CONTINUOUS YONEDA REPRESENTATION
OF A SMALL CATEGORY

Anders Kock

October, 1966
CONTINUOUS YONEDA REPRESENTATION
OF A SMALL CATEGORY

by Anders Kock

1. Introduction

The well-known full and faithful embedding of a small category $A$ into the category $\mathcal{C}^{A^{\text{opp}}}$ of contravariant functors from $A$ to $\mathcal{C}$ (the category of sets) is left continuous (preserves inverse limits), but in general not right continuous. Also, the embedding functor is dense in the sense of Ulmer [7], or, equivalently, adequate in the sense of Isbell [3]; but it is not in general codense. In Section 2 we define a "codensity monad" for any functor from a small category to a left complete category. The word "monad" means "triple" as defined by Eilenberg and Moore [2]. In particular, the Yoneda embedding defines a codensity monad $T$ on $\mathcal{C}^{A^{\text{opp}}}$. We study the universal generator $[2]

\begin{align*}
\mathcal{C}^{A^{\text{opp}}} & \xrightarrow{F} (\mathcal{C}^{A^{\text{opp}}})^T \xrightarrow{U} \mathcal{C}^{A^{\text{opp}}} \\
(F \text{ left adjoint to } U, \text{ } UF = T)
\end{align*}

for $T$ in Section 3, and prove that the Yoneda embedding followed by $F$ gives a full and faithful embedding which is as well left as right continuous. (Corollary 3.2; this is what we refer to in the headline.) In Section 4 we show that the monad which Isbells adjoint conjugation functors $([3], \ast \text{ and } +)$

\begin{align*}
\mathcal{C}^{A^{\text{opp}}} & \xrightarrow{\ast} (\mathcal{C}^{A^{\text{opp}}})^{\ast} \xrightarrow{+} \mathcal{C}^{A^{\text{opp}}}
\end{align*}

determine is precisely the codensity monad $T$. Finally, in Section 5 we define generalized (direct and inverse) limit functors and show that the duality functors (1) as well as codensity monads and density comonads come in that way. Also, if $J: C \rightarrow D$ is a functor
between small categories, the obvious functor

$$D \rightarrow C^{\text{opp}}$$

is a generalized limit.

If $A$ and $B$ are objects in a category $A$, we denote
\(\text{hom}(A,B)\) by $A(A,B)$. This is assumed to be a set throughout. If
$F: A \rightarrow B$ is a functor whose inverse limit exists ("left root"),
we denote this limit by $\operatorname{lim}^{-1}(F)$. Similarly $\operatorname{lim}^{-1}(F)$ for direct
limit. A category $B$ is called left complete if all functors from
small categories to $B$ have inverse limits. Similarly for right
complete and direct limits.

Denote the identity morphism of an object $B$ by $I_B$.

2. Codensity monads

We recall Lawvere's definition [6] of a comma category. Given
functors $F, G$ with common codomain

$$A_0 \xrightarrow{F} B \xleftarrow{G} A_1.$$ 

Then the comma category $[F,G]$ has as objects triples $(A_0,b,A_1)$,
where $A_i$ is an object in $A_i$ and $b$ is a morphism in $B$:

$$FA_0 \xrightarrow{b} GA_1.$$ 

A morphism from $(A_0,b,A_1)$ to $(A'_0,b',A'_1)$ is a pair $(a_0,a_1)$
where $a_i: A_i \rightarrow A'_i$ in $A_i$ such that the diagram

$$\begin{array}{ccc}
FA_0 & \xrightarrow{b} & GA_1 \\
Fa_0 \downarrow & & \downarrow Ga_1 \\
FA'_0 & \xrightarrow{b'} & GA'_1
\end{array}$$

commutes. There are obvious functors

(1) $[F,G] \xrightarrow{a_0} A_0, \quad [F,G] \xrightarrow{a_1} A_1$. 
In the important special case, where \( A_0 \) is the category \( 1 \) (the one-morphism category), the functor \( F \) determines and is determined by an object \( B \in \mathcal{C} \). In that case \([F,G]\) will be denoted \([\varepsilon_B,G]\). Similarly, if \( A_1 = 1 \) and \( G \) determines and is determined by \( B \in \mathcal{C} \), we denote \([F,G]\) by \([F,\varepsilon_B]\).

Now, let \( \mathcal{B} \) be a left complete category, and let \( A \) be small. We shall assign to a functor \( G: A \to \mathcal{B} \) a monad \( T_G: \mathcal{C} \to \mathcal{B} \) on \( \mathcal{B} \). Given an object \( B \in \mathcal{C} \). Put

\[
T_G(B) = \varprojlim \left( [\varepsilon_B,G] \xrightarrow{\varepsilon_B} A \xrightarrow{G} B \right).
\]

One easily sees that \( T_G \) is a covariant functor. If \((B,b,A)\) is an object in \([\varepsilon_B,G]\), (i.e. \( B \xrightarrow{b} GA \) a morphism in \( \mathcal{B} \)), denote by \( t(B,b,A) \) (or short, but not precise, by \( t_b \)) the canonical projection

\[
t(B,b,A): T_G(B) = \varprojlim \left( [\varepsilon_B,G] \xrightarrow{\varepsilon_B} A \xrightarrow{G} B \right) \xrightarrow{G} GA.
\]

We define a functor transformation \( \gamma: I_{\mathcal{B}} \to T_G \) by

\[
t(B,b,A) \circ \gamma_B = b.
\]

Finally, define a functor transformation \( \mu: (T_G)^2 \to T_G \) by

\[
t(B,b,A) \circ \mu_B = t(T_G(B), t(N,b,A), A)
\]

(in the short notation \( t_b \circ \mu_B = t_b \)).

It is now easy to check that (2), (4), and (5) together determine a monad on \( \mathcal{B} \) (monad meaning triple, [2]). Ulmer's definition [6] of codensity of the functor \( G \) can be stated: \( \gamma \) is an equivalence. We therefore call \((T_G, \gamma, \mu)\) the codensity monad for \( G: A \to \mathcal{B} \). The same monad (or rather its dual) appear in Tierney's work and is called model cotriple.
3. A right continuous embedding

Recall [2] that a monad \((T, \gamma, \mu)\) on a category \(\mathcal{C}\) has a universal generator

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{C}^T \\
& \xrightarrow{\xi} & \\
& \xrightarrow{T} & \mathcal{C}
\end{array}
\]

where \(F\) is left adjoint to \(U\). The objects of \(\mathcal{C}^T\) are morphisms \(T(B) \xrightarrow{\xi} B\) with

\[
(2) \quad \xi_\circ \tau(\xi) = \xi_\circ \mu_B \quad \text{and} \quad \xi_\circ \eta_B = I_B.
\]

The morphisms are commutative diagrams

\[
\begin{array}{ccc}
T(B) & \xrightarrow{T(b)} & T(B') \\
\downarrow{\xi} & & \downarrow{\xi'} \\
B & \xrightarrow{b} & B'
\end{array}
\]

and \(F\) and \(U\) are defined by \(F(B) = (T^2(B) \xrightarrow{\mu} T(B)); U(TB) \xrightarrow{\xi} B = B\).

**Theorem 3.1.** Let \(y: A \rightarrow \mathcal{B}\) be a full and faithful embedding of a small category into a left complete category. Let \((T, \gamma, \mu)\) be its codensity monad, and \((F, \gamma, \mu)\) (as in (1)) its universal generator. Then the functor \(F \circ y\) is a full and faithful right continuous embedding.

**Proof.** Using the definition (2.2) of \(T_y\), one easily gets \(T_y(yA) \cong yA\) by means of \(\gamma(yA);\) the inverse for \(\gamma(yA)\) is \(t(yA, yA, A)\) also \(\mu_{yA}\) is an isomorphism. So it follows immediately from the definition of \(F\) and \(\mathcal{C}^T\) that \(F \circ y\) is full and faithful; it is equivalent to the functor \(y\)' which sends \(A\) to the object \(T_yA \xrightarrow{t(IyA)} yA\) in \(\mathcal{C}^T\). We now prove right continuity of \(y\)' Let \(R: D \rightarrow A\) be a functor with a direct limit \(A_\infty\) in \(A\). Let the morphisms in the limit diagram be denoted

\[
R(D) = A_D \xrightarrow{i_D} A_\infty.
\]
Let a transformation \( f \) from \( y^\circ R \) to a constant functor \( \varepsilon(X, \xi) \) be given, i.e. a family of morphisms in \( \mathcal{C}^T \)

\[
y^A_D \xrightarrow{f_D}(X, \xi),
\]

where \( TX \xrightarrow{\xi} X \) is an object in \( \mathcal{C}^T \). We are required to produce a morphism

\[
f_\infty : y^A_\infty \longrightarrow (X, \xi).
\]

Let us first produce a morphism \( \alpha \) in \( \mathcal{C} \)

\[
\alpha : yA_\infty \longrightarrow TX = \lim_{\xi}(\varepsilon_X, y) \xrightarrow{\varepsilon_1} \mathcal{A} \xrightarrow{y} \mathcal{C},
\]

Put

\[
t(X, g, A) \circ \alpha \circ y(i_D) = g^\circ Uf_D : yA_D \longrightarrow yA;
\]

\( \alpha \) is determined by this, since \( yA_\infty \) is the direct limit of the \( yA_D \)'s in the subcategory \( \mathcal{A} \).

We see that

\[
\xi \circ \alpha \circ y(i_D) = Uf_D;
\]

for \( U(f_D) \) can be described as \( \xi \circ TU(f_D) \circ \varphi_D \), so to see (5) it suffices to show

\[
\alpha \circ y(i_D) = TU(f_D) \circ \varphi_D
\]

or

\[
tg \circ \alpha \circ y(i_D) = tg \circ TU(f_D) \circ \varphi_D
\]

for all \( g : X \rightarrow yA \). But this follows easily from (4) and the definition of the transformation \( \varphi \); so (5) is proved. Next we prove commutativity of

\[
Ty(A_\infty) \xrightarrow{T(\xi \circ \alpha)} TX
\]

Precede the diagram by \( \dot{\varphi} y(A_\infty) = (t_A^I(A_\infty))^{-1} \); we then have to prove

\[
\xi \circ \alpha = \xi \circ T(\xi \alpha) \circ \varphi yA_\infty. \quad \text{This follows easily, using naturality of } \varphi,
\]
and the equations
\[ \xi \cdot T\xi = \xi \cdot \mu; \quad \mu \cdot \zeta_{TX} = \mathbb{I}_{TX}. \]

Commutativity of (6) is proved, and therefore the diagram is a morphism \( f_\infty \) in \( B^T \) from \( y' A_\infty \) to \( (X, \xi) \). We have
\[ f_\infty \circ y'(i_D) = f_D; \]
for \( U \) is faithful [2], and acting with \( U \) on (7) gives (5) since \( U(f_\infty) = \xi \circ \alpha \).

Finally, \( f_\infty \) is the unique morphism satisfying (7). For \( U f_\infty \) must satisfy
\[ U(f_\infty) \circ y(i_D) = U(f_D), \]
and hence \( T U f_\infty \) satisfy (the right hand equality sign in)
\[ t_g \circ T U(f_\infty) \circ y(A_\infty) \circ y(i_D) = t_g \circ T U(f_\infty) \circ T y(i_D) \circ y(A_D) \]
(9)
\[ = t_g \circ T U(f_D) \circ y(A_D) \]
for all \( g: X \to yA \) and all \( D \in \mathcal{D} \). Since \( y(A_\infty) \) is an isomorphism and the morphism in (9) ends up in an object in \( y(A) \), we can use the direct limit property of \( A_\infty \) to get that \( t_g \circ T U(f_\infty) \) is the only morphism in \( B \) which satisfies (9) for all \( D \in \mathcal{D} \). Using next the inverse limit property of \( T X \), we get that \( T U(f_\infty) \) is the only morphism in \( B \) which satisfy (9) for all \( g \) and \( D \).

Since \( t_{I(A_\infty)} \) is an isomorphism, there is at most one morphism \( \varphi \) in \( B \) with \( T \varphi = T U(f_\infty) \) and \( \varphi \circ t_{I(A_\infty)} = \xi \cdot T \varphi \). This proves the uniqueness of \( f_\infty \), and therefore the theorem.

**Corollary 3.2.** Let \( y: A \to \mathcal{A}^{\text{opp}} \) be the Yoneda embedding for the small category \( A \). Let \( (T, \gamma, \mu) \) be its codensity monad, and \( (F, U) \) (as in (1)) its universal generator. Then the functor \( Fy \) is a full and faithful left and right continuous embedding.
Proof. The preceding theorem gives everything except left continuity. Now every \( U \) appearing in a universal generator (1) reflects and preserves inverse limits, i.e., if \( R: D \rightarrow C^T \) is any functor, then \( \lim (U \circ R) \) exists iff \( \lim (R) \) exists, and
\[
U \lim (R) = \lim (U \circ R).
\]
Using this and the fact that \( U \cdot y' \) equals the left continuous \( y \), we get left continuity of \( y' \). But \( F \cdot y \) is equivalent to \( y' \).

4. The conjugation monad

Isbell, in [3], defined the conjugate of a set valued functor \( A^{\text{opp}} \rightarrow \mathcal{C} \). In the case where \( A \) is small, the conjugation procedure gives (covariant) functors \( + \) and \( * \) so that the diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{y} & y' \\
\downarrow & & \downarrow \\
\mathcal{C}^{A^{\text{opp}}} & \xrightarrow{+} & \mathcal{C}^{A^{\text{opp}}} \\
\mathcal{C}^{A^{\text{opp}}} & \xleftarrow{*} & \mathcal{C}^{A^{\text{opp}}} \\
\end{array}
\]

commute. Here \( y' \) is the (full and faithful) co-Yoneda embedding functor \( A \sim \mathcal{C}(A,-) \). The definition of \( + \) and \( * \) is as follows.

Let \( F \in \mathcal{C}^{A^{\text{opp}}} \). Then \( F^+ \in (\mathcal{C}^{A^{\text{opp}}})^{\text{opp}} \) is the functor
\[
A \xrightarrow{\mathcal{C}^{A^{\text{opp}}}} (F,yA).
\]

Let \( G \in (\mathcal{C}^{A^{\text{opp}}})^{\text{opp}} \). Then \( G^* \in \mathcal{C}^{A^{\text{opp}}} \) is the functor
\[
A \xrightarrow{\mathcal{C}^{A^{\text{opp}}}} (y'A,G).
\]

Lambek [5] noticed that \( + \) is left adjoint to \( * \). So they give rise to a monad \( (T', \varphi', \kappa') \) on \( \mathcal{C}^{A^{\text{opp}}} \).

Theorem 4.1. The monad on \( \mathcal{C}^{A^{\text{opp}}} \) coming from the conjugation functors equals the codensity monad for \( y \).
Proof. We denote as in the preceding sections the codensity monad for \( y \) by \((T, \gamma, \mu)\). We first prove \( T = T' \), i.e. that \( T'F \) can be used as the inverse limit of \( ([\epsilon_F, y] \xrightarrow{\partial} A \xrightarrow{y} S_A^{\text{opp}}) \). Let \((F, g, A') \in [\epsilon_F, y] \), i.e. \( g: F \to yA' \). Define

\[
 t_{(F, g, A')} : T'F \to yA
\]

by letting \( t_{(F, g, A')} (A) \) be the map

\[
 \left( S_A \right)^{\text{opp}}(y'(A), F^+) = T'F(A) \to yA'(A) = A(A, A')
\]

sending \( \tau \) on the left hand side to \( \tau_A'(g) \). (Notice that since \( \tau \in \left( S_A^{\text{opp}} \right)(y'(A), F^+) \), \( \tau_A' \) as a set mapping goes from \( F^+(A') \) to \( y'(A)(A') = A(A, A') \).) In other words

\[
 (2) \quad t_{(F, g, A')} (A)(\tau) = \tau_A'(g) \in A(A, A')
\]

or short \( t_g (\tau) = \tau_A'(g) \) for \( g: F \to yA' \) and \( \tau: y'(A) \to F^* \). We leave it for the reader to check that \( t_{(F, g, A')} \) is a morphism in \( S_A^{\text{opp}} \). Next we have to prove a universal property for \( TF \) and \( t_g \).

Let \( R \in S_A^{\text{opp}} \) be given together with a transformation \( k \) from the functor constant \( R \) to the functor \( ([\epsilon_F, y] \xrightarrow{\partial} A \xrightarrow{y} S_A^{\text{opp}}) \). We define a morphism \( u: R \to T'F \) in \( S_A^{\text{opp}} \) by the formula

\[
 u_A(r)(A')(g) = k_{(F, g, A')}(A)(r)
\]

or short \( u(r)(g) = k_g (r) \) for \( r \in R(A) \), \( g \in F^+(A') \). One easily checks that

\[
 (3) \quad k_g = t_g \circ u \quad \text{for} \quad g = (F, g, A') \in [\epsilon_F, y],
\]

and that \( u \) is a morphism in \( S_A^{\text{opp}} \). Also, \( u \) is the unique morphism satisfying (3) for all \( g \); for if \( v \) also satisfies (3), then by (2)

\[
 k_g (r) = t_g (v(r)) = (v(r))_{A'}(g).
\]

Hence \( T'F = TF \).

The \( \gamma' \) is given on \( x \in F(A) \), \( g \in F^+(A') \) by \( \gamma'(x)(g) = g(x) \). Hence

\[
 t_g (\gamma'(x)) = g(x)
\]

and so \( \gamma = \gamma' \). Just as obvious is \( \mu = \mu' \). The theorem is proved.
5. Generalized limits

It is well-known (see e.g. [1]) that if $A$ is a small category and $E$ a right complete category, and $J: A \rightarrow B$ is any functor, then the induced functor

$$E^J: E^B \rightarrow E^A$$

has a left adjoint $S_J$. Similarly, if $E$ is left complete, $E^J$ has a right adjoint $S^J$. Note that if $B$ is not small, then $E^B$ is in general a category whose hom classes are proper classes.

We shall use the notation $\lim^{(J)}(F)$ for the value of $S_J$ at $F \in E^A$. Similarly we use $\lim^{(J)}(F)$ for $S^J(F)$. If $B$ is the category 1, then $\lim^{(J)}(F)$ is $\lim(F)$ and $\lim^{(J)}(F) = \lim(F)$.

For this reason we call $\lim^{(J)}(F)$ a generalized inverse limit.

Many functors arise that way, e.g. the conjugation functors $+$ and $*$ in Section 4, and the codensity monad.

Theorem 5.1. The conjugation functor $+$ equals $\lim^{(y)}(y')$.

Proof. We have to show that $+$ has a certain universal property. First we have to produce a transformation $+ y \rightarrow y'$; but (4.1) commutes up to equivalence, so take that equivalence as $\varepsilon$.

Next, if $G': (\mathcal{C}^A)^{\text{opp}} \rightarrow (\mathcal{C}^A)^{\text{opp}}$ is a functor and $\varepsilon': G' y \rightarrow y'$ a transformation, we have to find a unique transformation $\Phi: G' \rightarrow+$ so that

$$\varepsilon'_A = \varepsilon _A \circ \Phi _A.$$ 

The morphism $\Phi _F$ for $F \in (\mathcal{C}^A)^{\text{opp}}$ is constructed as follows. It is as a set mapping $F^A \rightarrow G'FA$, given by

$$\Phi _F(A)(g) = \lambda^{-1}(G'F \rightarrow G'yA \rightarrow y'A),$$

where $\lambda$ is the Yoneda isomorphism

$$\lambda: X(A) \rightarrow (\mathcal{C}^A)^{\text{opp}}(X,y'A)$$

($X \in (\mathcal{C}^A)^{\text{opp}}$). It is easy to check that $\Phi _F$ is a morphism in
\((\mathbb{S}^A)^{\text{opp}}\), and that (1) holds. It is natural in \(F\). If \(\psi\) were another such transformation satisfying (1), we would by naturality have commutativity of the two diagrams in \((\mathbb{S}^A)^{\text{opp}}\):

\[
\begin{array}{ccc}
G'F & \xrightarrow{\phi_F} & F' \\
\downarrow \psi_F & & \downarrow t^+ \\
G't & \xrightarrow{\psi} & (tA)^+
\end{array}
\]

(3)

for any \(t\). Use the contravariant functor: evaluation at \(A\). Again, by the isomorphism (2), if \(\phi_F \neq \psi_F\) there would be a \(t\) such that

\[
\phi_F(A) \cdot t'(A) \neq \psi_F(A) \cdot t'(A),
\]

contradicting the commutativity of (3). This proves the theorem.

The proofs of the following theorems will be omitted since they are similar to the preceding one: checking a universal property.

**Theorem 5.2.** The conjugation functor \(*\) equals \(\lim(y^*)(y)\).

**Theorem 5.3.** Let \(y: A \to \mathcal{B}\) be any functor, with \(A\) small and \(\mathcal{B}\) left complete. Then

\[
\lim(y)(y): \mathcal{B} \to \mathcal{B}
\]

is the codensity monad for \(y\).

**Theorem 5.4.** Let \(J: A \to \mathcal{B}\) be any functor between small categories. Then the composite functor

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{y_B} & \mathbb{S}^B^{\text{opp}} & \xrightarrow{\mathbb{S}J} & \mathbb{S}^A^{\text{opp}} \\
\downarrow \lim(J)(y_A) & & & & \\
(y_A) & & \text{the Yoneda embedding } & & \mathbb{S}^A^{\text{opp}}
\end{array}
\]

equals.

(Proofs of the two last theorems were given in [4].) It seems plausible that a combination of the theorems in this section will give another way of proving the connection between the codensity monad for the Yoneda embedding and the duality monad.
REFERENCES


University of Aarhus, Denmark

October, 1966.