Journal of Pure and Applied Algebra 154 (2000) 257–264

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Differential forms as infinitesimal cochains

Anders Kock

Department of Mathematical Science, University of Aarhus, Ny Munkegade, Building 530, 8000 Aarhus C, Denmark

Communicated by M. Barr

Dedicated to Bill Lawvere

Abstract

In the context of synthetic differential geometry (SDG), we provide, for any manifold, a homotopy equivalence between its de Rham complex, and a complex of infinitesimal singular cochains. The equivalence takes wedge product of forms to cup product of singular cochains.
© 2000 Elsevier Science B.V. All rights reserved.

MSC: 58A03; 58A12; 18F99; 14F40

The purpose of the present note is to identify the de Rham complex of differential forms on a manifold M with a certain cochain complex related to the singular complex of M. In fact, this cochain complex is dual to a certain simplicial subcomplex of the singular complex, consisting of "infinitesimal simplices". The notions make sense in the context of an embedding of the category of smooth manifolds into a suitable topos, more precisely, into a "model for synthetic differential geometry" (SDG).

Our comparison is based on some results from [4], and is inspired by Felix and Lavendhomme's [2]. They also provide an identification of the de Rham complex with a complex related to the singular one; they, however, use cubical rather than simplicial theory, and, more importantly, their cochains are finite, not infinitesimal [1].

Recall that if ω is a differential k-form on a manifold M, and $s : \Delta_k \to M$ is a smooth map from the standard k-dimensional simplex, we may form the real number $\int_s \omega$, "the integral of the k form ω along the smooth singular k-simplex s".

We thus get the classical map

$$\Omega^*(M) \xrightarrow{\int} \mathrm{Hom}(Sing_*(M), \boldsymbol{R}),$$

[1] In fact, in spite of their statement to the contrary, it seems to me that their identification of differential forms with certain functionals on smooth singular cubes does not need the context of SDG.

0022-4049/00/\$ - see front matter © 2000 Elsevier Science B.V. All rights reserved.
PII: S0022-4049(00)00068-2

where the left-hand side is the de Rham complex of differential forms on M, and the right-hand side is the cochain complex dual to the simplicial complex $Sing_*(M)$ of smooth singular simplices on M. Stokes' Theorem asserts that this map commutes with coboundary formation; so it is a map of cochain complexes.

Recall that a differential k form on M is a law which to each point $x \in M$ and each k-tuple of tangent vectors at x associates a number, in a k-linear alternating way (and everything smooth). On the other hand, in the context of SDG, there is another notion of differential form, which is already of the type of simplicial cochain; it was considered in the late 1970s by Bkouche, Joyal, and Kock (cf. [4, I.18]). It hinges on the notion of *the first neighbourhood of the diagonal*, $M_{[1]}$, of a manifold, or scheme, M. This notion originated in algebraic geometry, where one was never afraid of nilpotent elements in rings, such nilpotent elements being the basis of SDG. Recall that the set-up of SDG is a topos $\underline{E}$ containing the category $\underline{Mf}$ of smooth manifolds as a full subcategory, $i : \underline{Mf} \subseteq \underline{E}$. The ring $\boldsymbol{R}$ of reals goes by i to a ring object R, which has many more subobjects than $\boldsymbol{R}$ has, in particular, we require it to have a sufficiently large subobject $D \subseteq R$ of "elements of square zero", as well as many other similar sufficiently large objects of nilpotent elements. (What is meant by "sufficiently large" will be elaborated below; this is the "Kock–Lawvere axiom scheme".) Of course, we nowadays know well how to speak of the objects of a topos "as if they were sets", and we shall do this, and we shall call $R \in \underline{E}$ "the Reals", and so we are led to allowing ourselves to speak of all the nilpotent elements in the reals. This we shall consistently do. So when we say "let d be a real number whose square is zero", we are not necessarily talking about the number 0. (Likewise, working in $\underline{E}$, we do not have to say that the functions we consider or construct are *smooth*, this is built into the context.) Now, the first neighbourhood of the diagonal of the manifold R^m (m-dimensional coordinate vector space) is easy to describe; it is the subset of $R^m \times R^m$ consisting of pairs of vectors $\boldsymbol{x}, \boldsymbol{y}$ such that $\boldsymbol{x} - \boldsymbol{y}$ belongs to $D(m) \subseteq R^m$, where $D(m) \subseteq R^m$ in turn consists of vectors $\underline{d} = (d_1, \ldots, d_m)$ with $d_i \cdot d_j = 0$ for all i and j (in particular $d_i^2 = 0$ for all i). The notion of first neighbourhood $M_{[1]}$ of the diagonal of an m-dimensional manifold M can now be defined, using charts from R^m, and the notion is independent of choice of charts. We say that $x \in M$ and $y \in M$ are *neighbours* if $(x, y) \in M_{[1]}$, and then we write $x \sim_1 y$ or just $x \sim y$.

The neighbour relation is reflexive and symmetric, but not transitive. A certain amount of the differential geometry of M can be expressed entirely in terms of this combinatorial structure $M_{[1]}$ on M. For instance, one immediately gets a simplicial complex $M_{[*]}$ (the "coskeleton of $M_{[1]}$"), with $M_{[n]}$ being the "set of $n+1$-tuples of mutual neigbours", so an n-simplex $\boldsymbol{x} \in M_{[n]}$ is an $n+1$-tuple $x_0, x_1, \ldots, x_n$ of elements in M with $x_i \sim x_j$ for all $i, j = 0, \ldots, n$. The elements $\boldsymbol{x} = (x_0, \ldots, x_k)$ in $M_{[k]}$ we may call *infinitesimal k-simplices* in M, and the x_i's the *vertices* of $\boldsymbol{x}$. A k-simplex $\boldsymbol{x}$ is called *degenerate* if two of its vertices are equal.

Maps $\omega : M_{[k]} \to R$ are now naturally called *infinitesimal k-cochains* on M, and ω is called a *normalized* cochain if it vanishes on all degenerate k-simplices. Since $M_{[*]}$

is a simplicial complex, the collection $\mathrm{Hom}(M_{[*]}, R)$ of all cochains form a cochain complex.

In [4], I presented in terms of coordinates, (using the basic axiom scheme of SDG), an isomorphism between the object $\Omega^k(M)$ of "classical" k-forms, on the one side, and the set of *normalized* k-cochains on $M_{[*]}$ on the other. Such isomorphism b will be recalled below, differing from the one of [4], though, by a factor $k!$. Also it is an easy fact (cf. e.g. [5, Proposition 2]) that normalized cochains are *alternating*, i.e. the value of such cochain changes sign if two of the vertices of the simplex are interchanged. From this follows easily that the subset of normalized cochains form a subcomplex of the cochain complex of all infinitesimal cochains. Denoting it $\mathrm{Hom}^{norm}(M_{[k]}, R)$, we thus have the situation of the following diagram (the dotted arrows yet to be filled in):

$$\begin{array}{ccccc} \Omega^*(M) & \xrightarrow{\int} & \mathrm{Hom}(Sing_*(M), R) & & Sing_*(M) \\ \cong \big\downarrow b & & \vdots\, a^* \downarrow & & \uparrow\vdots\, a \\ \mathrm{Hom}^{norm}(M_{[*]}, R) & \xrightarrow{\subseteq} & \mathrm{Hom}(M_{[*]}, R) & & M_{[*]} \end{array} \qquad (1)$$

We shall provide a map a of simplicial sets (far right). It induces clearly a map a^* of cochain complexes. The horizontal maps in (1) are likewise maps of cochain complexes, as observed; from the commutativity of (1) (to be proved), it then follows that b is also a cochain map.

The result of the present note is summarized in:

Theorem 1. *The diagram commutes; and it establishes a homotopy equivalence between the cochain complexes* $\Omega^*(M)$ *and* $\mathrm{Hom}(M_{[*]}, R)$. *The equivalence* $\Omega^*(M) \to \mathrm{Hom}(M_{[*]}, R)$ *preserves the product structure.*

(The product structure in $\mathrm{Hom}(M_{[*]}, R)$ is the well-known cup product, which exists by virtue of the fact that the complex in question is the dual of a simplicial set. The product structure on $\Omega^*(M)$ is the classical wedge product of forms (possibly modulo some factors of type $p!q!/(p+q)!$, depending on the conventions).)

Proof/Construction. There are two new constructions to be performed, and several equations to be checked. The constructions are: (1) the construction of the simplicial map a, and (2) the construction of a homotopy inverse of the inclusion of cochain complexes (bottom line of the diagram).

(1) The construction of a is essentially contained in Theorem 1 in [6], which I quote. Recall that an *affine combination* is a linear combination where the sum of the coefficients is 1 (and it is a *convex* combination if further the coefficients are non-negative).

Theorem 2. *Given an infinitesimal k-simplex $\boldsymbol{x}=(x_0,\dots,x_k)$ in a manifold M, affine combinations of the x_i's can be formed by choice of coordinates, but the result does not depend on the choice. All affine combinations thus obtained are mutual neighbours.*

Thus there is a canonical map $R^k \to M$, taking an affine combination of 0 and the k unit vectors $e_1,\dots,e_k$ in R^k into the corresponding affine combination of $x_0,x_1,\dots,x_k$. Restricting this map to the affine combinations with non-negative coefficients (= the convex combinations) provides a map $[\![\boldsymbol{x}]\!] : \Delta_k \to M$, where Δ_k is the k-simplex in R^k consisting of the convex combinations of $0,e_1,\dots,e_k$. We put $a(\boldsymbol{x}):=[\![\boldsymbol{x}]\!] : \Delta_k \to M$. The construction a is evidently compatible with face- and degeneracy formation of simplices, thus collectively, a provides a simplicial map as displayed in (1). The name 'a' is to suggest '*affine* singular simplex'. The map a is a monic map of simplicial sets, so we may think of the infinitesimal simplices as forming a subcomplex of the complex of singular ones. It follows that a^* is a homomorphism of cochain complexes.

(2) The construction of a homotopy inverse of the inclusion map

$$\mathrm{Hom}^{norm}(M_{[*]},R) \to \mathrm{Hom}(M_{[*]},R)$$

can essentially be derived from a recent proof of Barr [1]. Barr first noted that a simplicial complex of the form $Sing_*(M)$ has some further structure: the permutation group in $k+1$ letters acts on the set of singular k-simplices in an evident way. This action restricts to an action on the sub-simplicial complex of infinitesimal k-simplices. Secondly, barycentric subdivisions can be made on singular simplices. This subdivision process likewise restricts to infinitesimal simplices (as will be argued).

I shall sketch a version of Barr's theory, which is explicit enough to handle our case of infinitesimal simplices.

Let $C(X)$ denote the free R-module-object on an object X. (Actually, it suffices that "R perceives $C(X)$ to be free", which sometimes will be the case for $C(X)=$ the "internal object of distributions on X", a much more concrete object than the abstractly-free R-module $F(X)$ on X.) We get in any case

$$\mathrm{Hom}(M_{[k]},R) \cong \mathrm{Hom}_R(C(M_{[k]}),R), \tag{2}$$

$\mathrm{Hom}_R(-,-)$ denoting the set of R- linear maps. The $C(M_{[k]})$'s jointly form a chain complex $C(M_{[*]})$, since $M_{[*]}$ is a simplicial set, and its R-dual cochain complex may be identified, by (2), with the $\mathrm{Hom}(M_{[*]},R)$ considered above. Now we may, like in [1], form the subcomplex [2] $U_{[*]}(M) \subseteq C(M_{[*]})$, where $U_{[k]}$ is the R-submodule of $C(M_{[k]})$ generated by "alternating chains", meaning expressions $\boldsymbol{x}-sign(\sigma)\boldsymbol{x}^\sigma$ (where $\boldsymbol{x}^\sigma$ denotes the result of acting with the permutation σ on the simplex $\boldsymbol{x}$). Thus, a k-cochain ω is alternating (which is the same as normalized, see above) precisely when its extension to an R-linear map $C(M_{[k]}) \to R$ annihilates $U_{[k]}$. In other words, the inclusion of cochain complexes $\mathrm{Hom}^{norm}(M_{[*]},R) \subseteq \mathrm{Hom}(M_{[*]},R)$ is induced by the quotient map

[2] The proof that this is a subcomplex is identical to that of [1].

of chain complexes

$$C(M_{[*]}) \to C(M_{[*]})/U_{[*]}. \tag{3}$$

The following is a version of Barr's result.

Theorem 3. *The chain map* (3) *is a chain equivalence.*

We sketch the proof, because we need it in a more explicit form than in loc.cit. to see that the homotopy inverse, constructed by Barr on the level of singular chains, restricts to the infinitesimal ones. To construct a candidate for a homotopy inverse

$$C(M_{[*]})/U_{[*]} \to C(M_{[*]})$$

for (3), one needs a chain map $C(M_{[*]}) \to C(M_{[*]})$ annihilating $U_{[*]}$. The "explicit computation" of Barr in loc.cit. carried out in low dimensions (≤ 2), shows that here, the classical barycentric subdivision chain map χ (cf. e.g. [3, p. 331]) will do the job. We claim that it will do so in all dimensions. For simplicity, we only document this for infinitesimal chains, which is all we need. To see that χ annihilates the alternating chains (and thus $U_{[*]}$), it is better to give an explicit description of χ, rather than the usual inductive one: If $\boldsymbol{x} = (x_0, \dots, x_k)$ is an infinitesimal k-simplex, $\chi(\boldsymbol{x})$ is a linear combination of $(k+1)!$ infinitesimal k-simplices. These simplices may be labelled by the permutations σ of the $k+1$ symbols $0, 1, \dots, k$ as follows: to the permutation σ corresponds the k-simplex

$$(x_{\sigma(0)}, [x_{\sigma(0)}, x_{\sigma(1)}], [x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}], \dots, [x_{\sigma(0)}, x_{\sigma(1)}, \dots, x_{\sigma(k)}]),$$

where square brackets denote 'barycenters of', thus for instance $[x_{\sigma(0)}, x_{\sigma(1)}]$ denotes the affine combination (midpoint) $\frac{1}{2}x_{\sigma(0)} + \frac{1}{2}x_{\sigma(1)}$. All the barycenters in question are mutual neighbours, by the last clause in Theorem 2, so the above $(k+1)$-tuple is indeed an infinitesimal k-simplex. The coefficients in the linear combination are $+1$ or -1 according to whether the σ is even or odd.

From the description in terms of permutations, it is almost immediate to prove that $\chi(\boldsymbol{x}) = sign(\sigma)\chi(\boldsymbol{x}^{\sigma})$, and in particular χ annihilates alternating chains. As in [3, p. 332], a chain homotopy Λ between χ and the identity map can be explicitly given also, and by inspecting the description of Λ in loc.cit., one sees by induction that (1) Λ restricts to infinitesimal chains, and (2) it maps $U_{[*]}$ into itself. It follows that $C(M_{[*]}) \to C(M_{[*]})/U_{[*]}$ is a chain equivalence.

We now prove the commutativity of diagram (1), for any manifold M. The clockwise composite is clearly invariantly defined (independent of choice of coordinates). The construction of the left-hand vertical map (which we shall recall from [4], but with the slight convention change already mentioned), is given in terms of coordinates, i.e. assuming that $M = R^m$, but from the commutativity it then follows that it, too, is independent of the choice of coordinate system.

So assume $M = R^m$. Then we may identify (classical) differential k-forms $\bar{\omega}$ on it with functions $\bar{\omega}(x_0; v_1, \dots, v_k)$ of $k+1$ variables from M, multilinear and alternating in the k arguments after the semicolon. In these terms, the infinitesimal cochain $\omega = b(\bar{\omega})$ is given by

$$\omega(x_0, x_1, \ldots, x_k) := \frac{1}{k!}\bar{\omega}(x_0; x_1 - x_0, \ldots, x_k - x_0)$$

whenever the x_i's form an infinitesimal k-simplex. (The factor $1/k!$ was not included in the description in [4], but is necessary to make the diagram commutative, with the conventions we otherwise have chosen.)

(The proof that this passage from differential k-forms to normalized infinitesimal cochains can be inverted is an application of one of the more subtle instances of the fundamental Axiom Scheme for SDG, see [4, I.18] (notably Corollary 18.4). For $k = 1$, $m = 1$, however, it is just the statement that, given x_0, any map to R from the set $x_0 + D$ of neighbours x_1 of x_0, taking x_0 to 0, extends uniquely to a map, linear in $x_1 - x_0$, defined for *all* $x_1 \in R$; this statement is essentially the original Kock–Lawvere Axiom.)

To prove the commutativity of (1), let $\bar{\omega} \in \Omega^k(R^m)$, and let $(x_0, \ldots, x_k)$ be an infinitesimal k-simplex. Proving the commutativity thus means proving

$$\int_{[\![x_0,\ldots,x_k]\!]} \bar{\omega} = 1/k!\,\bar{\omega}(x_0; x_1 - x_0, \ldots, x_k - x_0). \tag{4}$$

For this, we need some generalities. Recall the following standard facts from multilinear algebra: (1) a k-linear alternating function on R^k is entirely determined by its value on the k-tuple of standard basis vectors $e_1, \ldots, e_k$; and (2) $(k+1)$-linear alternating functions on R^k are zero. Finally, since we assume that (2) is invertible, (3) a multilinear function is alternating iff it has the property that it vanishes whenever two of its arguments are equal.

Lemma 1. *Let $L : R^k \to R^m$ be a linear function with the property that all its values are ~ 0, and let H be a $(k+1)$-linear function on R^m. Then the function $H \circ (L \times \cdots \times L)$ is zero.*

Proof. The function in question is a $(k+1)$-linear function on R^k. It is alternating, for if two arguments are equal, H gets supplied with two equal arguments which furthermore are ~ 0; but any multilinear expression returns the value 0 when supplied with two equal vectors ~ 0 as some of its arguments. Thus the lemma follows from facts (2) and (3) above.

To describe the k-form $[\![x_0, \ldots, x_k]\!]^*(\bar{\omega})$ on R^k, it suffices, by fact (1), to calculate, for each $\underline{t} \in R^k$, its value at $\underline{t}$ on the k-tuple of standard basis vectors; for this, we have

Proposition 1. *For any $\underline{t} \in R^k$, we have*

$$[\![x_0, \ldots, x_k]\!]^*(\bar{\omega})(\underline{t}; e_1, \ldots, e_k) = \bar{\omega}(x_0; x_1 - x_0, \ldots, x_k - x_0),$$

(*and*, *in particular*, *the expression does not depend on* $\underline{t}$).

Proof. The map $[\![x_0,\dots,x_k]\!]$ is affine, with all its values being neighbours, and 0 goes to x_0. Therefore, it is of form $\underline{t} \mapsto x_0 + L(\underline{t})$, where L is a linear map, all of whose values are ~ 0. By construction of $[\![x_0,\dots,x_k]\!]$, L takes e_j to $x_j - x_0$. Pulling back a form along a smooth map involves the differential of the map, and in our case, this differential is the map L, so that

$$[\![x_0,\dots,x_k]\!]^*(\bar{\omega})(\underline{t}; e_1,\dots,e_k) = \bar{\omega}(x_0 + L(\underline{t}); x_1 - x_0,\dots,x_k - x_0).$$

Now, we Taylor expand the function $\bar{\omega}$ in its non-linear argument (before the semicolon). We get

$$\bar{\omega}(x_0; x_1 - x_0,\dots,x_k - x_0) + D_{L(\underline{t})}\bar{\omega}(x_0; L(e_1),\dots,L(e_k)),$$

and no more terms since $L(\underline{t})\sim 0$. The second term is linear in all arguments (except possibly the one just before the semicolon), and these $(k+1)$-linear arguments are filled with vectors of form "L of something". Therefore, by the lemma, this term vanishes, and the proposition now follows.

Recall that if $\bar{\theta}$ is a k-form on R^k, it is of form $f(\underline{t})$vol where vol denotes the volume form (taking value 1 on the k-tuple of standard basis vectors). The proposition above says that the form $[\![x_0,\dots,x_k]\!]^*(\bar{\omega})$ equals Kvol where K is the constant $\bar{\omega}(x_0; x_1 - x_0, \dots, x_k - x_0)$. Recall also that if $\Delta \subseteq R^k$ is an affine k-simplex, then $\int_\Delta \bar{\theta}$ is defined as the "measure-theoretic" integral $\int_\Delta f(\underline{t})\,\mathrm{d}\mu$ (where μ is Lebesgue measure), so

$$\int_{\Delta_k} [\![x_0,\dots,x_k]\!]^*(\bar{\omega}) = \bar{\omega}(x_0; x_1 - x_0,\dots,x_k - x_0)(\text{volume of } \Delta_k),$$

where Δ_k is the standard k-simplex used for singular cochains previously, i.e. the set of convex combinations of 0 and $e_1,\dots,e_k$. It has volume $1/k!$.

So $\int_{[\![x_0,\dots,x_k]\!]} \bar{\omega} = 1/k!\,\bar{\omega}(x_0; x_1 - x_0,\dots,x_k - x_0)$; but the cochain $\omega = b(\bar{\omega})$ associated to the form $\bar{\omega}$ was defined to have this latter value on $x_0,\dots,x_k$. This proves the commutativity of square (1).

We shall finally prove the assertion about products. So consider again the total map $b : \Omega^*(M) \to \mathrm{Hom}(M_{[*]}, R)$ of (1). Let $\bar{\omega}$ and $\bar{\theta}$ be a p- and a q-form, respectively. Again it suffices to consider the case where $M = R^m$, so, as before, we exhibit the values of $\bar{\omega}$ in the style $\bar{\omega}(x_0; v_1,\dots,v_p)$, and similarly for $\bar{\theta}$. Let ω and θ denote the cochains $b(\bar{\omega})$ and $b(\bar{\theta})$, respectively. The $p+q$ form $\bar{\omega} \wedge \bar{\theta}$ is defined by

$$(\bar{\omega} \wedge \bar{\theta})(x_0; v_1,\dots,v_{p+q})$$
$$= \sum sign(\sigma)\bar{\omega}(x_0; v_{\sigma(1)},\dots,v_{\sigma(p)}) \cdot \bar{\theta}(x_0; v_{\sigma(p+1)},\dots,v_{\sigma(p+q)}), \tag{5}$$

where the summation is ranging over the set of shuffles σ of $p+q$, i.e. the permutations of $p+q$ letters which do not change the relative order of the p first letters, nor change the relative order of the last q letters. Now consider the value of $b(\bar{\omega} \wedge \bar{\theta})$ on a $p+q$ simplex $x_0,\dots,x_{p+q}$. Write $x_i = x_0 + v_i$; then the value $b(\bar{\omega} \wedge \bar{\theta})(x_0,\dots,x_{p+q})$ is given by $1/(p+q)!$ times the expression in (5) above, but because the v_i's now are mutual neighbours, as well as neighbours of 0, it follows that any multilinear function of the arguments $v_1,\dots,v_{p+q}$ behaves as if it were alternating (cf. [4, I.18]), so the terms in (5) are alternating, and hence all the terms in the sum are equal, so are all equal to

$$\bar{\omega}(x_0;v_1,\dots,v_p)\cdot\bar{\theta}(x_0;v_{p+1},\dots,v_{p+q}).$$

Since there are $(p+q)!/p!q!$ shuffles, hence also this many terms in the sum, we get that

$$b(\bar{\omega}\wedge\bar{\theta})(x_0,\dots,x_{p+q})=\frac{1}{p!q!}\bar{\omega}(x_0;v_1,\dots,v_p)\cdot\bar{\theta}(x_0;v_{p+1},\dots,v_{p+q}).$$

On the other hand, the classical simplicial cup product $\omega\cup\theta$ is defined by

$$(\omega\cup\theta)(x_0,\dots,x_{p+q})=\omega(x_0,\dots,x_p)\cdot\theta(x_p,x_{p+1},\dots,x_{p+q}).$$

Now as argued in [5] Section 5, this value does not change if we replace the first argument x_p in the θ-factor by x_0 (using the infinitesimal nature of the simplex $x_0,\dots,x_{p+q}$). If we do this, and recall the factorials $1/p!$ and $1/q!$ that we supply in the passage b from forms $\bar{\omega}$, resp. $\bar{\theta}$, to cochains ω, θ, the preservation of the product structure follows.

References

[1] M. Barr, Oriented singular homology, Theory Appl. Categories 1 (1) (1995) 1–9.
[2] Y. Felix, R. Lavendhomme, On de Rham's theorem in synthetic differential geometry, J. Pure Appl. Algebra 69 (1990) 21–31.
[3] P. Hilton, S. Wylie, Homology Theory, Cambridge University Press, Cambridge, 1960.
[4] A. Kock, Synthetic Differential Geometry, Cambridge University Press, Cambridge, 1981.
[5] A. Kock, Combinatorics of curvature, and the Bianchi identity, Theory Appl. Categories 2 (7) (1996) 69–89.
[6] A. Kock, Geometric construction of the Levi–Civita parallelism, Theory Appl. Categories 4 (9) (1998) 195–207.