Classifying surjective equivalences

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An equivalence of groupoids $P : \mathbf{G}' \to \mathbf{G}$ is called *surjective* if it surjective on objects. If $p : G'_0 \to G_0$ is the object-part of P, we say that P *covers* p. The classification we give is more precisely: given a groupoid $\mathbf{G}'$ and a fixed surjection $p : G'_0 \to G_0$; then we describe the category of equivalences $\mathbf{G}' \to \mathbf{G}$ covering p in terms of a category of cocycles.

The study was motivated by the desire to construct the "gauge" groupoid XX^{-1} of a principal G-bundle X directly out of a G-valued Čech cocycle for the bundle X; such desription is essentially given in [1], Theorem II.2.19. We reproduce this in Section 2. There we also discuss briefly the two extreme cases, where p is the identity map, and where p has codomain 1.

We present the construction and theory in the category of sets. But it is clear that it applies in any topos, and hence also, via a Yoneda embedding, in any category, provided the notion of "surjection" is replaced by "universal effective descent epi". For the case of principal bundles, the choice of surjection is often "etale surjection", like $\coprod U_i \to M$, where $(U_i)_{i \in I}$ is an open cover of a space M.

Most of the present note was circulated as a pamphlet entitled "Descending groupoids" in Feb. 2001. We used there a superfluous hypothesis, namely that the groupoids in question were *transitive* (connected).

1 Groupoids under a groupoid, vs. cocycles

For a fixed groupoid $\mathbf{G}'$ with object set G'_0, and for a fixed surjection $p : G'_0 \to G_0$, we organize the class of equivalences $P : \mathbf{G}' \to \mathbf{G}$, covering p, as objects of a category (a groupoid, in fact); the morphisms from $P : \mathbf{G}' \to \mathbf{G}$ to $Q : \mathbf{G}' \to \mathbf{H}$ are pairs T, t, where $T : \mathbf{G} \to \mathbf{H}$ is a full and faithful functor which is the identity map on the set G_0 of objects, and where t is a natural transformation from $T \circ P$ to Q. Call the category thus described $Eq(p, \mathbf{G}')$ (for "equivalences covering p"). It may be considered as a subcategory of groupoids *under* $\mathbf{G}'$.

We let R denote the kernel pair of $p : G'_0 \to G_0$. It is a groupoid with G'_0 as set of objects, so we may consider the functor category, whose objets are those functors ∇ from R to $\mathbf{G}'$, which are the identity map on the set G'_0 of objects. Morphisms are the natural transformations between such functors. Call the category thus described $Cocycl(p, \mathbf{G}')$. (Its objects are kind of cocycles, cf. Example 1 in the next Section; and in a certain sense, they are descent data).

There is a canonical functor $Eq(p, \mathbf{G}') \to Cocycl(p, \mathbf{G}')^{op}$ which we now describe. First, given an object $P : \mathbf{G}' \to \mathbf{G}$ of $Eq(p, \mathbf{G}')$, we describe the cocycle ∇ as follows. Let $(x', x'') \in R$, so $p(x') = p(x'')$ ($= x$, say). We put $\nabla(x', x'')$ equal to the unique arrow $x' \to x''$ which by P goes to the identity arrow 1_x of x. (Recall that P is assumed to be full and faithful.) By uniqueness, it is clear that ∇ is indeed a functor. Next, given a morphism T, t of $Eq(p, \mathbf{G}')$. Let $\tilde{\nabla}$ be the cocycle associated to its codomain $Q : \mathbf{G}' \to \mathbf{H}$. To describe a natural transformation $\tau : \tilde{\nabla} \to \nabla$, let $x' \in G'_0$. Then $\tau_{x'}$ is taken to be the unique arrow $x' \to x'$ which by Q maps to $t_{x'}$. Note that $t_{x'}$ is an endo-arrow, since $T(p(x')) = q(x')$, by assumption. (Note that T does not enter in the description; in fact, T is redundant information, it can be reconstructed from t.) To prove naturality of τ, consider the naturality square for t w.r.to the arrow $\nabla(x', x'')$. This naturality square is (composing forwards)

$$t_{x'}.Q(\nabla(x', x'')) = T(P(\nabla(x', x''))).t_{x''}.$$

(All four corners of this naturality square are equal, namely $T(p(x')) = q(x') = q(x'') = T(p(x''))$ ($= x$, say). This equation may be written

$$Q(\tau_{x'}).Q(\nabla(x', x'')) = 1_x.Q(\tau_{x''}),$$

and so since Q is a functor, we conclude

$$Q(\tau_{x'}.\nabla(x', x'').\tau_{x''}^{-1}) = 1_x$$

from which we conclude that $\tau_{x'}.\nabla(x', x'').\tau_{x''}^{-1} = \tilde{\nabla}(x', x'')$, and this is exactly the naturality condition for $\tau : \tilde{\nabla} \to \nabla$ with respect to the arrow (x', x'') in R. So we have defined a functor

$$Eq(p, \mathbf{G}') \to Cocycl(p, \mathbf{G}')^{op} \tag{1}$$

Theorem 1 *The functor (1) is an equivalence of categories.*

We construct an inverse for it, up to isomorphism; in other words, given a cocycle $\nabla : R \to \mathbf{G}'$, we construct a surjective equivalence $P : \mathbf{G}' \to \mathbf{G}$ (we "descend $\mathbf{G}'$ along p"); and to a natural transformation $\tau : \tilde{\nabla} \to \nabla$ ("cohomologous cocycles"), we define a morphism (T, t) between the descended groupoids.

So given $\nabla : R \to \mathbf{G}'$, we construct the groupoid $\mathbf{G}$ as follows. The objects are those of the given G_0, of course; if $x, y \in G_0$, then an arrow $x \to y$ is an equivalence class of triples $< x', \phi, y' >$ where $p(x') = x$ and $p(y') = y$, and $\phi : x' \to y'$, under an equivalence relation yet to be described. To compose

$$< x', \phi, y' > \text{ and } < y'', \psi, z' >,$$

where $p(y') = p(y'')$, we take the triple

$$< x', \phi.\nabla(y', y'').\psi, z' >, \tag{2}$$

and then it is clear that the composition is associative and has triples of form $< x', 1_{x'}, x' >$ as identities (using $\nabla(x', x') = 1_{x'}$). The equivalence relation is given by identifying

$$< x', \phi, y' > \text{ with } < x'', \nabla(x'', x').\phi.\nabla(y', y''), y'' > .$$

To prove that the composition (2) is well defined, one uses the cocycle condition $\nabla(x', x'').\nabla(x'', x''') = \nabla(x', x''')$. The functor P which takes ϕ to the equivalence class of $< x', \phi, y' >$, where x' and y' are the domain and codomain of ϕ, is then clearly full and faithful, and the construction of $\mathbf{G}' \to \mathbf{G}$ is then provided. It is clear that the cocycle arising from P is the given ∇.

This describes the functor $Cocycl(p, \mathbf{G}') \to Eq(p, \mathbf{G}')$ in so far as objects are concerned. We now consider it on morphisms, so assume that τ is a natural transformation from $\tilde{\nabla}$ to ∇. Let $P : \mathbf{G}' \to \mathbf{G}$ and $Q : \mathbf{G}' \to \mathbf{H}$ be the surjective equivalences constructed from ∇ and $\tilde{\nabla}$, respectively. We construct a morphism $(T, t) : P \to Q$ in $Eq(p, \mathbf{G}')$ as follows

First, we produce an (iso)morpism of groupoids $T : \mathbf{G} \to \mathbf{H}$ (identity on the set G_0 of objects); for arrows: to $< x', \phi, y' >$, associate $\{x', \tau_{x'}.\phi.\tau_{y'}^{-1}, y'\}$, where curly brackets denote the (equivalence classes of) triples that define $\mathbf{H}$ out of $\tilde{\nabla}$).

Both well-definedness and preservation of composition depend in a purely equational way on the the naturality of τ, and we shall do only the proof for composition in detail. Given $< x', \phi, y' >$ and $< y'', \psi, z' >$, their composite

$$< x', \phi.\nabla(y', y'').\psi, z' >$$

goes by T to $\{x', \tau_{x'}.\phi.\nabla(y', y'').\psi.\tau_{z'}^{-1}, z'\}$, whereas the composite of

$$\{x', \tau_{x'}.\phi.\tau_{y'}^{-1}, y'\} \text{ and } \{y'', \tau_{y''}.\psi.\tau_{z'}^{-1}, z'\}$$

is

$$\{x', \tau_{x'}.\phi.\tau_{y'}^{-1}.\tilde{\nabla}(y', y'').\tau_{y''}.\psi.\tau_{z'}^{-1}, z'\}.$$

Now the product of three factors $\tau_{y'}^{-1}.\tilde{\nabla}(y', y'').\tau_{y''}$ may by naturality of τ with respect to (y', y'') be replaced by $\nabla(y', y'')$, and then we are back at the previous expression.

We finally have to produce $t_{x'} : T(P(x')) \to Q(x')$, for any $x' \in G_0'$. We put $t_{x'} := \{x', \tau_{x'}, x'\}$. Let us prove naturality of t w.r. to $\phi : x' \to y'$. This means to prove the equation

$$t_{x'}.Q(\phi) = T(P(\phi)).t_{y'},$$

and by the definitions, this means

$$\{x'.\tau_{x'}, x'\}.\{x', \phi, y'\} = T(< x', \phi, y' >).\{y', \tau_{y'}, y'\}.$$

The left hand side is $\{x', \tau_{x'}.\phi, y'\}$, since $\tilde{\nabla}(x', x') = 1$; the right hand side is

$$\{x', \tau_{x'}.\phi.\tau_{y'}^{-1}, y'\}.\{y', \tau_{y'}, y'\},$$

and using $\tilde{\nabla}(y', y') = 1$, the $\tau_{y'}^{-1}$ and $\tau_{y'}$ cancel, and we are again left with $\{x', \tau_{x'}.\phi, y'\}$, proving naturality of the t constructed.

It is easy to prove that the composite of the two constructions

$$Cocycl(\mathbf{G}', p) \to Eq(\mathbf{G}', p) \to Coycl(\mathbf{G}', p)$$

is in fact the identity functor. Conversely, consider the effect of

$$Eq(\mathbf{G}', p) \to Coycl(\mathbf{G}', p) \to Eq(\mathbf{G}', p)$$

on $P : \mathbf{G}' \to \mathbf{G}$, call the result $Q : \mathbf{G}' \to \mathbf{H}$. There is a canonical isomorphism from Q to P: there is a canonical functor $T : \mathbf{H} \to \mathbf{G}$ given on morphisms by

$$< x', \phi, y' > \mapsto P(\phi).$$

This functor evidently satisfies $T \circ Q = P$, and taking the 2-cell t to be the identity, we obtain the desired canonical isomorphism (T, t) from Q to P.

This concludes the proof/construction of the Theorem.

2 Special cases

Example 1. Suppose $U_i \to M$ is an open cover of a space M, and let G be a group. A G-valued Čech cocycle γ_{ij} defined on this cover may be construed as a cocycle in the sense of the previous section; the surjection p in question is $U \to M$, where $U = \coprod_i U_i$. We have a groupoid $\mathbf{G}'$ with object set U, namely the product of the "chaotic" groupoid $U \times U$ on U, and the group G (considered as a 1-object groupoid). (This is what in [1] is called the *trivial groupoid on U with group G*.) An element (x', x'') of the kernel pair R of $U \to M$ is the same thing as a triple (x, i, j) where $x \in U_i \cap U_j \subseteq M$, and we put

$$\nabla(x', x'') := (x', x'', \gamma_{ij}(x)) \in U \times U \times G.$$

The construction then furnishes a surjective equivalence from the groupoid $U \times U \times G$ to a groupoid on M. If $X$is the principal G-bundle on M constructed by the Čech cocycle, the groupoid constructed is what is usually denoted XX^{-1}. - This is essentially the construction given in [1], Theorem II.2.19.

To give a natural transformation $\tau : \nabla \Rightarrow \tilde{\nabla}$ is equivalent to exhibiting the two corresponding Čech cocycles γ and $\tilde{\gamma}$ as cohomologous.

Example 2. The map $p : G'_0 \to G_0$ is the identity. Then R is the diagonal (the minimal equivalence relation on G'_0), and there is only one cocycle ∇. So the category (groupoid) $Cycl(\mathbf{G}', p)$ has only one object, i.e. is a group. It is easy to see that this group is $\prod_x \mathbf{G}'(x, x)$ where x ranges over G'_0. The category of equivalences out of $\mathbf{G}'$ covering the identity map is therefore, by the Theorem, equivalent to this group.

Example 3. The map p is $p : G'_0 \to 1$. Then R is the maximal equivalence relation on G'_0, and a cocycle ∇ is the same thing as a trivialization of the groupoid $\mathbf{G}'$. Of course, there are only such trivializations if the groupoid is transitive. An equivalence covering $p : G'_0 \to 1$ is the same thing as an equivalence of $\mathbf{G}'$ with a group. But such equivalences are now organized in a category (groupoid). Because of the Theorem, we may analyze this groupoid by means of the groupoid of natural transformations τ between trivializations ∇ of $\mathbf{G}'$. Because there now are so many arrows in R, there are many naturality constraints, and in fact it is clear that such a natural transformation τ is completely described by its component τ_x at any fixed $x \in G'_0$.

So the groupoid of equivalences covering $G'_0 \to 1$ is equivalent (not canonically, though) to any vertex group $\mathbf{G}'(x,x)$.

References

[1] K. Mackenzie, Lie Groupoids and Lie Algebroids in Differential Geometry, London Math. Soc. Lecture Note Series 124, Cambridge University Press 1987.