which is well-known for the case where $\mathbb{T}$ is the theory of K-algebras.

Theorem 2.3. Let A be a $\mathbb{T}$-algebra. Consider the codiagonal $\nabla: A \otimes A \to A$, and let its kernel be I. Then

$$\begin{aligned} A &\longrightarrow I/I^2 \\ a &\longmapsto a \otimes 1 - 1 \otimes a \end{aligned}$$

is a universal derivation, where I/I^2 is made into an A-module by $a \cdot [s] = [1 \otimes a \cdot s]$ (square brackets denoting residue classes mod I^2). So

$$\Omega_A = I/I^2.$$

The proof of the Theorem will be given in §4, after certain synthetic/geometric considerations that at the same time provide the geometric meaning of the construction I/I^2.

In the case where A is a finitely generated $\mathbb{T}$-algebra, $A = K\{X_1,\ldots,X_n\}/((f_i)_{i\in I})$, we can give an alternative description of Ω_A; it is constructed out of the $\mathbb{T}$-algebra $K\{X_1,\ldots,X_n, Y_1,\ldots,Y_n\}$. Let J denote the ideal in $K\{X_1,\ldots,X_n\}$ generated by the f_i's, and let $\bar{J}$ denote the ideal generated by the $f_j(X_1,\ldots,X_n)$ in $K\{X_1,\ldots,X_n,Y_1,\ldots,Y_n\}$. Also, let dJ denote the ideal in $K\{X_1,\ldots,X_n,Y_1,\ldots,Y_n\}$ generated by

$$\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(X_1,\ldots,X_n)\cdot Y_i, \qquad j \in I.$$

Finally, let $(\underline{Y})^2$ denote the ideal in $K\{X_1,\ldots,X_n,Y_1,\ldots,Y_n\}$ generated by all $Y_i \cdot Y_j$. Consider the diagram

$$\begin{array}{ccccc} K\{\underline{X}\} & \longrightarrow & K\{\underline{X},\underline{Y}\} & \longrightarrow\!\!\!\!\rightarrow & K\{\underline{X},\underline{Y}\}/(\underline{Y})^2 \\ \downarrow & & \downarrow & & \downarrow \\ K\{\underline{X}\}/J & \longrightarrow & K\{\underline{X},\underline{Y}\}/(\bar{J}+dJ) & \longrightarrow & K\{\underline{X},\underline{Y}\}/(\bar{J}+dJ+(\underline{Y})^2), \\ = A & & = A^* & & = A^{\#} \end{array}$$

where $K\{\underline{X}\}$ denotes $K\{X_1,\ldots,X_n\}$, etc. Also, use square brackets to denote residue classes modulo ideals. Then

Theorem 2.4. The submodule of $A^{\#}$ generated by the classes of $Y_1,\dots,Y_n$ is Ω_A, with $d\colon A \to \Omega_A$ given by

$$d([\varphi(X_1,\dots,X_n)]) = [\sum_i \frac{\partial\varphi}{\partial x_i}(X_1,\dots,X_n)\cdot Y_i].$$

Furthermore $A^{\#} = A[\varepsilon\Omega_A]$ as a $\mathbb{T}$-algebra.

The proof of this theorem is likewise given after the synthetic considerations; see §7.

We proceed by describing some auxiliary results concerning "commutative algebra" of $\mathbb{T}$-algebras. We shall assume that K $(=\mathbb{T}(0,1))$ has the property that 2 is invertible. For an ideal I in a $\mathbb{T}$-algebra B, we may consider the ideal $I^2 \subseteq B$ as in commutative algebra (it consists of finite sums of products of form $b\cdot b'$ with $b \in I$, $b' \in I$). Since

$$b\cdot b' = \tfrac{1}{2}[(b+b')^2 - b^2 - b'^2],$$

we have, as in commutative algebra

Proposition 2.5. If $\varphi\colon B \to C$ is a $\mathbb{T}$-homomorphism, then φ annihilates I^2 iff it annihilates b^2 for all $b \in I$.

Proposition 2.6. Let $I \subseteq B$ be an ideal. Then we have the following equality of ideals in $B \otimes C$ (C any $\mathbb{T}$-algebra)

$$(I \otimes C)^2 = I^2 \otimes C.$$

Proof. To prove $(I \otimes C)^2 \subseteq I^2 \otimes C$, it suffices, by Proposition 2.5 to prove $s^2 \in I^2 \otimes C$ for any $s \in I \otimes C$. Let $s = \Sigma(u_i \otimes 1)\cdot t_i$ with $u_i \in I$. Then

$$s^2 = \sum_{ij}(u_i \otimes 1)\cdot(u_j \otimes 1)\cdot t_i\cdot t_j = \sum_{ij}(u_iu_j \otimes 1)\cdot t_i\cdot t_j,$$

but $u_iu_j \in I^2$ $\forall i,j$. - The other inclusion is clear.

Proposition 2.7. Let $I \subseteq B$ be an ideal, and $\beta: B \to C$ a $\mathbb{T}$-homomorphism. Then the following are equivalent:

i) β annihilates I

ii) $\binom{\beta}{1}: B \otimes C \to C$ annihilates $I \otimes_K C$

iii) $\binom{\beta}{1}: B \otimes C \to C$ annihilates $I \otimes C$.

Proof. Assume i). Since the composite $I \to B \to C$ is zero, then so is

$$I \otimes_K C \longrightarrow B \otimes_K C \xrightarrow{\beta \otimes_K C} C \otimes_K C \xrightarrow{\nabla} C$$

(∇ = codiagonal = multiplication map). But, clearly, the composite of the two last arrows here equals the composite

$$B \otimes_K C \longrightarrow B \otimes C \xrightarrow{\binom{\beta}{1}} C,$$

so that this latter composite takes $I \otimes_K C$ to zero. Conversely, assume ii). Then i) follows by observing that

$$\binom{\beta}{1}(b \otimes 1) = \beta(b),$$

and $b \otimes 1 \in I \otimes_K C$ if $b \in I$. Finally, (ii) $\Leftrightarrow$ (iii) by Proposition 1.3.

The three last propositions may be used to derive

Proposition 2.8. Let $I \subseteq B$ be an ideal, and $\beta: B \to C$ a $\mathbb{T}$-homomorphism. Then the following three conditions are equivalent:

i) β annihilates I^2

ii) $\binom{\beta}{1}: B \otimes C \to C$ takes every element of $I \otimes C$ into an element of square zero

iii) For every $\mathbb{T}$-homomorphism $\varepsilon: C \to C'$, the homomorphism $\binom{\varepsilon\circ\beta}{1}: B \otimes C' \to C'$ takes every element of $I \otimes C'$ into an element of square zero.

Proof. We first prove i) $\Leftrightarrow$ ii). By Proposition 2.5, condition ii) is equivalent to

$$\binom{\beta}{1}((I \otimes C)^2) = 0,$$

which in turn, by Proposition 2.6, is equivalent to

$$\binom{\beta}{1}(I^2 \otimes C) = 0.$$

This, by Proposition 2.7 (applied to I^2) is equivalent to i). - To prove i) $\Rightarrow$ iii), if β annihilates I^2, then so does $\varepsilon\circ\beta$ for any $\varepsilon: C \to C'$; now apply the implication "i) $\Rightarrow$ ii)" for the case of $\varepsilon\circ\beta$. Finally iii) $\Rightarrow$ ii) is trivial.

§3. First neighbourhood of the diagonal

We remind the reader of some of the basic notions of synthetic differential geometry (or, consult [6]): one considers a commutative ring object R in a topos $\mathcal{E}$, about which we talk as if it were the category of sets. For every natural number $n \geq 1$, we consider $D(n) \subseteq R^n$ defined by

$$D(n) := [\![(x_1,\dots,x_n) \in R^n \mid x_i . x_j = 0 \quad \forall i,j]\!]$$

We assume that R satisfies the following form of "Axiom 1" for synthetic differential geometry:

"For every $n \geq 1$, the canonical map

$$R^{n+1} \xrightarrow{\alpha} R^{D(n)}$$

with description

$$\alpha(a_0, a_1, \dots a_n) = [(x_1,\dots,x_n) \mapsto a_0 + \sum_{i=1}^{n} a_i . x_i]$$

is invertible",

or, verbally

"Every map $D(n) \to R$ extends uniquely to an affine map $R^n \to R$".

(For the verbal formulation to be equivalent to the statement about invertibility of α via the semantics of categorical logic, as exposed in [6] Part II, say, it should be formulated with " $\vdash_1 \forall f \in R^{D(n)} \dots$" , cf footnote p. 3 in [6]).

<u>Definition</u>. We say that $\underline{x} \in R^n$, $\underline{y} \in R^n$ are 1-<u>neighbours</u> if $\underline{x}-\underline{y} \in D(n)$. We then write $\underline{x} \sim_1 \underline{y}$, (or just $\underline{x} \sim \underline{y}$).

Note that since $D(n)$ is not stable under addition in R^n, the neighbour relation thus defined is not transitive. However, it is symmetric and reflexive. It has been studied and utilized in [4].

The following Proposition yields equivalent descriptions of it, which will allow us to generalize it to a relation defined on arbitrary objects in E. We let $\Delta: R^n \to R^n \times R^n$ denote the diagonal map; also it assumed throughout the present § that 2 is invertible in R.

Proposition 3.1. Let $\underline{x}=(x_1,\ldots,x_n)$ and $\underline{y}=(y_1,\ldots y_n)\in R^n$. Then the following conditions are equivalent

i) $\underline{x}\sim_1\underline{y}$, i.e. $(\underline{x}-\underline{y})\in D(n)$, i.e. $(x_i-y_i)\cdot(x_j-y_j)=0\ \forall_{i,j}$

ii) $\forall f\colon R^n\times R^n\to R$ with $f\circ\Delta\equiv 0$: $(f(\underline{x},\underline{y}))^2=0$

iii) $\forall g\colon R^n\to R$: $(g(\underline{x})-g(\underline{y}))^2=0$

iv) $\forall\varphi\colon R^n\to R$ with φ homogeneous: $(\varphi(\underline{x}-\underline{y}))^2=0$.

(Recall that a map φ between R-modules is homogeneous if $\varphi(\lambda\cdot\underline{v})=\lambda\cdot\varphi(\underline{v})\quad\forall\lambda\in R\quad\forall\underline{v}$.

Proof. Assume i), so write $\underline{y}=\underline{x}+\underline{d}$ with $\underline{d}\in D(n)$. Then

$$f(\underline{x},\underline{y})=f(\underline{x},\underline{x}+\underline{d})=f(\underline{x},\underline{x})+\sum d_i\cdot\frac{\partial f}{\partial y_i}(\underline{x},\underline{x}),$$

by Taylor's formula (which holds for 'increments' $\underline{d}\in D(n)$, by the assumed form of "Axiom 1" which also allows us to define the partial derivatives in question, cf. e.g. [6] §§ I.1-I.4). By assumption, $f(\underline{x},\underline{x})=0$, and the sum Σ has square zero because $d_i\cdot d_j=0\quad\forall i,j$. This proves ii). Assume ii), and let $g\colon R^n\to R$ be given. Define $f\colon R^n\times R^n\to R$ by $f(\underline{s},\underline{t}):=g(\underline{s})-g(\underline{t})$. It has $f\circ\Delta\equiv 0$, so apply the assumption ii) on this f to get iii). Assume iii), and let $\varphi\colon R^n\to R$ be homogeneous. Then by [6], Proposition I.10.2, φ is R-linear, so $\varphi(\underline{x}-\underline{y})=\varphi(\underline{x})-\varphi(\underline{y})$. So apply the assumption iii) with $g=\varphi$. Finally, assume iv). First, consider the homogeneous $\mathrm{proj}_i\colon R^n\to R$ to conclude $(x_i-y_i)^2=0$, and similarly for x_j-y_j. Then consider the homogeneous $\mathrm{proj}_i+\mathrm{proj}_j$ to conclude

$$0=((x_i-y_i)+(x_j-y_j))^2=(x_i-y_i)^2+(x_j-y_j)^2+2(x_i-y_i)\cdot(x_j-y_j).$$

The two first terms vanish, by what has alreasy been proved. Hence so does the third, and since 2 was assumed invertible, we conclude $(x_i - y_i)\cdot(x_j - y_j) = 0$. This proves i).

We note that the equivalence of ii) and iii) is of more general nature: the domain R^n could be replaced by an arbitrary object M:

<u>Proposition 3.2</u>. For any object M, and any $\underline{x}$, $\underline{y} \in M$, the following are equivalent

ii) $\forall f\colon M\times M \to R$ with $f\circ\Delta \equiv 0$: $(f(\underline{x},\underline{y}))^2 = 0$

iii) $\forall g\colon M \to R$, $(g(\underline{x}) - g(\underline{y}))^2 = 0$

<u>Proof</u>. The proof of ii) $\Rightarrow$ iii) is as above. Conversely, assume iii) and let $f\colon M\times M \to R$ be given with $f\circ\Delta \equiv 0$. Define $g\colon M \to R$ by $g(\underline{s}) := f(\underline{x},\underline{s})$. Then

$$0 = (g(\underline{x}) - g(\underline{y}))^2 = (f(\underline{x},\underline{x}) - f(\underline{x},\underline{y}))^2 = (-f(\underline{x},\underline{y}))^2,$$

since $f(\underline{x},\underline{x}) = 0$.

<u>Definition 3.3</u>. For any object M in E and any $\underline{x},\underline{y} \in M$, we say that $\underline{x}$ and $\underline{y}$ are <u>1-neighbours</u>, and write $\underline{x} \sim_1 \underline{y}$, if the two equivalent conditions in Proposition 3.2 hold.

This is consistent with the previous definition because of Proposition 3.1. We immediately note that $\sim_1$ is symmetric and reflexive. And it is functorial: if $h\colon M \to N$ is any map, then

$$\underline{x} \sim_1 \underline{y} \text{ in } M \text{ implies } h(\underline{x}) \sim_1 h(\underline{y}) \text{ in } N.$$

We define $M_{(1)} \subseteq M\times M$ by

(3.1) $$M_{(1)} := [\![(x,y) \in M\times M \mid \underline{x} \sim_1 \underline{y}]\!] ;$$

because $\sim_1$ is reflexive, $M_{(1)}$ contains the diagonal $\Delta: M \to M\times M$, and we call it the <u>first neighbourhood of the diagonal</u>. By the functorality of $\sim_1$, $M \mapsto M_{(1)}$ is a functor $E \to E$.

<u>Remark</u>. The definition of $\sim_1$, given above for general M works for our purposes (because we are mainly interested in affine objects), but is not "morally" correct in general. For instance, in the context of algebraic geometry, if M is is projective n-space, the object R^M of R-valued functions on M is $\cong R$, so that $x \sim_1 y$ for <u>all</u> x,y in M. A correct version of Definition 3.3 would say: $x \sim_1 y$ iff there exists and "open" $U \subseteq M$ containing x and y, such that for all $g: U \to R$, $(g(x)-g(y))^2 = 0$. The word "open" here would then have to be interpreted in a suitable way. Some "open-ness" notions are discussed in [6] I§17 and I§19. Other, more intrinsic ones are found in recent works of Penon (cf. [9]).

Leaving the naive (set theoretic) way of talking about E aside for a moment, now that the main object of study, namely $M_{(1)}$, has been introduced, we put

<u>Definition 3.4</u>. A <u>1-form on</u> M is a map in E

$$\omega: M_{(1)} \to R$$

with $\omega\circ\Delta \equiv 0$ (where $\Delta: M \to M_{(1)} \subseteq M\times M$ is the diagonal). The <u>set</u> of all 1-forms on M is denoted $\Lambda^1(M)$.

We could of course also have defined an internal object in E "of 1-forms on M"; its set of global sections would then be $\Lambda^1(M)$, but for the purpose of the present article, we shall not do this; see remarks at the end of §8.

Because of the functorality of $M_{(1)}$ in M, and the naturality of Δ, we immediately get that $M \mapsto \Lambda^1 M$ is a contravariant functor $E \to \underline{\text{Set}}$: if $h: M \to N$, and if ω is a 1-form on N, we get a 1-form $h^*\omega$ on M by putting $(h^*\omega)(x,y) := \omega(h(x),h(y))$ for $x \sim_1 y$ in M.

We have a map d ($=d_M$, natural in M)

$$(3.2) \qquad \hom_E(M,R) \xrightarrow{d} \Lambda^1 M$$

given by

$$g \longmapsto [(x,y) \longmapsto g(y)-g(x)],$$

where $x \sim_1 y$ is in M. (Note that x and y denote generalized (or parametrized) elements of M, cf. [6] II, whereas g denotes an ordinary element of $\hom_E(M,R)$.)

All this generalizes (a special case of) the 1-form notion considered by Bkouche and Joyal, and by the second author [5].

Clearly, $\Lambda^1 M$ carries the structure of abelian group, by valuewise addition, $(\omega+\varphi)(x,y) = \omega(x,y)+\varphi(x,y)$. We may even make it a module over the ring $\hom_E(M,R)$, by putting

$$(3.3) \qquad (g\cdot\omega)(x,y) = g(y)\cdot\omega(x,y)$$

for $g \in \hom_E(M,R)$, $\omega \in \Lambda^1(M)$. This algebraic structure is natural in an obvious sense. (For a 1-form ω on a general object M, it is not clear whether in general $\omega(x,y) \in D$ $(x \sim y)$, nor whether the lack of symmetry in the definition (3.3) is only apparent.) However, for the affine objects M to be considered later, it is so.)

It is immediate to prove that d in (3.2) above is additive. Further, with the $\hom_E(M,R)$ module structure defined on $\Lambda^1 M$, it is even a derivation:

$$(3.4) \qquad d(g_1 \cdot g_2) = g_1 \cdot dg_2 + g_2 \cdot dg_1 .$$

For, if $x \sim_1 y$ in M

$$d(g_1 \cdot g_2)(x,y) = g_1(y) \cdot g_2(y) - g_1(x) \cdot g_2(x)$$

whereas

$$(g_1 \cdot dg_2 + g_2 \cdot dg_1)(x,y) = g_1(y) \cdot g_2(y) - g_1(y) \cdot g_2(x) + g_2(y) \cdot g_1(y) - g_2(y) \cdot g_1(x).$$

The difference between these two expressions is by simple calculation

$$(3.5) \qquad (g_2(y) - g_2(x)) \cdot (g_1(y) - g_1(x)) .$$

But $(g_1, g_2): M \to R \times R$ preserves the relation $\sim_1$ so that, by Proposition 3.1

$$(g_1(x) - g_1(y), g_2(x) - g_2(y)) \in D(2),$$

and so (3.5) is zero.

We have, however, a stronger result (Theorem 3.8 below), under an assumption satisfied in most good models:

<u>Definition 3.5</u>. We say that the ring object R is a <u>Fermat ring</u> if we have validity of

$$(3.6) \qquad \forall f \in R^R \quad \exists! \; g \in R^{R \times R} : \forall x, y \in R \times R \qquad f(x+y) = f(x) + y \cdot g(x,y).$$

(This notion has been studied in [11].)

<u>Proposition 3.6</u>. If R is a Fermat ring, then the algebraic theory $\mathbb{T}_R$ given by $\mathbb{T}_R(n,1) = \hom_E(R^n, R)$ is a Fermat theory.

Proof. The functor $(-)_M = -\times M : E \to E/M$ (for any $M \in E$) preserves algebraic structure, formation of exponential objects, as well as logical structure. So if R is a Fermat ring in E, then so is R_M in E/M. Apply this to the case $M = R^p$ to get validity of (3.6) for R_{R^p}; this reinterprets in E as validity of: $\forall f: R^p \times R \to R \ \exists! \ g: R^p \times R \times R \to R$ s.t. ... which is precisely the property of $\mathbb{T}_R$ required.

Proposition 3.7. If R satisfies Axiom 1, and is also a Fermat ring, then, for any $f: R^n \to R$ and any $i = 1,\dots,n$, the $\frac{\partial f}{\partial x_i} : R^n \to R$ defined using Axiom 1 agrees with the ones arising from the Fermat property of $\mathbb{T}_R$.

Proof. For simplicity of notation, we do the case $n=1$ only. Then $f': R \to R$ arising from the Fermat property is characterized by validity of

$$\forall x,y : f(x+y) = f(x) + y \cdot f'(x) + y^2 \cdot g(x,y)$$

for some $g: R^2 \to R$ (cf. Proposition 1.7). In particular

$$\forall x \in R \ \forall d \in D : f(x+d) = f(x) + d \cdot f'(x).$$

But this characterizes the f' defined from Axiom 1.

Theorem 3.8. Assume R satisfies Axiom 1 (as above) and is a Fermat ring. Then for any $M \in E$, the map

$$d: \hom_E(M,R) \to \Lambda^1 M$$

is a $\mathbb{T}_R$-derivation.

Proof. Let $\varphi: R^n \to R$, and consider $\underline{g} = (g_1, \ldots, g_n): M \to R^n$. Then, for $s \sim_1 t$ in M, $\underline{g}(t) = \underline{g}(s) + \underline{d}$ for some $\underline{d} = (d_1, \ldots, d_n) \in D(n)$, by Proposition 3.1 and the fact that $\underline{g}$ preserves the relation $\sim_1$. So

$$\begin{aligned} d(\varphi \circ \underline{g})(s,t) &= \varphi(\underline{g}(t)) - \varphi(\underline{g}(s)) = \varphi(\underline{g}(t)) - \varphi(\underline{g}(t) - \underline{d}) \\ &= \sum d_i \cdot \frac{\partial \varphi}{\partial x_i}(\underline{g}(t)) \end{aligned}$$

by Taylor's formula (available because of Axiom 1). On the other hand

$$d_i = g_i(t) - g_i(s) = (dg_i)(s,t),$$

so that the sum above becomes

$$\sum \frac{\partial \varphi}{\partial x_i} \underline{g}(t) \cdot dg_i(s,t) = \sum \left(\frac{\partial \varphi}{\partial x_i}(\underline{g}) \cdot dg_i \right)(s,t)$$

proving the Theorem.

§4. A topos model

Let $\mathbb{T}$ be a Fermat theory such that 2 is invertible in the commutative ring $K = \mathbb{T}(0,1)$. Let A be the category of finitely generated $\mathbb{T}$-algebras (= $\mathbb{T}$-algebras of form $K\{X_1, \ldots, X_n\}/I$ (I some ideal)). (It could also be taken to be some other suitable category of $\mathbb{T}$-algebras, stable under those constructions we perform; the notion of "internal affine variety", to be given below must then be modified accordingly. - In particular, it could be the category of finitely presented $\mathbb{T}$-algebras.)

We let E be the topos $\underline{\text{Set}}^A$ (= category of covariant functors from A so $\underline{\text{Set}}$). We have the Yoneda embedding

$$A^{op} \xrightarrow{y} \text{Set}^A = E.$$

It turns $K\{X\} \in A$ into a $\mathbb{T}$-algebra object in E (namely the forgetful functor $A \to \underline{\text{Set}}$), which we denote R . By the fullness of y ,

$$\hom_E(R^n,R) = \hom_A(K\{X\},K\{X_1,\dots,X_n\}) = K\{X_1,\dots,X_n\}$$
$$=\mathbb{T}(n,1),$$

which one may express by saying that the full algebraic theory $\mathbb{T}_R$ of R is just the given $\mathbb{T}$.

The category E should be thought of as a category of "geometric" or smooth" objects. For the case where $\mathbb{T}$ is the theory of commutative rings, it is a category whose use is well established in algebraic geometry, cf. e.g. [1], or [6] III.1. Any $\mathbb{T}$-algebra $C \in A$ represents a geometric object, namely $y(C) \in E$, which we also denote $\overline{C}$. The $\mathbb{T}$-algebra C can be reconstructed as the set of functions (maps in E) from $\overline{C}$ to R; for,

$$\hom_E(\overline{C},R) = \hom_E(y(C),y(K\{X\})) = \hom_A(K\{X\}, C) = C,$$

the middle equality sign by Yoneda. So $\mathbb{T}$-algebras will now appear as sets of R-valued functions on geometric objects; and modules will appear as subsets of such function sets, determined by further equational conditions.

More generally, functions $f: \overline{C} \to R^{\overline{B}}$ correspond by exponential adjointness to maps $\overline{C}\times\overline{B} = \overline{C\otimes B} \to R$, i.e. to elements $\check{f} \in C\otimes B$. Thinking of f as a generalized element (cf.[6], Part II) of the functionspace object $R^{\overline{B}}$, and if $b: \overline{C} \to \overline{B}$ is a generalized element of $\overline{B}$ (defined at same stage, and corresponding under y to a $\mathbb{T}$-algebra map $\check{b}: B \to C$), then we have the generalized element

$$f(b) \in_{\overline{C}} R,$$

thus $f(b): \overline{C} \to R$, and it corresponds to an element $(f(b))^{\vee} \in C$;

we have

$$(f(b))^{\vee} = \binom{1}{b^{\vee}}(f^{\vee})$$

where $\overset{\vee}{f} \in C\otimes B$, and where $\binom{1}{b^{\vee}}: C\otimes B \to C$ is that map from the coproduct, which has the identity of C , and $b^{\vee}$, as its two components. We shall generally omit the $^{\vee}$ from notation, and just use terminology like

$$\text{"}f \in_{\overline{C}} R^{\overline{B}} \qquad \text{reinterprets as} \quad f \in C\otimes B\text{"}$$

and

$$\text{"}f(b) \in_{\overline{C}} R \quad \text{reinterprets as} \quad \binom{1}{b}(f) \in C\text{"}$$

Being a $\mathbb{T}$-algebra , R is in particular a K-algebra , and the following Proposition refers to that structure.

Proposition 4.1. The K-algebra R satisfies Axiom 1^{W}_{K} (cf. [6] I.16), and it is a Fermat ring. In particular, it satisfies the form of Axiom 1 utilized in §3.

Proof. Axiom 1^{W}_{K} is proved like Corollary III. 1.3 in [6] (using Proposition 1.8 3) to derive Axiom 1^{W}_{K} from Axiom 2_K , to use the terminology of loc. cit.). To prove that R is a Fermat ring, let $f \in_{\overline{C}} R^{R}$. Then f reinterprets as $f \in C\otimes K\{X\} = C\{X\}$, and the unique existence to be proved reinterprets as proving unique existence of $g \in C\{X,Y\}$ so that

$$(4.1) \qquad f(X+Y) = f(X) + Y\cdot g(X,Y) \quad \text{in} \quad C\{X,Y\}.$$

But we know by Proposition 1.5 that $\mathbb{T}_C$ is a Fermat theory, which implies, for the given $f \in \mathbb{T}_C(1,1) = C\{X\}$ the unique existence of $g \in \mathbb{T}_C(2,1) = C\{X,Y\}$, as in (4.1).

The point about working in $Set^A = E$ instead of in $A^{op} \subseteq Set^A$ is that E is a topos, so that any logical formula $\varphi(x)$ about elements of an object M has an <u>extension</u>:

$$(4.2) \qquad [\![x \in M \mid \varphi(x)]\!] \subseteq M \ ;$$

this is not true for A^{op} , unless φ is purely equational, in general. The following key proposition deals with an object of form (4.2) with a φ which is not purely equational, and it so happens that, for the special case dealt with in this Proposition, the extension in question <u>is</u> in A^{op} (i.e is representable) if M is.

Let B be a finitely generated $\mathbb{T}$-algebra , and $I \subseteq B$ an ideal. Denote B/I by A , and the surjective homomorphism $B \to B/I = A$ by q . It goes by y to a monic $\bar{q}: \bar{A} \rightarrowtail \bar{B}$. - In the following proposition, our main interest is assertion ii), assertion i) being included for completeness.

<u>Proposition 4.2.</u> We have the following equality of subobjects of $\bar{B}$:

$$\text{(i)} \qquad \bar{A} \ (= \overline{B/I}) = [\![b \in \bar{B} \mid \forall f \in R^{\bar{B}}: f/_{\bar{A}} = 0 \Rightarrow f(b) = 0]\!]$$

Likewise, we have the following equality of subobjects of $\bar{B}$:

$$\text{(ii)} \qquad \overline{B/I^2} = [\![b \in \bar{B} \mid \forall f \in R^{\bar{B}}: f/_{\bar{A}} \equiv 0 \Rightarrow (f(b))^2 = 0]\!] \ .$$

<u>Proof.</u> Let $\beta \in_C \bar{B}$. It reinterprets as a $\mathbb{T}$-homomorphism $\beta: B \to C$. To prove the equality of the two subobjects in (i) is tantamount to proving, for any such β , the equivalence of the following (4.3) and (4.4):

(4.3) β factors across $B \to A$, i.e. β annihilates I

and

(4.4) for every $\varepsilon: C \to C'$ and every $f \in B\otimes C'$ which goes to 0 by $q\otimes C'$, $\binom{\varepsilon 0 \beta}{1}(f) = 0$.

Similarly, to prove the equality of the two subobjects in (ii) is tantamount to proving, for any such β, the equivalence of the following (4.5) and (4.6):

(4.5) β annihilates I^2

and

(4.6) for every $\varepsilon: C \to C'$ and every $f \in B\otimes C'$ which goes to 0 by $q\otimes C'$, $\left(\binom{\varepsilon 0 \beta}{1}(f)\right)^2 = 0$.

The equivalence of the assertions (4.5) and (4.6) was proved in Proposition 2.8, noting that the assumption on f is equivalent to $f \in I \otimes C$ by Proposition 1.3. The proof of the equivalence of (4.3) and (4.4) is similar.

The Proposition will now be applied to the case where $B = A\otimes A$ and $q: A\otimes A \to A$ the codiagonal. Then $\overline{B} = \overline{A}\times\overline{A}$, and $\overline{q}$ is the diagonal $\Delta: \overline{A} \to \overline{A}\times\overline{A}$. Note that in this case, the subobject of $\overline{B} = \overline{A}\times\overline{A}$ described on the right hand side of (ii) in the Proposition is $\overline{A}_{(1)}$. So we conclude

<u>Corollary 4.3</u>. For a finitely generated $\mathbb{T}$-algebra, the subobject $\overline{A}_{(1)}$, i.e.

$$[\![(a_1,a_2) \in \overline{A}\times\overline{A} \mid \forall g \in R^{\overline{A}} : (g(a_2)-g(a_1))^2 = 0]\!]$$

equals y applied to $A\otimes A \to (A\otimes A)/I^2$, where I is the kernel of the codiagonal $A\otimes A \to A$.

To state one further Corollary of Proposition 4.2., let us agree to call two $\mathbb{T}$-homomorphisms $a_1, a_2 : A \to B$ 1-close to each other, if for every $\alpha \in A$ $(a_1(\alpha)-a_2(\alpha))^2 = 0$. Let A and B be in A.

Corollary 4.4. Let $a_1, a_2 : A \longrightarrow B$. Consider the following conditions:

i) a_1, a_2 are 1-close

ii) $a_1 \otimes C, a_2 \otimes C : A\otimes C \longrightarrow B\otimes C$ are 1-close for any C ("a_1 and a_2 are stably 1-close")

iii) $(a_1, a_2) \in_{\bar{B}} \bar{A}_{(1)}$.

We have (iii) $\Longleftrightarrow$ (ii) $\Rightarrow$ (i) (and, for the purely algebraic case, also (i) $\Rightarrow$ (ii)).

Proof. Assume (ii). To prove (iii) means

$$\vdash_{\bar{B}} \forall g \in R^{\bar{A}} : (g(a_1) - g(a_2))^2 = 0.$$

So let $\beta : \bar{C} \to \bar{B}$ be a later stage, and $g \in_{\bar{C}} R^{\bar{A}}$. Then β and g reinterpret as

$$\beta : B \to C \quad \text{and} \quad g \in A\otimes C, \tag{4.7}$$

and the conclusion $(g(a_1) - g(a_2))^2$ reinterprets as saying that the difference between the $\mathbb{T}$-homomorphismus $\binom{\beta\circ a_i}{1} : A\otimes B \to C$ $(i = 1,2)$ takes g into an element of square zero in C. However,

$$\binom{\beta\circ a_i}{1} = \binom{\beta}{1} \circ (a_i \otimes C), \quad i = 1,2; \tag{4.8}$$

since $a_1\otimes C$ and $a_2\otimes C$ are 1-close, their difference takes g into an element of square zero. Since $\binom{\beta}{1}$ is a ring homomorphism (being a $\mathbb{T}$-homomorphism), the composites in (4.8) have the same property as well.

Assume (iii). We first prove (i), let $\alpha \in A$. Now

$$(a_1(\alpha) - a_2(\alpha))^2 = \left(\binom{a_1}{a_2}(\alpha\otimes 1 - 1\otimes\alpha)\right)^2$$
$$= \binom{a_1}{a_2}((\alpha\otimes 1 - 1\otimes\alpha)^2);$$

but $(\alpha\otimes 1 - 1\otimes\alpha)^2 \in I^2$, which is annihilated by $\binom{a_1}{a_2}$, by the assumption and Corollary 4.3, so (i) is proved. However, since (iii) is a synthetic statement, its validity is preserved by the functor $(-)_{\bar{C}} = -\times\bar{C}: \mathcal{E} \to \mathcal{E}/\bar{C}$. The argument just carried out for iii) $\Rightarrow$ i) is valid in $\mathcal{E}/\bar{C}$ for the Fermat theory $\mathbb{T}_C$ as well, and $(-)_{\bar{C}}$ take a_1, a_2 to elements which reinterpret as $a_1\otimes C$, $a_2\otimes C$, which thus are 1-close, by i) applied in $\mathcal{E}/_{\bar{C}}$. We omit the proof of the parenthetical remark in the Corollary.

The following analysis now provides a proof of Theorem 2.3. for the case of a finitely generated A. By Corollary 4.3, a map $\omega: \bar{A}_{(1)} \to R$ in $\mathcal{E}$ reinterprets as an element $\bar{\omega}$ of $A\otimes A/I^2$. To say $\omega/\Delta \equiv 0$ is equivalent to saying that $\bar{\omega}$ belongs to the subset $I/I^2 \subseteq A\otimes A/I^2$. Thus, recalling the Definition 3.4 of 1-forms, we see that the set $\Lambda^1(\bar{A})$ of 1-forms on $\bar{A}$, i.e. the set of those $\omega: \bar{A}_{(1)} \to R$ which vanish on $\Delta: \bar{A} \rightarrowtail \bar{A}_{(1)} \subseteq \bar{A}\times\bar{A}$ is identified with the set I/I^2. Together with identifications of the type $\hom_{\mathcal{E}}(\bar{C},R) = C$, it fits into the following communative diagram, in which only the map $d: A \to I/I^2$ has still to be described:

$$\begin{array}{ccccccc}
\hom_{\mathcal{E}}(\bar{A},R) & \longrightarrow & \Lambda^1(\bar{A}) \subseteq \hom_{\mathcal{E}}(\bar{A}_{(1)},R) & & \twoheadleftarrow & \hom_{\mathcal{E}}(\bar{A}\times\bar{A},R) \\
\big| \wr & & \big| \wr \qquad\qquad \big| \wr & & & \big| \wr \\
A & \xrightarrow[d]{\;-\,-\,-\;} & I/I^2 \quad \subseteq \quad (A\otimes A)/I^2 & & \twoheadleftarrow & A\otimes A.
\end{array}$$

The unique map d (dotted arrow) which will make the left hand square commutative, is given by $a \longmapsto 1\otimes a - a\otimes 1$. To see this, it suffices to see the commutativity of

$$\begin{array}{ccc} \hom_E(\bar{A},R) & \longrightarrow & \hom_E(\bar{A}\times\bar{A},R) \\ \wr\wr & & \wr\wr \\ A & \xrightarrow[a \mapsto 1\otimes a - a\otimes 1]{} & A\otimes A \ . \end{array}$$

Now $g \mapsto g\circ proj_2$ corresponds to $a \mapsto 1\otimes a$, and similarly for $proj_1$, so $g \mapsto g(y)-g(x)$ $(=dg(x,y))$ corresponds to $a \mapsto 1\otimes a - a\otimes 1$, since the identifications preserve the ring structure.

Furthermore, we have seen in §3 that $\Lambda^1(\bar{A})$ carries a $\hom(\bar{A},R)$-module structure, given by

$$(g\cdot\omega)(x,y) = g(y)\cdot\omega(x,y).$$

The unique A-module structure on I/I^2 corresponding to this under the identification is easily analyzed as the one with

$$a\cdot(t+I^2) = (1\otimes a)\cdot t + I^2$$

for $a \in A$, $t \in I \subseteq A\otimes A$. From the fact (Theorem 3.8) that d is a $\mathbb{T}_R$ $(=\mathbb{T})$ derivation, it therefore follows that d is a $\mathbb{T}$-derivation. We shall finally prove that d is a <u>universal</u> $\mathbb{T}$-derivation into A-modules. Let V be a finitely generated A $(=\hom(\bar{A},R))$ - module, and let $\delta: A \to V$ be a $\mathbb{T}$-derivation. Then we get a $\mathbb{T}$-homomorphism $\alpha_1: A \to A[\varepsilon V]$, $\alpha_1(a) = a+\varepsilon\cdot\delta(a)$ (cf. Proposition 2.2). We also have a $\mathbb{T}$-homomorphism $\alpha_0: A \to A[\varepsilon V]$, $\alpha_0(a) = a+\varepsilon\cdot 0$, corresponding to the zero derivation. Note $A[\varepsilon V]$ is finitely generated. Clearly α_0 and α_1 are 1-close. In fact, they are <u>stably</u> 1-close. To see this let C be a $\mathbb{T}$-algebra. Any element t in $A\otimes C$ can be written

$$t = \varphi(a_1,\dots,a_k,c_{k+1},\dots,c_n)$$

(with a_i short for $a_i\otimes 1$, c_j for $1\otimes c_j$, $a_i \in A$, $c_j \in C$), for some $\varphi \in \mathbb{T}(n,1)$. Then

$$(\alpha_1 \otimes C)(t) = \varphi(a_1+\varepsilon\cdot\delta(a_1),\ldots,a_k+\varepsilon\cdot\delta(a_k),\ c_{k+1},\ldots,c_n)$$
$$= \varphi(a_1,\ldots,a_k,c_{k+1},\ldots,c_n)+\sum_{i=1}^{k}\frac{\partial\varphi}{\partial x_i}(a_1,\ldots,a_k,c_{k+1},\ldots c_n)\cdot\varepsilon\cdot\delta(a_i).$$

Also $(\alpha_0 \otimes C)(t)$ equals the first term in this expression, so the difference $\alpha_1 \otimes C - \alpha_0 \otimes C$ takes t to the $\sum^k$-sum here, which has square zero because of the ε's.

Since α_0 and α_1 are stably 1-close, we get from Corollary 4.4 that

$$\binom{\alpha_0}{\alpha_1}: A\otimes A \longrightarrow A[\varepsilon V]$$

annihilates I^2, so factors across the projection $A\otimes A \to A\otimes A/I^2$ via a map $\hat{\delta}: A\otimes A/I^2 \to A[\varepsilon V]$. The composite of $\binom{\alpha_0}{\alpha_1}$ with the projection $p: A[\varepsilon V] \to A$ is the codiagonal, so takes I into 0, so that $\binom{\alpha_0}{\alpha_1}$ takes I into the submodule $\varepsilon V \subseteq A[\varepsilon V]$. It follows that $\hat{\delta}$ maps I/I^2 into $\varepsilon V \cong V$, and clearly it is an A-module map. To see

$$\hat{\delta}\circ d = \delta, \tag{4.9}$$

let $a \in A$. Then $d(a) = 1\otimes a - a\otimes 1$, so

$$\binom{\alpha_0}{\alpha_1}(d(a)) = \alpha_1(a)-\alpha_0(a)$$
$$= (a+\varepsilon\cdot\delta(a))-a = \varepsilon\cdot\delta(a),$$

so since $\hat{\delta}$ was defined as the restriction of $\binom{\alpha_0}{\alpha_1}$, (4.9) follows. We now prove the uniqueness of a map $\hat{\delta}$ which makes (4.9) hold for the given δ. If $\hat{\delta}'$ has $\hat{\delta}'\circ d = \delta$, we may subtract to get an A-module map $\nu: I/I^2 \to V$ with $\nu\circ d = 0$; we want to prove $\nu = 0$. Let

$$s = \varphi(a_1\otimes 1,\ldots,a_k\otimes 1,\ 1\otimes a_{k+1},\ldots,1\otimes a_n) \in I \subseteq A\otimes A$$

Since $a_i\otimes 1 - 1\otimes a_i = -da_i$, this may be written

$$s = \varphi(1\otimes a_1 - da_1, \dots, 1\otimes a_k - da_k, 1\otimes a_{k+1}, \dots, 1\otimes a_n).$$

Then we have by Proposition/Definition 1.7 that

$$s \equiv \varphi(1\otimes a_1, \dots, 1\otimes a_n) - \sum_{i=1}^{k} \frac{\partial\varphi}{\partial x_i}(1\otimes a_1, \dots, 1\otimes a_n)\cdot da_i \mod I^2$$

since $da_i \in I$. Since $s \in I$, $\varphi(a_1, \dots, a_n) = 0$; so also

$$\varphi(1\otimes a_1, \dots, 1\otimes a_n) = 1\otimes\varphi(a_1, \dots a_n) = 0$$

so

$$s \equiv -\sum_{i=1}^{k} \frac{\partial\varphi}{\partial x_i}(1\otimes a_1, \dots, 1\otimes a_n)\cdot da_i = -\sum_{i=1}^{k} (1\otimes \frac{\partial\varphi}{\partial x_i}(a_1, \dots, a_n))\cdot da_i \mod I^2.$$

Since ν vanishes on I^2, and is an A-module map

$$\nu(s) = -\sum \frac{\partial\varphi}{\partial x_i}(a_1, \dots, a_n)\cdot \nu(da_i)$$

(recalling how the A-module structure was defined on I/I^2). But this is 0 since $\nu\circ d$ was assumed to be 0.

We have proved that $d: A \to I/I^2$ is a universal $\mathbb{T}$-derivation with respect to finitely generated A-modules. But since a $\mathbb{T}$-derivation $\delta: A \to V$ into any A-module factors through the submodule generated by the $\delta(a_1), \dots, \delta(a_m)$ (where $a_1, \dots, a_m$ is a set of $\mathbb{T}$-algebra generators for A), it follows that d has the universal property for derivations into <u>any</u> A-module. This proves Theorem 2.3 for the case A finitely generated.

For a general $\mathbb{T}$-algebra, write $A = \varinjlim A_i$, a filtered union of finitely generated $\mathbb{T}$-algebras. Then $I = \varinjlim(I_i)$, $I^2 = \varinjlim(I_i{}^2)$, $I/I^2 = \varinjlim I_i/I_i{}^2$, so the result easily follows from the finitely generated case.

We shall in a later § need the following Lemma (which in a terminology to be introduced there, expresses that R is <u>soft</u>).

<u>Lemma 4.5</u>. If $\beta: B \twoheadrightarrow A$ is a surjective $\mathbb{T}$-homomorphism in A, then, in $E = \underline{Set}^{A}$, $R^{i}: R^{\overline{B}} \to R^{\overline{A}}$ (where $\overline{A} \rightarrowtail \overline{B}$ is $\overline{\beta}$) is epic.

<u>Proof</u>. It suffices to show that for any $C \in A$, $R^{\overline{B}}(C) \to R^{\overline{A}}(C)$ is surjective. This map reinterprets as the map $\beta\otimes A: B\otimes C \to A\otimes C$, so the Lemma just amounts to : β surjective $\Rightarrow$ $\beta\otimes C$ surjective. Now in the category of $\mathbb{T}$-algebras (for any algebraic theory), surjective is the same as regular epi, and regular epis in any category are preserved by taking coproduct with a fixed object.

<u>§5. Linear algebra and the $\mathcal{D}$-construction</u>

Let F be a topos, and R a commutative ring object in it, satisfying "Axiom 1", as in §3. In the application, F will be E/M, where M is a suitably nice object in another topos E, and R will be $\tilde{R} \times M \to M$, where $\tilde{R}$ is a ring object in E with good properties. In the present §, we shall talk about F as if it were the category of sets.

We shall be dealing with R-module objects V (typically, $V = R^{m}$, or, for the case $F = E/M$, the tangent bundle $TM \to M$ of M), but many properties of these modules depend only on that part of the structure on V which consists in multiplication by elements ("scalars") from R, so we find it convenient to put

Definition. An R-object is an object V with a specified (global) element $\underline{0}: 1 \to V$, and with an associative action of $(R,\cdot)$,

$$R \times V \to V$$

denoted $(\lambda,\underline{v}) \longmapsto \lambda\underline{v}$, such that $0\underline{v} = \underline{0}$ and $\lambda\underline{0} = \underline{0}$ for all $\underline{v} \in V$ and $\lambda \in R$. A map $\varphi: V \to V'$ between two R-objects is homogeneous if $\lambda\varphi(\underline{v}) = \varphi(\lambda\underline{v})$ for all $\underline{v} \in V$ and $\lambda \in R$.

Examples include R^n, as well as the $D(n)$ of §3 (with $E = F$). Furthermore, if V is an R-object, the following sub-object $\mathcal{D}(V)$ of it is also an R-object:

$$\mathcal{D}(V) := [\![\underline{v} \in V \mid (\varphi(\underline{v}))^2 = 0 \text{ for all homogeneous } \varphi: V \to R]\!].$$

(The condition here must be read: "$\forall\varphi \in R^V$: φ homogeneous $\Rightarrow$ $(\varphi(\underline{v}))^2 = 0 \ \forall\underline{v}$" in order to yield the correct interpretation for the semantics of categorical logic, [6],II).

Clearly, $\mathcal{D}$ is functorial with respect to homogeneous maps $V \to V'$. Also, for all $d \in D = [\![x \in R \mid x^2 = 0]\!]$, and all $\underline{v} \in V$, $d\underline{v} \in \mathcal{D}(V)$. In particular, $\underline{0} \in \mathcal{D}(V)$. From the equivalence i) $\Leftrightarrow$ iv) in Proposition 3.1, we see that

$$\mathcal{D}(R^n) = D(n). \tag{5.1}$$

For any 0-preserving map $\omega: \mathcal{D}(V) \to R$, and any $\underline{v} \in V$, the map $D \to R$ given by $d \mapsto \omega(d\underline{v})$ is of form $d \mapsto \lambda d$ for some unique $\lambda \in R$, by Axiom 1. We denote this λ by $\lambda = \hat{\omega}(\underline{v})$. This defines a map $\hat{\omega}: V \to R$, characterized by the equation

$$\omega(d\underline{v}) = d\hat{\omega}(\underline{v}) \qquad \forall d \in D,\ \underline{v} \in V.$$

(Note that the uniqueness assertion in Axiom 1 for $n = 1$ may be stated: $\forall \lambda \in R \ ((\forall d \in D: d\lambda = 0) \Rightarrow \lambda = 0)$, which we express verbally by saying "universally quantified d's may be cancelled"; cf. e.g. [6], §I.1.)

Proposition 5.1. $\hat{\omega}: V \to R$ is homogeneous for all 0-preserving $\omega: \mathcal{D}(V) \to R$.

Proof. $d \cdot \hat{\omega}(\lambda v) = \omega(d\lambda \underline{v}) = d\lambda\hat{\omega}(\underline{v}) \ \forall d \in D$. Cancelling the universally quantified d, we get $\hat{\omega}(\lambda \underline{v}) = \lambda\hat{\omega}(\underline{v})$, for any $\lambda \in R$, $\underline{v} \in V$.

Proposition 5.2. a) Given any 0-preserving $\omega: \mathcal{D}(V) \to R$, we have $\hat{\omega}(d\underline{v}) = \omega(d\underline{v}) \quad \forall \underline{v} \in V \quad \forall d \in D$.

b) Given a pair of homogeneous maps $\varphi, \varphi': V \to R$, we have

$$(\forall \underline{v} \in V \quad \forall d \in D: \quad \varphi(d\underline{v}) = \varphi'(d\underline{v})) \Rightarrow (\varphi = \varphi').$$

Proof..For a), $\hat{\omega}(d\underline{v}) = d\hat{\omega}(\underline{v}) = \omega(d\underline{v})$. For b)

$$d\varphi(\underline{v}) = \varphi(d\underline{v}) = \varphi'(d\underline{v}) = d\varphi'(\underline{v})$$

for all $d \in D$, so the result follows by cancelling the universally quantified d.

This shows that homogeneous maps are characterized by their value on "vectors" of form $d\underline{v}$, and that $\hat{\omega}$ extends ω for vectors of this form. The following Proposition gives equivalent conditions on V for $\hat{\omega}$ to extend ω for all elements in $\mathcal{D}(V)$.

Proposition 5.3. The following conditions on an R-object V are equivalent:

i) all 0-preserving maps $\omega: \mathcal{D}(V) \to R$ are homogeneous

ii) given any pair ω, ω' of maps $\mathcal{D}(V) \to R$, then

$$(\forall \underline{v} \in V \quad \forall d \in D\colon \omega(d\underline{v}) = \omega'(d\underline{v})) \Rightarrow (\omega = \omega')$$

iii) for any 0-preserving $\omega\colon \mathcal{D}(V) \to R$, $\hat{\omega}$ extends ω.

iv) "$\mathcal{D}(V)$ classifies homogeneous maps", i.e., to any 0-preserving $\omega\colon \mathcal{D}(V) \longrightarrow R$, there exists a unique homogeneous $\varphi\colon V \to R$ extending ω.

Proof. Assume i). The condition in ii) implies $\omega(0) = \omega'(0)$, so we may as well assume both ω and ω' zero-preserving; the conclusion in ii) now follows by Proposition 5.2b). Next ii) implies iii) by Proposition 5.2a); iii) implies iv) by Proposition 5.2b) and 5.1. Finally, iv) $\Rightarrow$ i) is clear.

We shall need some general concepts which refer to the fixed ring object R. An internal affine variety is a subobject of some R^n, carved out by a family of equations, i.e. a joint equalizer of a family of maps $R^n \to R$.

Also, we say that a monic map $M \rightarrowtail N$ has the extension property if the induced $R^N \longrightarrow R^M$ is epic. Talking about the topos F as if it were the category of sets, this reads: "every map $M \to R$ extends to a map $N \to R$", whence the name "extensions property".

Finally, we shall say that R is soft if every internal affine variety $M \rightarrowtail R^n$ has the extension property.

Note that if A is the category of finitely generated $\mathbb{T}$-algebras, and $E = \mathrm{Set}^A$ (as in §4), then the internal affine varieties are exactly the representables, i.e. objects of form $\bar{B}$ where $B \in A$, and $\bar{B}$ is an internal finitary affine variety iff B is finitely presented. If A = finitely presented $\mathbb{T}$-algebras, the internal finitary affine varieties are exactly the representables.

We now present some general theory about R-modules V, where R is a Fermat-ring object in the sense of §3. We first remark that if R is a Fermat ring in F, then the algebraic theory $\mathbb{T}$ defined by

$$\mathbb{T}(n,1) = \hom_F(R^n,R)$$

is a Fermat theory (with $K = \hom_F(1,R)$). We even have the stronger result that all the "Taylor" developments of §1 (e.g. Proposition 1.7 and 1.7') are true when interpreted internally; thus in the logical notation of [6], II, we have

$$\Vdash_1 \forall f \in R^{R^n} \quad \exists!\, k_1,\ldots,k_n \in R^{R^n} \quad \exists h_{ij} \in R^{R^{2n}} \quad :$$

$$\forall \underline{x} \in R^n,\ \underline{y} \in R^n\colon\ f(\underline{x}+\underline{y}) = f(\underline{x}) + \Sigma y_i \cdot k_i(\underline{x})$$

$$+ \Sigma y_i y_j h_{ij}(\underline{x},\underline{y}).$$

This is due to the fact that the property of being a Fermat ring is preserved by logical functors. We need this <u>internal</u> validity of Proposition 1.7, because this is precisely what allows us to use it as if F were the category of sets.

<u>Proposition 5.4</u>. Let $V \subset R^n$ be a submodule, and suppose the inclusion has the extension property. Then every homogeneous $\varphi\colon V \to R$ extends to a homogeneous $R^n \to R$.

<u>Proof</u>. Let $\varphi\colon R^n \to R$ be an extension of the given φ; such exists by the assumed extension property, but the extended φ is not necessarily homogeneous, except on V. Since $\varphi(0) = 0$,

we may use the Fermat property, as expressed in Proposition/ Definition 1.7, to write

$$\varphi(\underline{x}) = \sum_i \alpha_i \cdot x_i + \sum_{ij} h_{ij}(\underline{x}) \cdot x_i \cdot x_j \quad \forall \underline{x} = (x_1, \dots, x_n) \in R^n.$$

For any $\underline{x} \in V$, we have $\lambda\varphi(\underline{x}) = \varphi(\lambda\underline{x})$ so that

$$\lambda(\Sigma\alpha_i \cdot x_i + \Sigma h_{ij}(x)\ x_i \cdot x_j) = \Sigma\alpha_i \lambda x_i + \Sigma h_{ij}(\lambda\underline{x})\lambda x_i \lambda x_j$$

$\forall \underline{x} \in V$. Subtracting, the linear terms cancel, and we are left with

$$\lambda \cdot [\Sigma h_{ij}(\underline{x}) \cdot x_i \cdot x_j - \lambda h_{ij}(\lambda\underline{x}) \cdot x_i \cdot x_j)] = 0, \qquad \forall \underline{x} \in V .$$

Fix $\underline{x} \in V$, and consider the square bracket as a function $f_{\underline{x}}(\lambda)$ of λ. We then have

$$\forall \lambda:\ \lambda \cdot f_{\underline{x}}(\lambda) = 0 \quad .$$

The uniqueness assertion (applied to 0) in the assumption that R is a Fermat ring then gives $f_{\underline{x}}(\lambda) \equiv 0$. In particular $f_{\underline{x}}(0) = 0$. So

$$\Sigma h_{ij}(\underline{x}) \cdot x_i \cdot x_j = 0 \qquad \forall \underline{x} \in V.$$

Therefore, $\varphi(\underline{x})$ and $\Sigma\alpha_i x_i$ agree on V, and the latter is clearly a linear function defined on all of R^n. This proves the Proposition.

<u>Corollary 5.5</u>. If $V \subseteq R^n$ is a submodule with the extension property, then

$$\mathcal{D}(V) = \mathcal{D}(R^n) \cap V \qquad (= D(n) \cap V).$$

Proof. The inclusion $\subseteq$ is obvious, so let $\underline{v} \in D(n) \cap V$, and let $\varphi: V \to R$ be homogeneous. By the Proposition, φ extends to a homogeneous $\varphi: R^n \to R$, and then $(\varphi(\underline{v}))^2 = 0$ since $\underline{v} \in \mathcal{D}(R^n)$.

§6. First neighbourhood of the zero section of the tangent bundle

We remind the reader of some of the category theory involved in the relationship between E and $F = E/M$. First, we have the functor $(\ )_M: E \longrightarrow E/M$ given by

$$X_M = (M \times X \xrightarrow{\text{proj}} M)$$

It is a logical functor, and in particular, it preserves exponentiation, and it has adjoints on both sides, denoted Σ_M and Π_M, respectively, or Σ and Π, for short, since M will be kept fixed. The functor Σ, left adjoint to $(\ )_M$, is given by

$$\Sigma(Z \longrightarrow M) = Z$$

whereas the right adjoint Π to $(\ )_M$ can be described the following way (if we talk about E as if it were the category of sets):

$$\begin{aligned} \Pi(Z \to M) &= \text{set of sections of } Z \to M \\ &= \prod_{m \in M} Z_m, \end{aligned}$$

where Z_m is the fibre of $Z \to M$ over $m \in M$. The following Lemma is probably well known.

Lemma 6.1. The functor $\Pi: E/M \to E$ reflects epimorphisms.

Proof. Let $f: \eta \to \zeta$ be a morphism in $\mathcal{E}/M$, displayed in $\mathcal{E}$ as a commutative triangle

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "\eta"'] & & Z \arrow[dl, "\zeta"] \\ & M & \end{tikzcd}$$

and assume Π_f epi in $\mathcal{E}$. We must prove f epi in $\mathcal{E}/M$. It suffices to prove f epi in $\mathcal{E}$. We prove this as if $\mathcal{E}$ were the category of sets. So let $z \in Z$ be given. We construct a section σ of ζ by putting $\sigma(m) = z \quad \forall m \in M$. Since Π_f is epi, this means that there exists a section τ of η with $\Pi_f(\tau) = \sigma$, so $f \circ \tau = \sigma$. Then $f(\tau(m)) = \sigma(m) = z$, so that $\tau(m)$ maps to z by f, (for any $m \in M$, in fact; in particular for $m = \zeta(z)$, say). This proves f surjective.

Proposition 6.2. If

$$\begin{tikzcd} Y \arrow[rr, tail, "f"] \arrow[dr, "\eta"'] & & M \times P \arrow[dl, "\mathrm{proj}"] \\ & M & \end{tikzcd}$$

is a monic in $\mathcal{E}/M$ such that $f: Y \rightarrowtail M \times P$ has the extension property w.r.to R in $\mathcal{E}$, then $f: \eta \longrightarrow P_M$ has the extension property w.r.to R_M in $\mathcal{E}/M$.

Proof. Let us temporarily denote exponentiation in $\mathcal{E}$ and $\mathcal{E}/M$ by $\pitchfork$ and $\pitchfork_M$ respectively (so $A \pitchfork B$ denotes B^A, for A and B in $\mathcal{E}$). To prove $f \pitchfork_M R_M$ epi in $\mathcal{E}/M$, it suffices, by the Lemma, to prove

$$\Pi(f \pitchfork_M R_M) \quad \text{epi in } \mathcal{E}.$$

But the adjunction $\Sigma \dashv (\ )_M$ is $\mathcal{E}$-enriched (where $\mathcal{E}/M$ is enriched over $\mathcal{E}$ via Π), so that

$$\Pi(f \pitchfork_M R_M) \cong \Sigma f \pitchfork R$$

which is epic by assumption on $f = \Sigma f$.

The results of the previous § will now be applied to the case where $\mathcal{F} = \mathcal{E}/M$, and where $\mathcal{E}$ in turn is a topos in which there is given a ring object R; M will be an internal affine variety (relative to R). The "R" in $\mathcal{F}$ will be R_M. We assume that $R \in \mathcal{E}$ satisfies Axiom 1 of synthetic differential geometry (as in §3, say), and is a Fermat ring. Since $(\ )_M$ is a logical functor, these same things will hold for R_M in $\mathcal{F}$. Also, R will be assumed to be soft. It is not clear to us whether this implies R_M soft.

We denote by $\mathcal{T}M \in \mathcal{F}$ the tangent bundle $TM \to M$ (or $M^D \to M$) of M. Since M is infinitesimally linear, it is canonically an R_M-module object in $\mathcal{F}$. Talking set-theoretically about $\mathcal{E}$, this might be expressed: the fibres of $M^D \to M$ are R-modules.

Applying the $\mathcal{D}$-construction (§5) to this R_M-module in $\mathcal{F}$ yields an object $\mathcal{D}(\mathcal{T}M) \subseteq \mathcal{T}M$. Applying Σ to this again, yields a subobject

$$\Sigma\,\mathcal{D}\mathcal{T}M \subseteq \Sigma\,\mathcal{T}M = TM = M^D$$

which we denote $D(M)$. Thus

$$D(M) \subseteq M^D.$$

The fact that $0_M \in \mathcal{D}(\mathcal{T}M)$ gives, by application of Σ, that the 0-section $M \rightarrowtail TM$ of the tangent bundle factors through $D(M)$. We call $D(M)$ the <u>first neighbourhood of the zero section</u> (of the tangent bundle).

Let us identify $T(R^n)$ with $R^n \times R^n$ in the standard way: any $t \in T(R^n)$ is of form $t(d) = \underline{x} + d \cdot \underline{y}$ $\forall d \in D$ for unique $(\underline{x},\underline{y}) \in R^n \times R^n$, and may be identified with $(\underline{x},\underline{y}) \in R^n \times R^n$. Under this identification $\lambda \cdot t$ (for $\lambda \in R$) becomes identified with $(\underline{x}, \lambda \cdot \underline{y})$.

Now suppose M is an internal affine variety, defined by the family of equations $(f_i(\underline{x}) = 0 \mid i \in I)$, where $f_i : R^n \to R$. Under the identification $T(R^n) = R^{2n}$, we have

$$TM \subseteq T(R^n) = R^{2n},$$

and it is clear that $TM \subseteq R^{2n}$ is carved out by a family of equations: $(\underline{x},\underline{y})$ belongs to TM iff $\underline{x} \in M$ (i.e. $f_i(\underline{x}) = 0$ $\forall i \in I$) and $df_i(\underline{x})(y) = 0$ $\forall i \in I$. So TM is an internal affine variety, and finitary if M is. We may see $TM \subseteq M \times R^n$ as Σ applied to

$$\mathcal{T}M \subseteq (R_M)^n \quad \text{in } \mathcal{F} = \mathcal{E}/M.$$

This subobject is a submodule for the ring R_M, since (talking about $\mathcal{E}$ as if it were the category of sets) each fibre $(TM)_m$ of TM is a sub-R-module of $\{m\} \times R^n$ $(m \in M)$. (Actually, $\mathcal{T}M \subseteq (R_M)^n$ is an internal affine variety, relative to R_M, carved out by R_M-linear equations only.)

<u>Proposition 6.3</u>. $\mathcal{D}\mathcal{T}M \subseteq \mathcal{T}M$ classifies homogeneous maps $\mathcal{T}M \to R_M$, i.e. the equivalent conditions of Proposition 5.3 hold (with $V = \mathcal{T}M$, $R = R_M$).

<u>Proof</u>. Since R in $\mathcal{E}$ is soft, and $TM \subseteq R^{2n}$ is an internal affine variety, $TM \subseteq R^{2n}$ has the extension property, and hence so does $TM \subseteq M \times R^n$. Since $TM = \Sigma\,\mathcal{T}M$, it follows from Proposition 6.2 that $\mathcal{T}M \subseteq (R^n)_M = (R_M)^n$ has the extension property, relative to R_M. Since it is furthermore an R_M-

module, we may apply Corollary 5.5 to conclude that

$$\mathcal{D}(TM) = \mathcal{D}(R^n_M) \cap TM \qquad (\subseteq TM).$$

But

$$\mathcal{D}(R^n_M) = (\mathcal{D}(R^n))_M = (D(n))_M,$$

using that $(\ )_M$ is a logical functor, and using (5.1). So

$$\mathcal{D}(TM) = D(n)_M \cap TM \subseteq (R^n)_M.$$

Applying Σ, we get

$$\Sigma\mathcal{D}TM = (M \times D(n)) \cap TM \subseteq M \times R^n \subseteq R^{2n}. \tag{6.1}$$

Since $TM \subseteq R^{2n}$, and clearly also $M \times D(n) \subseteq R^{2n}$, are internal affine varieties, we conclude from softness of R that $\Sigma\mathcal{D}TM \subseteq R^{2n}$ has the extension property. But hence so does $\Sigma\mathcal{D}TM \subseteq M \times R^n$. By Proposition 6.2, we therefore conclude that $\mathcal{D}TM \subseteq R^n_M$ has the extension property relative to R_M. To prove condition i) in Proposition 5.3, let $\omega: \mathcal{D}TM \to R_M$ be 0-preserving. By the extension property just proved, we extend it to a 0-preserving map $\omega': R^n_M \to R_M$. By the Fermat property for R_M, ω' may be written

$$\omega'(\underline{x}) = \Sigma\alpha_i x_i + \Sigma\alpha_{ij}(\underline{x})\cdot x_i \cdot x_j \quad .$$

The last sum is zero on $\mathcal{D}(R^n_M)$, and hence on $\mathcal{D}(TM)$, so $\omega'(\underline{x}) = \Sigma\alpha_i x_i$ here, so is homogeneous on $\mathcal{D}(TM)$. But ω' agrees with ω on $\mathcal{D}(TM)$, hence ω itself is homogeneous. This proves the proposition.

By a <u>linear</u>-1-form on an object M in $\mathcal{E}$, we understand a fibrewise R-homogeneous map $TM \to R$. (If M is infinitesimally linear, in particular if it is an internal affine variety, this is equivalent to: fibrewise R-linear). We let DM denote $\Sigma \mathcal{D} \mathcal{T} M$, so if M is an internal affine variety, (6.1) expresses

$$DM = (M \times D(n)) \cap TM \subseteq R^{2n} . \tag{6.2}$$

As a Corollary of the above Proposition, we shall obtain, with assumptions on R as above:

Theorem 6.4. Let $M \subseteq R^n$ be an internal affine variety. Then restriction produces a bijective correspondence between linear-1-forms on M, and 0-preserving maps $DM \to R$, in other words, "DM classifies linear-1-forms on M".

Proof. This proceeds via a string of natural bijective correspondences:

$$\begin{array}{ll}
DM = \Sigma \mathcal{D} \mathcal{T} M \longrightarrow R & \text{0-preserving} \\
\hline
\mathcal{D} \mathcal{T} M \longrightarrow R_M & \text{0-preserving} \\
\hline
\mathcal{T} M \longrightarrow R_M & R_M\text{-linear} \\
\hline
TM = \Sigma \mathcal{T} M \longrightarrow R & \text{fibrewise } R\text{-linear}
\end{array}$$

where the middle correspondence follows from Proposition 6.3, and the two other ones are by $\Sigma \dashv (\ )_M$. Since the middle correspondence is mediated via restriction, then so is the total one.

The reader may as an exercise derive the following description of $DM = \Sigma \mathcal{D} \mathcal{T} M$, in terms as if E were the category of sets: $t \in TM$ belongs to DM if for every "cotangent", i.e. for every R-linear $\varphi: T_{t(0)}M \to R$, we have $(\varphi(t))^2 = 0$. This implies that $(\omega(t))^2 = 0$ for every linear-1-form $\omega: TM \to R$, but not conversely, so far we can see.

The theorem just proved produces, for M an internal affine variety, an object DM which classifies linear 1-forms, i.e. such that a linear-1-form on M is just a map $\omega: DM \to R$ (with a normalization property $w \circ z = 0$ where z is the zero section). This should be compared with the situation for the "non-linear" forms studied in §3; they were classified by $M_{(1)}$, by the very definition. To complete the comparison, we should therefore give an equational description of $M_{(1)}$, analogous to the description (6.2) of DM. So let M, as above, be an internal affine variety, $M \subseteq R^n$, carved out by the family of equations $\{f_i(\underline{x}) = 0 \mid i \in I\}$.

Proposition 6.5. The subobject $M_{(1)} \subseteq M \times M \subseteq R^n \times R^n$ is given as $M_{(1)} = (M \times M) \cap (R^n)_{(1)}$, so that

$$M_{(1)} = [\![(\underline{x},\underline{y}) \in R^{2n} \mid \bigwedge_{i \in I} f_i(\underline{x}) = 0 \wedge \bigwedge_{i \in I} f_i(\underline{y}) = 0$$

$$\wedge\ (\underline{x}-\underline{y}) \in D(n)]\!] \ .$$

In particular, it is an internal affine variety.

Proof. Since any map, in particular $M \hookrightarrow R^n$, preserves the 1-neighbour relation, we immediately get that $M_{(1)} \subseteq M \times M \cap R^n_{(1)}$. Conversely, let $(\underline{x},\underline{y}) \in M \times M$, and let $\underline{x} \sim_1 \underline{y}$ in R^n, i.e. $\underline{x}-\underline{y} \in D(n)$. Let $g: M \to R$ be any function. Since $M \subseteq R^n$ is an internal affine variety and R is soft, g extends to a map $g: R^n \to R$, and so $(g(\underline{x}) - g(\underline{y}))^2 = 0$ because $\underline{x} \sim_1 \underline{y}$ in R^n. But this proves that $\underline{x} \sim_1 \underline{y}$ in M.

Corollary 6.6. If A is a finitely presented $\mathbb{T}$-algebra, then $A \otimes A/I^2$ is finitely presented.

Proof. Consider the topos Set^{A} of §4 (A = finitely generated $\mathbb{T}$-algebras), and let $M = \tilde{A}$. It is thus finitarily affine, and hence so is $M_{(1)}$, by Proposition 6.5. But $M_{(1)}$ is by Corollary 4.3 represented by $A \otimes A/I^2$, which thus must be finitely presented.

§7. Comparison between the two neighbourhoods: exponential map

Let M be any object in $\mathcal{E}$. If $v \in T_x M$, v is an infinitesimal curve $D \to M$. The infinitesimal beginning of an exponential map $TM \to M$, but defined only in DM,

$$DM \xrightarrow{e} M,$$

should clearly satisfy the equation

$$(7.1) \qquad e(d\cdot v) = v(d) \qquad \forall d \in D.$$

(Recall from classical differential geometry that the exponential map $e: TM \to M$ (relative to some connection on M) satisfies the equation

$$e(\lambda v) = \bar{v}(\lambda) \qquad \forall \lambda \in R,$$

where $\bar{v}: R \to M$ is the geodesic curve through v; in particular $\bar{v}(d) = v(d)$ $\forall d \in D$. Thus (7.1) is just the infinitesimal part of this equation.)

For arbitrary N, there may be no map $e: DM \to M$ satisfying equation (7.1), or there may be several. However, on the assumptions on $(\mathcal{E},R)$ as in §6, we have:

Proposition 7.1. Let M be an internal affine variety. Then there exists a unique map $e: DM \to M$ such that (7.1) holds for any $v \in TM$. Furthermore, considering $DM \subset M \times D(n)$ as in (6.1), let $v = (\underline{u},\underline{v})$; then $e(v) = \underline{u}+\underline{v}$.

Proof. We first prove uniqueness. Since M can be embedded in som R^n, it suffices to prove that maps $DM \to R$ are characterized by their values on elements in DM of form $d \cdot v$ ($d \in D$, $v \in TM$), meaning diagrammatically that the map

$$R^{DM} \xrightarrow{R^M} R^{D\times TM}, \tag{7.2}$$

induced by the multiplication $\mu: D \times TM \to DM$, is monic. By Proposition 6.3, the equivalent conditions in Proposition 5.3 hold for TM in E/M. In particular, ii) in this Proposition says that

$$DM ⋔_M R_M \xrightarrow{\mu ⋔_M 1} (D_M \times TM) ⋔_M R_M \tag{7.3}$$

is monic, $⋔_M$ denoting, as in the proof of Proposition 6.2, exponentiation in E/M. The functor $\Pi_M: E/M \to E$ preserves monics, and gives (7.2) when applied to (7.3), by the E-enrichment of $\Sigma \dashv (\ )_M$ (cf. the proof of Proposition 6.2). This proves the uniqueness.

To show the existence, let M be carved out of R^n by a family of equations $f = 0$, where $f: R^n \to R$. So $D(M) \subseteq M \times R^n$ is identified with a subset of $M \times D(n)$, as in (6.2). Let $v \in DM$, $v = (\underline{u},\underline{v})$. Define $e(v) = e(\underline{u},d\underline{v}) = \underline{u} + d\underline{v} = v(d)$. It only remains to be seen that for $v \in DM$, $\underline{u} + \underline{v} \in M$. Let one of the equations defining M be $f = 0$. Then

$$f(\underline{u}+\underline{v}) = f(\underline{u}) + df_{\underline{u}}(\underline{v}) + h(\underline{u})(\underline{v}), \tag{7.4}$$

where $h(\underline{u})(\underline{v})$ is of degree ≥ 2 in the coordinates of $\underline{v}$. But $f(\underline{u}) = 0$ since $\underline{u} \in M$, and $df_{\underline{u}}(\underline{v}) = 0$ since $(\underline{u},\underline{v}) \in TM$. Finally $h(\underline{u})(\underline{v}) = 0$ since $\underline{v} \in D(n)$. This finishes the proof.

For M an internal affine variety, consider the map

$$\bar{e}\colon DM \longrightarrow M \times M$$

defined by $\bar{e}(v) = (v(0), e(v))$.

Proposition 7.2. The map $\bar{e}$ maps DM bijectively onto $M_{(1)} \subseteq M \times M$ and takes the zero section $z\colon M \to DM$ to the diagonal $\Delta\colon M \to M_{(1)}$.

Proof. Let $M \subseteq R^n$, carved out by equations, as in the proof of the previous proposition. So, as there, $DM \subseteq M \times D(n)$. Let $v \in DM$, $v = (\underline{u},\underline{v})$, and $e(v) = \underline{u}+\underline{v} \in M$. Since $\underline{v} \in D(n)$, it follows from Proposition 6.5 that $(\underline{u},\underline{u}+\underline{v}) \in M_{(1)}$, so

$$\bar{e}(v) = (e(0), e(v)) = (\underline{u},\underline{u}+\underline{v}) \in M_{(1)}.$$

We define an inverse for $\bar{e}\colon DM \to M_{(1)}$ by considering the map

$$\sigma\colon M_{(1)} \longrightarrow R^{2n}$$

defined by $\sigma(\underline{x},\underline{y}) = (\underline{x},\underline{y}-\underline{x})$. We prove $\sigma(\underline{x},\underline{y}) \in DM$ by using (6.2) and Proposition 6.5. By the latter, $\underline{y}-\underline{x} \in D(n)$. Since $\underline{x}$ and $\underline{y}$ belong to M, and $\underline{y}-\underline{x} \in D(n)$, an expansion similar to (6.2) gives $(\underline{x},\underline{y}-\underline{x}) \in DM$. Clearly σ, as a map $M_{(1)} \to TM$, is a two-sided inverse for $\bar{e}\colon TM \to M_{(1)}$, and takes $(\underline{x},\underline{x})$ to $(\underline{x},\underline{0})$, proving the Proposition.

As a first Corollary of the Proposition, we may derive Theorem 2.4 from Theorem 2.3 (which has already been proved). For, let us return to the specific topos considered in §4. If M is carved out of R^n by $f_i = 0$ $(i \in I)$, we have

$$D(M) = [\![(\underline{x},\underline{y}) \in R^{2n} \mid \bigwedge_{i\in I} f_i(\underline{x}) = 0 \wedge \bigwedge_{i\in I} df_{\underline{x}}(\underline{y}) = 0 \tag{7.5}$$
$$\wedge \bigwedge_{j,k} (y_j \cdot y_k = 0)\,]\!]$$

by (6.2). We have $D(M) \cong M_{(1)}$ by Proposition 7.1, so

$$\hom_E(M,R) \xrightarrow{d} \Lambda^1 M = \{\omega\colon M_{(1)} \to R \mid \omega\circ\Delta \equiv 0\}$$

$$\cong \{\bar{\omega}\colon D(M) \to R \mid \bar{\omega}\cdot z \equiv 0\}$$

is a universal derivation. If $A = K\{X_1,\ldots,X_n\}/((f_i)_{i\in I})$, so that $\hom_E(M,R) = A$, this composite is

$$A \xrightarrow{d'} \{\bar{\omega}\colon D(M) \to R \mid \bar{\omega}\circ z \equiv 0\} \subseteq \hom_E(D(M),R); \tag{7.6}$$

by (7.5) the codomain here equals $A^{\#}$ as defined in Theorem 2.4, and the submodule Ω_A of those $\bar{\omega} \in \hom_E(D(M),R)$ with $\bar{\omega}\circ z \equiv 0$ corresponds to the ideal generated by the y_i's. Let us identify the d' of (7.6). It suffices to do this for the case $M = R^n$, in view of the naturality of the correspondence $D(M) \cong M_{(1)}$ with respect to the inclusion $M \subseteq R^n$. Then the d' of (7.6) equals the composite

$$K\{X_1,\ldots,X_n\} = \hom_E(R^n,R) \xrightarrow{d} \hom_E(R^n{}_{(1)},R) \longrightarrow$$

$$\xrightarrow{\hom(\bar{e},R)} \hom_E(D(R^n),R) = \hom_E(R^n \times D(n),R).$$

For $\varphi \in \hom(R^n,R)$, $(d\varphi)(\underline{x},\underline{y}) = \varphi(\underline{y}) - \varphi(\underline{x})$, so since $e(\underline{x},\underline{d}) = (\underline{x},\underline{x}+\underline{d})$, $\hom(\bar{e},R)\circ d$ sends φ to the map $R^n \times D(n) \to R$

given by

$$(\underline{x},\underline{x}+\underline{d}) \longmapsto \varphi(\underline{x}+\underline{d}) - \varphi(\underline{x}) = \Sigma \frac{\partial \varphi}{\partial x_i}(\underline{x}) \cdot d_i .$$

Rewriting $\underline{d}$ as $\underline{y}$ yields the formula of Theorem 2.4 (writing d instead of d').

Finally, let us prove $A^{\#} = A[\varepsilon\Omega_A]$. Let $f \in A^{\#} = \hom_E(D(M),R)$. We have the zero section $z: M \to D(M) \subseteq TM$ of the projection map $p: D(M) \to M$, and $z \circ p: D(M) \to D(M)$ is idempotent, $(z \circ p) \circ (z \circ p) = z \circ p$. So the R-linear endo-map Z on $A = \hom(D(M),R)$ given by $f \mapsto f \circ z \circ p$ decomposes A into a direct sum, whose factors are A and Ω_A, respectively (note $\Omega_A = \ker(Z)$, $\hom(M,R) \cong \operatorname{Im} Z$). It remains to be shown that the R-linear map

$$A \xrightarrow[\cong]{} A \oplus \Omega_A = A[\varepsilon\, \Omega_A]$$

preserves the $\mathbb{T}$-algebra structure. It suffices to see that the elements in $\operatorname{Ker}(Z) = \operatorname{Im}(\mathrm{Id}-Z)$ have square zero in $A = \hom(D(M),R)$. With $M \subseteq R^n$ as above, let $f \in \hom(D(M),R)$; f may be extended to a function on $R^n \times D(n) \supseteq D(M)$. Then

$$(\mathrm{Id}-Z)(f)(\underline{x},\underline{d}) = f(\underline{x},\underline{d}) - f(\underline{x},\underline{0}) = \Sigma d_i \frac{\partial f}{\partial x_i}(\underline{x},0),$$

which has square zero. This proves Theorem 2.4.

The following Theorem gives a synthetic comparison betweeen the two 1-form notions considered.

Theorem 7.3. Let M be an internal affine variety. Then there is a natural bijective correspondence between the set $\Lambda^1(M)$ of 1-forms on M (in the sense of §3) and the set of linear-1-forms on M; namely, to $\omega \in \Lambda^1 M$, we associate the linear-1-form $\bar{\omega}$ classified by

$$D(M) \xrightarrow{\bar{e}} M_{(1)} \xrightarrow{\omega} R.$$

Explicitely, $\bar{\omega}$ is determined by

$$(7.7) \qquad d \cdot \bar{\omega}(t) = \omega(t(0), t(d)) \qquad \forall d \in D$$

for any $t \in M^D = TM$.

Proof. In view of Proposition 7.2, it only remains to prove (7.7). Let $\bar{\omega}$ be the linear-1-form classified by $\omega \circ \bar{e}$. Then

$$\begin{aligned} d \cdot \bar{\omega}(t) = \bar{\omega}(dt) = \omega(\bar{e}(dt)) &= \omega((dt)(0), e(dt)) \\ &= \omega(t(0), t(d)). \end{aligned}$$

This shows (7.7).

Let us finally note that the considerations of §§3, 6 and 7 apply to some well adapted models for synthetic differential geometry. In particular, let A be the category of finitely presented algebras for the algebraic theory $\mathbb{T}_\infty$ of smooth functions

$$\mathbb{T}_\infty(n,1) = C^\infty(\mathbb{R}^n, \mathbb{R}).$$

It is known (cf. e.g. [6]) that if $M \in \underline{Mf}$ (= the category of smooth manifolds), then $C^\infty(M) \in A$. Also, there exists (cf.e.g.[6]) a subcanonical Grothendieck topology j on A^{op} ("the open-cover topology") such that the sheaf category $\widetilde{A^{op}}$ is a well-adapted model, via the functor $i: \underline{Mf} \to \widetilde{A^{op}}$ given by $M \to \overline{C^\infty(M)}$; note $C^\infty(M) \in Set^A$, but it lives in the subcategory $\widetilde{A^{op}} \subseteq Set^A$, because j is subcanonical. The $\mathbb{T}_\infty$-algebra object

$R = i(\mathbb{R}) = \overline{C^\infty(\mathbb{R})}$ is well known to be a Fermat ring and satisfy Axiom $1^W_{\mathbb{R}}$, cf. e.g.[6]. The inclusion $\widetilde{A^{op}} \subseteq \mathrm{Set}^A$ preserves extensions of universally quantified formulas, and it also preserves exponentials. Since, for a surjection $B \twoheadrightarrow A$, the map $R^{\bar{B}} \to R^{\bar{A}}$ is epic in Set^A (by softness of R), and lives in $\widetilde{A^{op}}$, it follows that R is soft when viewed in $\widetilde{A^{op}}$. The internal finitary affine schemes relative to R are exactly the representables $\bar{A} = yA$ with $A \in A$. Since $\bar{A}_{(1)} \subseteq \bar{A} \times \bar{A}$ is an extension of a universally quantified formula,

$$(7.10) \qquad [\![(a,b) \in \bar{A} \times \bar{A} \mid \forall g \quad R^{\bar{A}} : (g(a) - g(b))^2 = 0]\!] \quad ,$$

and in Set^A is an internal affine variety, it is representable, and hence lives in $\widetilde{A^{op}}$, and is the extension (7.10) there as well, since the inclusion $\widetilde{A^{op}} \subseteq \mathrm{Set}^A$ preserves such extensions.

In particular, let M be a manifold. Then

$$i(M) = \overline{C^\infty(M)} \quad ;$$

we have $C^\infty(M) \otimes C^\infty(M) = C^\infty(M \times M)$, and the codiagonal ∇: $C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$ becomes identified with $C^\infty(M \times M) \to C^\infty(M)$, "restriction along the diagonal $\Delta_M : M \to M \times M$", such that the kernel I of ∇ simply consists of smooth functions $M \times M \to \mathbb{R}$ vanishing on Δ_M. But, in Set^A, we have

$$\overline{C^\infty(M)}_{(1)} = \overline{C^\infty(M) \otimes C^\infty(M)/I^2}$$

(Corollary 4.3), and hence we have this in $\widetilde{A^{op}}$ as well, so that

$$(iM)_{(1)} = \overline{C^\infty(M)}_{(1)} = \overline{C^\infty(M \times M)/I^2}$$

Similarly for open subsets $U \subseteq M$. In fact, $(iM)_{(1)}$ may be described in terms of the topological space M, with the sheaf of rings

$$U \longmapsto C^{\infty}(U \times U)/(I_U)^2,$$

via the canonical way of associating a ringed space to a $\mathbb{T}_\infty$-algebra, cf. [3].

We remark that all these considerations also apply to the well-adapted model defined by all $\mathbb{T}_\infty$-algebras presented by an ideal of local character [3] (= germ determined [6]). Here we have infinitely presented internal affine varieties which are representable. Also the remarks apply to the "Smooth Zariski Topos" defined by all finitely generated $\mathbb{T}_\infty$-algebras, but with finite covers only (cf. [11]).

§8. Differential forms as Ω-valued quantities

We consider the topos $\mathcal{E} = \mathrm{Set}^{\mathcal{A}}$, as in §4. The Ω-construction $(A \mapsto \Omega_A)$ of §2 is a functor $\mathcal{A} \to \underline{\mathrm{Set}}$, hence an object of $\mathcal{E}$, which we denote Ω. Since each Ω_A is in a natural way an A-module, it follows that Ω has the structure of R-module object in $\mathcal{E}$. Also, the derivations $d: A \to \Omega_A$ gives rise to a $\mathbb{T}$-derivation

$$d: \mathbf{R} \to \Omega .$$

We shall give the following refomulation of some of the results proved. Let $A \in \mathcal{A}$, i.e. A is a finitely generated $\mathbb{T}$-algebra.

Theorem 8.1. There are natural bijective correspondences between the following sets

i) Ω_A

ii) $\Lambda^1(\overline{A})$ (=set of 1-forms on $\overline{A}$, =set of ω: $\overline{A}\times\overline{A} \to R$ with $\omega\circ\Delta \equiv 0$)

iii) the set of linear 1-forms $\overline{\omega}$ on $\overline{A}$ (=set of fibrewise R-homogeneous maps $T\overline{A} \to R$).

Proof. The bijection between i) and ii) was established in §4 during the proof of Theorem 2.3; the bijection between ii) and iii) follows from Theorem 7.3.

Theorem 8.2. The R-module object Ω classifies linear-1-forms, i.e. for any $M \in E$, there is a natural bijective correspondence between

i) $\hom_E(M,\Omega)$

ii) the set of linear-1-forms on M.

Proof. We first consider the case where M is representable, $M = \overline{A}$ for some $A \in A$, or equivalently, an internal affine variety. Then a map $M \to \Omega$ corresponds, by Yoneda's lemma, to an element $\in \Omega(A) = \Omega_A$. This, in turn, corresponds to a linear-1-form on $\overline{A} = M$, by Theorem 8.1.

For general M , we write $M = \varinjlim \overline{A}_i$ (colimit of representables). Then, writing hom for $\hom_E$, and $\hom_h$ for "set of fibrewise R-homogenous maps" , we have

$$\hom(M,\Omega) = \hom(\varinjlim \bar{A}_i,\Omega) = \varprojlim \hom(\bar{A}_i,\Omega)$$

$$= \varprojlim \hom_h(T\bar{A}_i,R) \quad \text{(by what has already been proved)}$$

$$= \hom_h(\varinjlim T\bar{A}_i,R) \;,$$

but T commutes with colimits, by "Axiom 3", [6] I§18; this axiom evidently holds for the present E,R. Therefore

$$\hom_h(\varinjlim T\bar{A}_i,R) = \hom_h(T \varinjlim \bar{A}_i,R) = \hom_h(TM,R) \;,$$

establishing the bijection claimed in the theorem.

We leave to the reader to verify that if $f\colon M \to R$ is a map, then the composite

$$M \xrightarrow{f} R \xrightarrow{d} \Omega$$

corresponds, under the bijection, to the 1-form $df\colon TM \to R$ where $df(t)$ = principal part of $f \circ t$, for $t\colon D \to M$).

The existence of an Ω, and $d\colon R \to \Omega$ with the property in Theorem 8.2 and in this last remark, was envisaged and proved by Lawvere [8] on the basis of Axiom 3. (He used the notation $\Lambda^1(R)$ for Ω).

The reader may wonder why we in §3 and 6 have introduced the (external) set of 1-forms $\Lambda^1 M$, respectively, the (external) set of linear-1-forms, rather than the evident internal versions of the same, e.g. the object $\underset{\sim}{\hom}_h(TM,R) \in E$ of linear-1-forms on M. One reason is that it is not clear whether this is "morally" correct, for, in general

$$\underset{\sim}{\mathrm{hom}}_h(TM,R) \quad \text{and} \quad \Omega^M$$

are different objects, even though they, by Theorem 8.2, have the same "externalization" (=set of global sections). To wit, take e.g. $M = \mathbb{1}$. Then $\mathrm{hom}_h(TM,R) = \mathbb{1}$, whereas $\Omega^{\mathbb{1}} = \Omega$, which is not equal to $\mathbb{1}$ (if it were, there would not be non-trivial linear-1-forms on any object, by Theorem 8.2). So Ω^M appears to be richer than $\underset{\sim}{\mathrm{hom}}_h(TM,R)$.

The reason for the headline of the present § may be found in [6], I§20, or in [8].

REFERENCES

1. M. Demazure and P. Gabriel, Groupes algébriques, North Holland Publ. Co. 1970.
2. E. Dubuc, Sur les modèles de la géométrie differentielle synthétiques, Cahiers de Top. et Géom. Diff. 20 (1979), 231-279.
3. E. Dubuc, C^∞-schemes, Amer. Journ. Math. 103 (1981), 683-690.
4. A. Kock, Formal manifolds and synthetic theory of jet bundles, Cahiers de Top. et Géom. Diff. 21 (1980), 227-246.
5. A. Kock, Differential Forms with values in groups, Bull. Austr. Math. Soc. 25 (1982), 357-386.
6. A. Kock, Synthetic differential geometry, London Mathematical Society Lecture Notes Series No. 51, Cambridge University Press 1981.
7. F.W. Lawvere, Some algebraic problems in the context of functorial semantics of algebraic theories, In Reports of the Midwest Category Seminar II, Springer Lecture Notes in Math. Vol. 61 (1968).
8. F.W. Lawvere, Toward the description in a smooth topos of the dynamically possible motions and deformations of a smooth body, Cahiers de Top. et Géom. Diff. 21 (1980), 377-392.

9. J. Penon, Topologie et intuitionnisme, Journees Faisceaux et Logique, Mai 1981, Pre-publication Paris-Nord No. 23 (1981)

10. NgoVan Quê and G.E. Reyes, Smooth functors and synthetic calculus, Preprint, U. de Montréal, D.M.S. 82-5, 1982.

11. G.E. Reyes, Analyse C^{∞}, in Géométrie Différentielle Synthétique, Exposé 4, Rapport de Recherches, U. de Montréal, D.M.S. 80-12, 1980.