CONTINUOUS YONEDA REPRESENTATION OF A SMALL CATEGORY

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1. Introduction

The well-known full and faithful embedding of a small category $\underline{A}$ into the category $\mathcal{S}^{\underline{A}^{opp}}$ of contravariant functors from $\underline{A}$ to $\mathcal{S}$ (the category of sets) is left continuous (preserves inverse limits), but in general not right continuous. Also, the embedding functor is dense in the sense of Ulmer [7], or, equivalently, adequate in the sense of Isbell [3]; but it is not in general codense. In Section 2 we define a "codensity monad" for any functor from a small category to a left complete category. The word "monad" means "triple" as defined by Eilenberg and Moore [2]. In particular, the Yoneda embedding defines a codensity monad T on $\mathcal{S}^{\underline{A}^{opp}}$. We study the universal generator [2]

$$\mathcal{S}^{\underline{A}^{opp}} \xrightarrow{\ F\ } (\mathcal{S}^{\underline{A}^{opp}})^{T} \xrightarrow{\ U\ } \mathcal{S}^{\underline{A}^{opp}}$$

(F left adjoint to U, UF = T)

for T in Section 3, and prove that the Yoneda embedding followed by F gives a full and faithful embedding which is as well left as right continuous. (Corollary 3.2; this is what we refer to in the headline.) In Section 4 we show that the monad which Isbells adjoint conjugation functors ([3], * and +)

$$(1) \qquad \mathcal{S}^{\underline{A}^{opp}} \xrightarrow[+]{} (\mathcal{S}^{\underline{A}})^{opp} \xrightarrow[*]{} \mathcal{S}^{\underline{A}^{opp}}$$

determine is precisely the codensity monad T. Finally, in Section 5 we define generalized (direct and inverse) limit functors and show that the duality functors (1) as well as codensity monads and density comonads come in that way. Also, if $J: \underline{C} \to \underline{D}$ is a functor between small categories, the obvious functor

$$\underline{D} \longrightarrow \mathcal{S}^{\underline{C}^{opp}}$$

is a generalized limit.

If A and B are objects in a category $\underline{A}$, we denote hom(A,B) by $\underline{A}(A,B)$. This is assumed to be a set throughout. If $F\colon \underline{A} \longrightarrow \underline{B}$ is a functor whose inverse limit exists ("left root"), we denote this limit by $\varprojlim(F)$. Similarly $\varinjlim(F)$ for direct limit. A category $\underline{B}$ is called left complete if all functors from small categories to $\underline{B}$ have inverse limits. Similarly for right complete and direct limits.

Denote the identity morphism of an object B by I_B.

2. Codensity monads

We recall Lawvere's definition [6] of a comma category. Given functors F,G with common codomain

$$\underline{A}_0 \xrightarrow{F} \underline{B} \xleftarrow{G} \underline{A}_1 .$$

Then the comma category $[F,G]$ has as objects triples (A_0,b,A_1), where A_i is an object in $\underline{A}_i$ and b is a morphism in $\underline{B}$:

$$FA_0 \xrightarrow{b} GA_1 .$$

A morphism from (A_0,b,A_1) to (A_0',b',A_1') is a pair (a_0,a_1) where $a_i\colon A_i \longrightarrow A_i'$ in $\underline{A}_i$ such that the diagram

$$\begin{array}{ccc} FA_0 & \xrightarrow{b} & GA_1 \\ {\scriptstyle Fa_0}\downarrow & & \downarrow{\scriptstyle Ga_1} \\ FA_0' & \xrightarrow[b']{} & GA_1' \end{array}$$

commutes. There are obvious functors

(1) $$[F,G] \xrightarrow{\partial_0} \underline{A}_0, \qquad [F,G] \xrightarrow{\partial_1} \underline{A}_1 .$$

In the important special case, where $\underline{A}_0$ is the category 1 (the one-morphism category), the functor F determines and is determined by an object $B \in \underline{B}$. In that case $[F,G]$ will be denoted $[\epsilon_B, G]$. Similarly, if $\underline{A}_1 = 1$ and G determines and is determined by $B \in \underline{B}$, we denote $[F,G]$ by $[F, \epsilon_B]$.

Now, let $\mathcal{B}$ be a left complete category, and let $\underline{A}$ be small. We shall assign to a functor $G\colon \underline{A} \longrightarrow \mathcal{B}$ a monad $T_G\colon \mathcal{B} \longrightarrow \mathcal{B}$ on $\mathcal{B}$. Given an object $B \in \mathcal{B}$. Put

$$T_G(B) = \varprojlim([\epsilon_B, G] \xrightarrow{\partial_1} \underline{A} \xrightarrow{G} \mathcal{B}). \tag{2}$$

One easily sees that T_G is a covariant functor. If (B,b,A) is an object in $[\epsilon_B, G]$, (i.e. $B \xrightarrow{b} GA$ a morphism in $\mathcal{B}$), denote by $t_{(B,b,A)}$ (or short, but not precise, by t_b) the canonical projection

$$t_{(B,b,A)}\colon T_G(B) = \varprojlim([\epsilon_B, G] \xrightarrow{\partial_1} \underline{A} \xrightarrow{G} \mathcal{B}) \longrightarrow GA. \tag{3}$$

We define a functor transformation $\eta\colon I_{\mathcal{B}} \longrightarrow T_G$ by

$$t_{(B,b,A)} \circ \eta_B = b. \tag{4}$$

Finally, define a functor transformation $\mu\colon (T_G)^2 \longrightarrow T_G$ by

$$t_{(B,b,A)} \circ \mu_B = t_{(T_G(B),\, t_{(N,b,A)},\, A)} \tag{5}$$

(in the short notation $t_b \circ \mu_B = t_{t_b}$).

It is now easy to check that (2), (4), and (5) together determine a monad on $\mathcal{B}$ (monad meaning triple, [2]). Ulmer's definition [6] of codensity of the functor G can be stated: η is an equivalence. We therefore call (T_G, η, μ) the <u>codensity monad</u> for $G\colon \underline{A} \longrightarrow \mathcal{B}$. The same monad (or rather its dual) appear in Tierney's work and is called <u>model cotriple</u>.

3. A right continuous embedding

Recall [2] that a monad (T,η,μ) on a category $\mathcal{B}$ has a universal generator

$$(1) \qquad \mathcal{B} \xrightarrow{F} \mathcal{B}^T \xrightarrow{U} \mathcal{B}, \quad U \circ F = T$$

where F is left adjoint to U. The objects of $\mathcal{B}^T$ are morphisms $T(B) \xrightarrow{\xi} B$ with

$$(2) \qquad \xi \circ T(\xi) = \xi \circ \mu_B \quad \text{and} \quad \xi \circ \eta_B = I_B.$$

The morphisms are commutative diagrams

$$(3) \qquad \begin{array}{ccc} T(B) & \xrightarrow{T(b)} & T(B') \\ \xi \downarrow & & \downarrow \xi' \\ B & \xrightarrow{b} & B' \end{array} \quad ;$$

and F and U are defined by $F(B) = (T^2(B) \xrightarrow{\mu} T(B))$; $U(TB \xrightarrow{\xi} B) = B$.

Theorem 3.1. Let $y: \underline{A} \to \mathcal{B}$ be a full and faithful embedding of a small category into a left complete category. Let (T,η,μ) be its codensity monad, and (F,u) (as in (1)) its universal generator. Then the functor $F \circ y$ is a full and faithful right continuous embedding.

Proof. Using the definition (2.2) of T_y, one easily gets $T_y(yA) \cong yA$ by means of $\eta_{(yA)}$; the inverse for $\eta_{(yA)}$ is $t_{(yA, I_{yA}, A)}$ also μ_{yA} is an isomorphism. So it follows immediately from the definition of F and $\mathcal{B}^T$ that $F \circ y$ is full and faithful; it is equivalent to the functor y' which sends A to the object $TyA \xrightarrow[t_{(IyA)}]{} yA$ in $\mathcal{B}^T$. We now prove right continuity of y'. Let $R: \underline{D} \to \underline{A}$ be a functor with a direct limit A_∞ in $\underline{A}$. Let the morphisms in the limit diagram be denoted

$$R(D) = A_D \xrightarrow{i_D} A_\infty .$$

Let a transformation f from $y' \circ R$ to a constant functor $\epsilon_{(X,\xi)}$ be given, i.e. a family of morphisms in $\mathcal{B}^T$

$$y'A_D \xrightarrow[f_D]{} (X,\xi),$$

where $TX \xrightarrow{\xi} X$ is an object in $\mathcal{B}^T$. We are required to produce a morphism

$$f_\infty : \; y'A_\infty \longrightarrow (X,\xi).$$

Let us first produce a morphism α in $\mathcal{B}$

$$\alpha : \; yA_\infty \longrightarrow TX = \varprojlim\left([\epsilon_X, y] \xrightarrow{\partial_1} \underline{A} \xrightarrow{y} \mathcal{B}\right).$$

Put

(4) $$t_{(X,g,A)} \circ \alpha \circ y\,(i_D) = g \circ Uf_D : \; yA_D \longrightarrow yA;$$

α is determined by this, since yA_∞ is the direct limit of the yA_D's in the subcategory $\underline{A}$.

We see that

(5) $$\xi \circ \alpha \circ y(i_D) = Uf_D;$$

for $U(f_D)$ can be described as $\xi \circ TU(f_D) \circ \eta_D$, so to see (5) it suffices to show

$$\alpha \circ y(i_D) = TU(f_D) \circ \eta_D$$

or

$$t_g \circ \alpha \circ y(i_D) = t_g \circ TU(f_D) \circ \eta_D$$

for all $g : X \longrightarrow yA$. But this follows easily from (4) and the definition of the transformation η; so (5) is proved. Next we prove commutativity of

(6) $$\begin{array}{ccc} Ty(A_\infty) & \xrightarrow[T(\xi \circ \alpha)]{} & TX \\ {\scriptstyle t_{I(A_\infty)}}\downarrow & & \downarrow{\scriptstyle \xi} \\ y(A_\infty) & \xrightarrow[\xi \circ \alpha]{} & X \end{array}.$$

Precede the diagram by $\eta_{y(A_\infty)} = (t_{I(A_\infty)})^{-1}$; we then have to prove $\xi \circ \alpha = \xi \circ T(\xi\alpha) \circ \eta_{yA_\infty}$. This follows easily, using naturality of η, and the equations

$$\xi \circ T\xi = \xi \circ \mu; \qquad \mu \circ \eta_{TX} = I_{TX}.$$

Commutativity of (6) is proved, and therefore the diagram is a morphism f_∞ in $\mathcal{B}^T$ from $y'A_\infty$ to (X,ξ). We have

(7) $$f_\infty \circ y'(i_D) = f_D;$$

for U is faithful [2], and acting with U on (7) gives (5) since $U(f_\infty) = \xi \circ \alpha$.

Finally, f_∞ is the unique morphism satisfying (7). For Uf_∞ must satisfy

(8) $$U(f_\infty) \circ y(i_D) = U(f_D),$$

and hence $TU\,f_\infty$ satisfy (the right hand equality sign in)

(9) $$\begin{aligned} t_g \circ TU(f_\infty) \circ \eta_{y(A_\infty)} \circ y(i_D) &= t_g \circ TU(f_\infty) \circ Ty(i_D) \circ \eta_{y(A_D)} \\ &= t_g \circ TU(f_D) \circ \eta_{y(A_D)} \end{aligned}$$

for all $g\colon X \longrightarrow yA$ and all $D \in \underline{D}$. Since $\eta_{y(A_\infty)}$ is an isomorphism and the morphism in (9) ends up in an object in $y(\underline{A})$, we can use the direct limit property of A_∞ to get that $t_g \circ TU(f_\infty)$ is the only morphism in $\mathcal{B}$ which satisfies (9) for all $D \in \underline{D}$. Using next the inverse limit property of TX, we get that $TU(f_\infty)$ is the only morphism in $\mathcal{B}$ which satisfy (9) for all g and D.

Since $t_{I_{(A_\infty)}}$ is an isomorphism, there is at most one morphism φ in $\mathcal{B}$ with $T\varphi = TU(f_\infty)$ and $\varphi \circ t_{I_{(A_\infty)}} = \xi \circ T\varphi$. This proves the uniqueness of f_∞, and therefore the theorem.

<u>Corollary 3.2</u>. Let $y\colon \underline{A} \to \mathcal{S}^{\underline{A}^{opp}}$ be the Yoneda embedding for the small category $\underline{A}$. Let (T,η,μ) be its codensity monad, and (F,U) (as in (1)) its universal generator. Then the functor $F \circ y$ is a full and faithful left and right continuous embedding.

<u>Proof</u>. The preceding theorem gives everything except left continuity. Now every U appearing in a universal generator (1) reflects and preserves inverse limits, i.e. if $R: \underline{D} \to \mathcal{B}^T$ is any functor, then $\varprojlim(U \circ R)$ exists iff $\varprojlim(R)$ exists, and

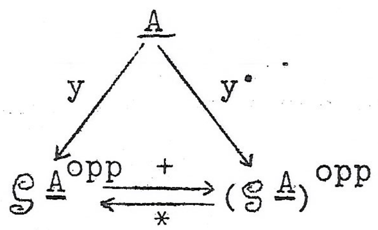
$$U \varprojlim(R) = \varprojlim(U \circ R).$$

Using this and the fact that $U \circ y'$ equals the left continuous y, we get left continuity of y'. But $F \circ y$ is equivalent to y'.

4. The conjugation monad

Isbell, in [3], defined the conjugate of a set valued functor $\underline{A}^{opp} \to \mathcal{S}$. In the case where $\underline{A}$ is small, the conjugation procedure gives (covariant) functors $+$ and $*$ so that the diagrams

(1)



commute. Here y° is the (full and faithful) co-Yoneda embedding functor $A \rightsquigarrow \underline{A}(A,-)$. The definition of $+$ and $*$ is as follows. Let $F \in \mathcal{S}^{\underline{A}^{opp}}$. Then $F^{+} \in (\mathcal{S}^{\underline{A}})^{opp}$ is the functor

$$A \rightsquigarrow \mathcal{S}^{\underline{A}^{opp}}(F, yA).$$

Let $G \in (\mathcal{S}^{\underline{A}})^{opp}$. Then $G^{*} \in \mathcal{S}^{\underline{A}^{opp}}$ is the functor

$$A \rightsquigarrow (\mathcal{S}^{\underline{A}})^{opp}(y^{\circ}A, G).$$

Lambek [5] noticed that $+$ is left adjoint to $*$. So they give rise to a monad (T', η', μ') on $\mathcal{S}^{\underline{A}^{opp}}$.

<u>Theorem 4.1</u>. The monad on $\mathcal{S}^{\underline{A}^{opp}}$ coming from the conjugation functors equals the codensity monad for y.

<u>Proof</u>. We denote as in the preceding sections the codensity monad for y by (T,η,μ). We first prove $T = T'$, i.e. that $T'F$ can be used as the inverse limit of $([\epsilon_F, y] \xrightarrow{\partial_1} \underline{A} \xrightarrow{y} \mathcal{S}^{\underline{A}^{opp}})$. Let $(F,g,A') \in [\epsilon_F, y]$, i.e. $g\colon F \longrightarrow yA'$. Define

$$t_{(F,g,A')}\colon T'F \longrightarrow yA$$

by letting $t_{(F,g,A')}(A)$ be the map

$$(\mathcal{S}^{\underline{A}})^{opp}(y^{\cdot}(A), F+) = T'F(A) \longrightarrow yA'(A) = \underline{A}(A,A')$$

sending τ on the left hand side to $\tau_{A'}(g)$. (Notice that since $\tau \in ((\mathcal{S}^{\underline{A}})^{opp})(y^{\cdot}(A), F^{+})$, $\tau_{A'}$ as a set mapping goes from $F^{+}(A')$ to $y^{\cdot}(A)(A') = \underline{A}(A,A')$.) In other words

(2) $$t_{(F,g,A')}(A)(\tau) = \tau_{A'}(g) \in \underline{A}(A,A')$$

or short $t_g(\tau) = \tau_{A'}(g)$ for $g\colon F \longrightarrow yA'$ and $\tau\colon y^{\cdot}(A) \longrightarrow F^{*}$. We leave it for the reader to check that $t_{(F,g,A')}$ is a morphism in $\mathcal{S}^{\underline{A}^{opp}}$. Next we have to prove a universal property for TF and t_g. Let $R \in \mathcal{S}^{\underline{A}^{opp}}$ be given together with a transformation k from the functor constant R to the functor $([\epsilon_F, y] \xrightarrow{\partial_1} \underline{A} \xrightarrow{y} \mathcal{S}^{\underline{A}^{opp}})$. We define a morphism $u\colon R \longrightarrow T'F$ in $\mathcal{S}^{\underline{A}^{opp}}$ by the formula

$$u_A(r)(A')(g) = k_{(F,g,A')}(A)(r)$$

or short $u(r)(g) = k_g(r)$ for $r \in R(A)$, $g \in F^{+}(A')$. One easily checks that

(3) $$k_g = t_g \circ u \quad \text{for } g = (F,g,A') \in [\epsilon_F, y],$$

and that u is a morphism in $\mathcal{S}^{\underline{A}^{opp}}$. Also, u is the unique morphism satisfying (3) for all g; for if v also satisfies (3), then by (2)

$$k_g(r) = t_g(v(r)) = (v(r))_{A'}(g).$$

Hence $T'F = TF$.

The η' is given on $x \in F(A)$, $g \in F^{+}(A')$ by $\eta'(x)(g) = g(x)$. Hence

$$t_g(\eta'(x)) = g(x)$$

and so $\eta = \eta'$. Just as obvious is $\mu = \mu'$. The theorem is proved.

5. Generalized limits

It is well-known (see e.g. [1]) that if $\underline{A}$ is a small category and $\underline{E}$ a right complete category, and $J: \underline{A} \to \underline{B}$ is any functor, then the induced functor

$$\underline{E}^J: \underline{E}^{\underline{B}} \to \underline{E}^{\underline{A}}$$

has a left adjoint S_J. Similarly, if $\underline{E}$ is left complete, $\underline{E}^J$ has a right adjoint $S_J^!$. Note that if $\underline{B}$ is not small, then $\underline{E}^{\underline{B}}$ is in general a category whose hom classes are proper classes. We shall use the notation $\varinjlim^{(J)}(F)$ for the value of S_J at $F \in \underline{E}^{\underline{A}}$. Similarly we use $\varprojlim^{(J)}(F)$ for $S_J^!(F)$. If $\underline{B}$ is the category 1, then $\varinjlim^{(J)}(F)$ is $\varinjlim(F)$ and $\varprojlim^{(J)}(F) = \varprojlim(F)$. For this reason we call $\varprojlim^{(J)}(F)$ a generalized inverse limit. Many functors arise that way, e.g. the conjugation functors + and * in Section 4, and the codensity monad.

Theorem 5.1. The conjugation functor + equals $\varprojlim^{(y)}(y^\cdot)$.

Proof. We have to show that + has a certain universal property. First we have to produce a transformation $+\circ y \xrightarrow{\tau} y^\cdot$; but (4.1) commutes up to equivalence, so take that equivalence as τ. Next, if $G': \mathcal{S}^{\underline{A}^{opp}} \longrightarrow (\mathcal{S}^{\underline{A}})^{opp}$ is a functor and $\tau': G'\circ y \to y^\cdot$ a transformation, we have to find a unique transformation $\varphi: G' \to +$ so that

(1) $$\tau'_A = \tau_A \circ \varphi_{yA}.$$

The morphism φ_F for $F \in \mathcal{S}^{\underline{A}^{opp}}$ is constructed as follows. It is as a set mapping $F^+A \to G'FA$, given by

$$\varphi_F(A)(g) = \lambda^{-1}(G'F \xrightarrow{G'g} G'yA \xrightarrow{\tau A} y^\cdot A),$$

where λ is the Yoneda isomorphism

(2) $$X(A) \longrightarrow (\mathcal{S}^{\underline{A}})^{opp}(X, y^\cdot A)$$

$(X \in (\mathcal{S}^{\underline{A}})^{opp})$. It is easy to check that φ_F is a morphism in

$(\mathcal{S}^{\underline{A}})^{opp}$, and that (1) holds. It is natural in F. If ψ were another such transformation satisfying (1), we would by naturality have commutativity of the two diagrams in $(\mathcal{S}^{\underline{A}})^{opp}$

$$
(3) \qquad \begin{array}{ccc}
G'F & \overset{\varphi_F}{\underset{\psi_F}{\rightrightarrows}} & F^{+} \\
{\scriptstyle G't}\downarrow & & \downarrow{\scriptstyle t^{+}} \\
G'yA & \xrightarrow[\varphi_{yA}]{} & (yA)^{+}
\end{array}
$$

for any t. Use the contravariant functor: evaluation at A. Again, by the isomorphism (2), if $\varphi_F \neq \psi_F$ there would be a t such that

$$\varphi_F(A)\circ t^{+}(A) \neq \psi_F(A)\circ t^{+}(A),$$

contradicting the commutativity of (3). This proves the theorem.

The proofs of the following theorems will be omitted since they are similar to the preceding one: checking a universal property.

Theorem 5.2. The conjugation functor $*$ equals $\underrightarrow{\lim}^{(y^{\cdot})}(y)$.

Theorem 5.3. Let $y: \underline{A} \rightarrow \mathcal{B}$ be any functor, with $\underline{A}$ small and $\mathcal{B}$ left complete. Then

$$\underleftarrow{\lim}^{(y)}(y): \mathcal{B} \rightarrow \mathcal{B}$$

is the codensity monad for y.

Theorem 5.4. Let $J: \underline{A} \longrightarrow \underline{B}$ be any functor between small categories. Then the composite functor

$$\underline{B} \xrightarrow{y_{\underline{B}}} \mathcal{S}^{\underline{B}^{opp}} \xrightarrow{\mathcal{S}^{J}} \mathcal{S}^{\underline{A}^{opp}}$$

equals

$$\underrightarrow{\lim}^{(J)}(y_{\underline{A}})$$

($y_{\underline{A}}$ the Yoneda embedding $\underline{A} \longrightarrow \mathcal{S}^{\underline{A}^{opp}}$).

(Proofs of the two last theorems were given in [4].) It seems plausible that a combination of the theorems in this section will give another way of proving the connection between the codensity monad for the Yoneda embedding and the duality monad.

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