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CONNECTIONS AND PATH CONNECTIONS IN GROUPOIDS

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Connections and path connections in groupoids

Anders Kock

Abstract We describe two gauge theoretic notions of connection in a (differentiable) groupoid. The two notions are related via the notion of holonomy. Holonomy formation (integration) is shown to be inverse of a certain differentiation process.

Introduction

The present note is an attempt to clarify (for myself, at least) some of the basic structures underlying gauge theory, in particular the holonomy construction. A more particular purpose of this attempt is to prepare the way for a synthetic rendering of "higher gauge theory", a subject intensively studied presently (cf. e.g. [1] and the references therein).

Gauge theory (i.e. 1-dimensional gauge theory) has traditionally as its basic structure that of a *connection in a principal fibre bundle*. Now there is a close relationship between principal fibre bundles, on the one hand, and groupoids on the other. The first choice made in the present note is to put the emphasis on the groupoid viewpoint, which seems to me more clean, and avoiding principal bundles altogether. (A functorial comparison between connection theory in groupoids and in principal bundles may be found in [13].) So we study connections in groupoids. This is not a novelty; it actually goes back to Ehresmann in the 1950s, and has also been an active viewpoint in modern gauge theory.

The second choice made is to utilize a certain "combinatorial" method from synthetic differential geometry (SDG), as expounded in e.g. [6], [7], [9], and in the work of Breen and Messing [2], [3]. As far as I understand, the latter work utilizes the synthetic method as one of the tools to lift 1-dimensional gauge theory (in principal bundles) one dimension up, replacing principal bundles by gerbes.

In the present work, we do not (yet) go into the next dimension, i.e. into dimension 2, where the notion of groupoid lifts to a variety of mutually related notions (crossed module, 2-groupoid, double groupoid,. . . .)

I would like to acknowledge an extensive e-mail exchange with Urs Schreiber in the winter 2005–2006. He called my attention to some of the problems in higher gauge theory, and in particular, to the problem of relating differential and integral formulations. See also [1], and [4] for various formulations of the problems (mainly in principal-bundle terms).

I assume that the reader is familiar with the basic technique in SDG, namely that one talks about the geometric objects ("spaces") (say, smooth manifolds, as in [6], or schemes, as in [2]) *as if* they were sets. Hence also "map" means "smooth map" or "scheme morphism". Some of the notions considered are purely algebraic, and make sense also for discrete sets. Some others, notably the crucial "first neighbourhood of the diagonal" make sense for (finite dimensional) manifolds, respectively schemes, only. – Some particular, less well known, aspects of SDG, are commented on in the Appendix.

Convention. Generally, concrete maps compose from right to left, and this composition is denoted by $\circ$: $(f \circ g)(x) = f(g(x))$, whereas composition in abstract groupoids is denoted by a lower dot, and composed from left to right.

1 Groupoids

We consider groupoids $\mathbf{\Phi} = (\Phi \rightrightarrows M)$, the two displayed maps being domain-formation d_0 and codomain-formation d_1. So M is the set ("space") of *objects* of the groupoid $\mathbf{\Phi}$, and Φ the space of *arrows* of the $\mathbf{\Phi}$. Often we identify $\mathbf{\Phi}$ and Φ notationally.

When two groupoids Φ and Ψ have the same set of objects M, we often tacitly assume that the functors $F : \Phi \to \Psi$ that we consider preserve the objects, i.e. $F(x) = x$ for all $x \in M$ (unless the contrary is explicitly stated).

Example 1. To any M, we have the *codiscrete* groupoid $M \times M \rightrightarrows M$ (also called *chaotic* or *banal* groupoid on M). More generally, to any group G, we have the "constant" groupoid $M \times G \times M \rightrightarrows M$ (with $(x, g, y).(y, g', z) := (x, g \cdot g', z)$.

Example 2. Let $E \to M$ be any map. We get a groupoid $\Phi \rightrightarrows M$, where the arrows $m_1 \to m_2$ are the bijections $E_{m_1} \to E_{m_2}$ (E_m denoting the fibre of $E \to M$ over $m \in M$). In the context of SDG, where "everything is smooth", the word "diffeomorphism" is more adequate than "bijection", although they are synonymous in the SDG context.

2 Graphs

We shall here consider graphs, in particular reflexive symmetric graphs. Every manifold gives, in the context of SDG, rise to such, and so does every groupoid, see below. Let us be explicit about the category of reflexive symmetric graphs. Its objects are pairs of sets (X_1, X_0), together with four maps: $d_0 : X_1 \to X_0$, $d_1 : X_1 \to X_0$, $i : X_0 \to X_1$ and $t : X_1 \to X_1$. The elements of X_0 are the *vertices* of the graph, the elements of X_1 the *edges*; for $u \in X_1$, $d_0(u)$ (resp. $d_1(u)$) is the *source* (resp. the *target*) vertex of u. The symmetry is a *structure*, namely a map $t : X_1 \to X_1$; it is assumed to be an involution ($t \circ t = \text{id}$), and to interchange source and target, i.e. $d_0 \circ t = d_1$ and $d_1 \circ t = d_0$. Also, reflexivity of the graph is a structure, given by the map $i : X_0 \to X_1$. We assume $d_0 \circ i = d_1 \circ i = \text{id}$ and $t \circ i = i$. The morphisms $\xi : X \to X'$ in the category of reflexive symmetric graphs are pairs of maps $\xi_0 : X_0 \to X_0'$, $\xi_1 : X_1 \to X_1'$ which commute with the four structural maps in an evident sense.

When two graphs have the same set X_0 of vertices, we often tacitly assume that the graph morphisms $\xi = (\xi_1, \xi_0)$ that we consider preserve the vertices, i.e. $\xi_0(x) = x$ for all $x \in X_0$.

We will often denote a graph X, as above, with the hieroglyph $X_1 \rightrightarrows X_0$.

The following example works in the context of SDG:

Example 1. If M is a manifold, we get a reflexive symmetric graph $M_{(1)} \rightrightarrows M$, whose vertices are the elements of M and whose edges are (ordered) pairs (x, y) of neighbour points $x \sim y$, $d_0((x, y)) = x$, $d_1((x, y)) = y$. Since the 1-neighbour relation is symmetric, we have an involution t given by $t(x, y) = (y, x)$, and i is given by $i(x) = (x, x)$. – This graph is often called *the first neighbourhood of the diagonal* of M.

For a graph arising as the first neighbourhood of the diagonal of a manifold M, the map $(d_0, d_1) : M_1 \to M_0 \times M_0$ is jointly mono; and the existence of the involution t is therefore a *property* rather than an added structure. For the graph arising from a groupoid to be described now, neither of these simplifications obtain.

Example 2. The reflexive symmetric graph arising from a groupoid has the objects of the groupoid as its vertices, the arrows for its edges, t is inversion in the groupoid $t(u) = u^{-1}$, and $i(x)$ is the identity arrow id_x at the object x. We also call the graph thus obtained *the underlying graph* of the groupoid.

The functor from manifolds to graphs thus described is full and faithful; the functor from groupoids to graphs is faithful, but not full. In particular, a map of reflexive symmetric graphs between the underlying graphs of groupoids preserves identities and inversion, but *does not* necessarily preserve composition. This is related to the notion of *curvature* or *flatness*.

Example 3. For any manifold M, we have the set $P(M)$ of (smooth) Moore paths $\gamma : [a, b] \to M$ where $[a, b]$ is a closed interval on the number line R ($a \leq b$). Then $P(M)$ carries a structure of graph with M as set of vertices, namely with $\gamma(a)$ and $\gamma(b)$ declared to be the domain and codomain of γ. This graph carries also a reflexive symmetric structure: for $x \in M$, i_x is taken to be the constant path $[0, 0] \to M$ with value x; and with γ as above, $t(\gamma)$ is taken to be $\gamma \circ s$ where $s : [a, b] \to [a, b]$ is reflection in the midpoint $(a + b)/2$.

We refer the reader to the Appendix for some subtleties of the notion of "closed interval" in the synthetic context.

Remark. Note that we do not attempt to define a *composition* of paths; this would lead to paths which are only *piecewise* smooth. (In the context of SDG, a piecewise smooth path is not really a path, but a suitable finite sequence of paths.) One could define composition of paths which are stationary in neigbourhoods of the endpoints; but that would exclude from consideration paths whose interval $[a, b]$ of definition are infinitesimal (say, with $a \sim b$), and such infinitesimal paths are of fundamental importance in the present study.

Example 4. For any manifold M, we have a canonical morphism of reflexive symmetric graphs,

$$M_{(1)} \to P(M);$$

it takes the pair $(x, y) \in M_{(1)}$ into the path $[[x, y]] : [0, 1] \to M$ given by $t \mapsto (1-t)x + ty$. Such affine combination of mutual neighbour points make sense in any manifold, cf. [10], or the Appendix below; we return to aspects of this construction in §10.

3 Base change (Full Image)

For groupoids as well as for graphs, one has the notion of *full image*: if $\mathbf{\Phi} = (\Phi \rightrightarrows M)$ is a groupoid, and $f : N \to M$ is a map, there is a groupoid $f^*(\mathbf{\Phi}) = (f^*(\Phi) \rightrightarrows N)$, where an arrow in $f^*(\Phi)$ from u to v ($u, v \in N$) is by definition an arrow in Φ from $f(u)$ to $f(v)$; and similarly for graphs (using the terms "vertex" and "edge", rather than "object" and "arrow"). For the groupoid case, there is an evident composition of arrows in $f^*(\mathbf{\Phi})$, making it into a groupoid; this is the *full image of* $\mathbf{\Phi}$ *under* f. (Thus, groupoids in fact form a fibered category over the category of manifolds $M, N, \ldots$)

A functor $\xi : (\Psi \rightrightarrows N) \to (\Phi \rightrightarrows M)$ given by $\xi_0 : N \to M$ and $\xi_1 : \Psi \to \Phi$ may be identified with a functor $(\Psi \rightrightarrows N) \to \xi_0^*(\Phi \rightrightarrows M)$.

Example. Let G be a group and M a set or manifold. Let 1 denote the one-element set. G may be viewed as a groupoid $G \rightrightarrows 1$. The full image of this groupoid along the unique map $M \to 1$ equals the constant groupoid $M \times G \times M \rightrightarrows M$ (cf. §1 Example 1).

4 Trivializations

Recall from §1 that for any set M, we have the "codiscrete" groupoid $M \times M \rightrightarrows M$.

Let $\mathbf{\Phi} = (\Phi \rightrightarrows M)$ be a groupoid. A functor $\overline{\nabla}$ from $M \times M \rightrightarrows M$ to $\mathbf{\Phi}$ is called a *(total) trivialization* of $\mathbf{\Phi}$.

So for x and y in M, $\overline{\nabla}(x, y)$ is an arrow $x \to y$ in Φ; $\overline{\nabla}(x, x)$ is id_x, and $\overline{\nabla}(x, y)$ is inverse of $\overline{\nabla}(y, x)$, (we shall meet equations similar to these two again when defining the notion of connection); but furthermore, the fact that $\overline{\nabla}$ is assumed to commute with composition (being a functor) reads

$$\overline{\nabla}(x, y).\overline{\nabla}(y, z) = \overline{\nabla}(x, z) \tag{1}$$

for all x, y, z in M.

Sometimes, we call such trivialization a *total* trivialization, because there is a more general notion, *partial trivialization* of a groupoid $\mathbf{\Phi} = (\Phi \rightrightarrows M)$ *along* a map $f : N \to M$; this is by definition a total trivialization $\overline{\nabla}$ of the groupoid $f^*(\mathbf{\Phi})$, thus for n_1 and n_2 in N, $\overline{\nabla}(n_1, n_2)$ is an arrow $f(n_1) \to f(n_2)$ in Φ, and $\overline{\nabla}(n, n) = \mathrm{id}_{f(n)}$, for all $n \in N$, and similarly for the other equations: $\overline{\nabla}(n_2, n_1) = \overline{\nabla}(n_1, n_2)^{-1}$ and $\overline{\nabla}(n_1, n_2).\overline{\nabla}(n_2, n_3) = \overline{\nabla}(n_1, n_3)$.

Trivializations pull back in an evident sense. If $\Phi \rightrightarrows M$ is a groupoid, and if $f : N \to M$ is an arbitrary map, a total trivialization $\overline{\nabla}$ of Φ gives rise to a total trivialization $f^*(\overline{\nabla})$ of $f^*(\Phi)$, i.e. a to a partial trivialization of Φ along f,

$f^*(\overline{\nabla})(n_1, n_2) = \overline{\nabla}(f(n_1), f(n_2))$ for $n_1, n_2 \in N$. More generally, if $\overline{\nabla}$ is a partial trivialization of $\Phi \rightrightarrows M$ along $f : N \to M$, and $g : P \to N$ is any map, we get an induced partial trivialization of Φ along $f \circ g : P \to M$, in an evident way.

Example. Recall from §1 Example 1 the groupoid $M \times G \times M \rightrightarrows M$ given by a set M and a group G. This groupoid carries a total trivialization $\overline{\nabla}$ given by $\overline{\nabla(x, y)} := (x, e, y)$ where $e \in G$ is the neutral element. But conversely:

Proposition 1 *Given a groupoid $\Phi \rightrightarrows M$ and a total trivialization $\overline{\nabla}$ of it. Then for each $z \in M$, there is a canonical isomorphism between the groupoids Φ and $M \times G \times M \rightrightarrows M$, where G is the group $\Phi(z, z)$.*

Proof/Construction. Let $\phi : m_1 \to m_2$ be an arrow in $\Phi \rightrightarrows M$. Then $(m_1, \overline{\nabla}(z, m_1).\phi.\overline{\nabla}(m_2, z), m_2)$ is an arrow $m_1 \to m_2$ in $M \times G \times M \rightrightarrows M$. Conversely, to an arrow (m_1, g, m_2) in $M \times G \times M \rightrightarrows M$, we associate the arrow $\overline{\nabla}(m_1, z).g.\overline{\nabla}(z, m_2)$ in $\Phi \rightrightarrows M$.

(Let us remark that under the correspondence between groupoids and principal bundles, constant groupoids correspond to trivial fibre bundles; (total) trivializations of groupoid correspond to (global) sections of principal bundles.)

5 Connections in groupoids

We can now describe the crucial notion of *connection in a groupoid*, as it may be rendered in the language of SDG, cf. [5] (Remark 6.4), and [8]. (There are several other synthetic renderings of connections in groupoids, see e.g. the references in Part I, Note 12 in the 2006 edition of [6].) More precisely, we are discussing the notion of *"infinitesimal connection"* or even more pedantically, *"first-order infinitesimal connection"*.

We consider a groupoid $\Phi \rightrightarrows M$ whose space of objects is a manifold. Recall from Section 2 the graph $M_{(1)} \rightrightarrows M$ of a manifold M ("first neighbourhood of the diagonal").

Definition 1 *A* connection ∇ *in a groupoid $\Phi \rightrightarrows M$ is a morphism of reflexive symmetric graphs from $M_{(1)}$ to (the underlying graph of) Φ.*

In other words, if $(x, y) \in M_{(1)}$ (i.e. if $x \sim y$), $\nabla(x, y)$ is an arrow $x \to y$ in Φ; and the following laws hold:

$$\nabla(x, x) = \mathrm{id}_x \tag{2}$$

for all $x \in M$, and, for all $x \sim y$,

$$\nabla(y, x) = (\nabla(x, y))^{-1}. \tag{3}$$

(In the context of SDG, it often happens that (3) is a consequence of (2).)

An important property which a connection may or may not have, is the following: A connection ∇ in $\Phi \rightrightarrows M$ is called *flat* or *curvature free* if

$$\nabla(x, y).\nabla(y, z) = \nabla(x, z) \tag{4}$$

whenever $x \sim y$, $y \sim z$, and $x \sim z$.

(We have elsewhere [11] used the terminology "infinitesimal 2-simplex" for such a triple x, y, z of points in a manifold. Note that the neighbour relation $\sim$ is not transitive, so that $x \sim z$ does not follow from $x \sim y$, $y \sim z$.)

If ∇ is a connection in $\Phi \rightrightarrows M$, and $\gamma : N \to M$ is a map between manifolds, we get an induced connection $\gamma^*(\nabla)$ in $\gamma^*(\Phi)$: if $(u, v) \in N_{(1)}$, $(\gamma(u), \gamma(v)) \in M_{(1)}$, and $\nabla(\gamma(u), \gamma(v)) : \gamma(u) \to \gamma(v)$ represents a arrow $u \to v$ in $\gamma^*(\Phi)$.

If $\gamma : N \to M$ is a map between manifolds, and $\Phi \rightrightarrows M$ a groupoid, a connection in the groupoid $\gamma^*(\Phi)$ is called a *connection in* Φ *along* γ, or a *partial connection in* Φ, if γ is understood from the context. This situation occurs often enough to be made explicit in more elementary terms: let $(\Phi \rightrightarrows M)$ be a groupoid. Then a connection in Φ along $\gamma : N \to M$ consists in giving, for each $x \sim y$ in N an arrow in Φ, $\nabla(x, y) : \gamma(x) \to \gamma(y)$; and this data is required to satisfy the laws

$$\nabla(x, x) = \mathrm{id}_{\gamma(x)} \tag{5}$$

for all $x \in N$, and

$$\nabla(y, x) = (\nabla(x, y))^{-1} \tag{6}$$

for all $x \sim y \in N$.

Given $\gamma : N \to M$ and a connection ∇ in $\Phi \rightrightarrows M$, the induced connection on $\gamma^*(\Phi) \rightrightarrows N$ may happen to be flat, even though ∇ itself may not be so. In this case, we say that ∇ is flat *along* γ. Under assumptions to be made later, any connection will be flat along any *path* $\gamma : [a, b] \to M$.

5.1 Integrals

A trivialization $\overline{\nabla}$ of a groupoid $\mathbf{\Phi} = (\Phi \rightrightarrows M)$ where M is a manifold gives rise to a connection ∇ in $\mathbf{\Phi}$ by restricting $\overline{\nabla} : M \times M \to \Phi$ to the subset $M_{(1)} \subseteq M \times M$, i.e. $\nabla(x, y) = \overline{\nabla}(x, y)$ for $x \sim y$ in M. Then (1) for $\overline{\nabla}$ implies that the connection ∇, obtained by restriction in this way, is flat in the sense of (4). An *complete integral* for a connection ∇ on a groupoid $\Phi \rightrightarrows M$ is a trivialization $\overline{\nabla}$ of Φ which extends the given ∇, $\overline{\nabla}(x, y) = \nabla(x, y)$ whenever $(x, y) \in M_{(1)}$. Clearly a necessary condition for ∇ to admit a complete integral is that it is flat.[1]

Complete integrals in this sense are rare. More common are "*partial integrals along maps*": a *partial integral* of the connection ∇ in Φ *along* $f : N \to M$ is a complete integral of $f^*(\nabla)$. (This is analogous to *partial trivializations* of a groupoid $\Phi \rightrightarrows M$ *along* $f : N \to M$, considered above in §4.)

If $g : N' \to N$ is a map, and $\Psi \rightrightarrows N$ a groupoid, then if $H : N \times N \to \Psi$ is a trivialization of Ψ, then $H \circ (g \times g)$ is a trivialization of $g^*(\Psi)$. And if H is a complete integral of a connection ∇ on Ψ, then $H \circ (g \times g)$ is a complete integral of $g^*(\nabla)$. We express these properties by saying that complete integrals, and trivializations, *pull back.*

[1] In fact, even the reflexivity law and symmetry law assumed for connections in groupoids, $\nabla(x, x) = \mathrm{id}_x$ and $\nabla(y, x) = (\nabla(x, y))^{-1}$ follow from existence of complete integrals.

6 Path connections

Recall the reflexive symmetric graph $P(M) \rightrightarrows M$ of paths in M of §2. – Let there be given a groupoid $\mathbf{\Phi} = (\Phi \rightrightarrows M)$.

Definition 2 *A* path connection *on a groupoid* $\Phi \rightrightarrows M$ *is a morphism* σ *of reflexive symmetric graphs* $P(M) \to \Phi$*, which satisfies a reparametrization law and a subdivision law (as spelled out below).*

Reparametrization: Let $f : [a, b] \to M$ be a path, and let $g : [a', b'] \to [a, b]$ be a map with $g(a') = a, g(b') = b$ ($a \leq b$ and $a' \leq b'$). Then the reparametrization law says

$$\sigma(f \circ g) = \sigma(f) \tag{7}$$

(note that both sides of this equation denote arrows $f(a) \to f(b)$ in Φ).

Subdivision: Let $\gamma : [a, b] \to M$ be a path, and let c be given with $a \leq c \leq b$. Let γ_1 and γ_2 denote the restriction of γ to the subintervals $[a, c]$ and $[c, b]$, respectively. Then the subdivision law says

$$\sigma(\gamma) = \sigma(\gamma_1).\sigma(\gamma_2), \tag{8}$$

(composing from left to right in Φ, compostion denoted by a lower dot).

Note that the assumption that σ preserves symmetry amounts to saying that for any path $\gamma : [a, b] \to M$

$$\sigma(\gamma \circ \tau) = (\sigma(\gamma))^{-1} \tag{9}$$

where $\tau : [a, b] \to [a, b]$ is "reflection in the midpoint", cf. §2.

This notion of path connection is essentially the one of Virsik ([15], Appendix); more precisely, Virsik axiomatizes this notion along the lines of the properties of Hol of the Remark 2 in §7 below.

We note that if $g : N \to M$ is a map, and σ is a path connection on a groupoid $\Phi \rightrightarrows M$, then we get a path connection $g^*(\sigma)$ on the groupoid $g^*(\Phi) \rightrightarrows N$ by putting

$$g^*(\sigma)(\gamma) := \sigma(g \circ \gamma)$$

for any path $\gamma : [a, b] \to N$.

7 Holonomy (Integrating connections along paths)

Let M be a manifold. Many groupoids with connection $(\Phi \rightrightarrows M, \nabla)$ have the property that unique partial integrals exist along any map (path) $[a, b] \to M$, where $[a, b]$ is a closed interval on the line; in this case, we say that the pair $(\Phi \rightrightarrows M, \nabla)$ *admits path integration.* If a groupoid $\Phi \rightrightarrows M$ has the property that $(\Phi \rightrightarrows M, \nabla)$ admits path integration for any connection ∇ on it, we say that the groupoid admits path integration.

If $(\Phi \rightrightarrows M, \nabla)$ admits path integration, we have thus for each path $\gamma : [a, b] \to M$ a unique complete integral for the induced connection $\gamma^*(\nabla)$ on $[a, b]$; we denote it $\int_\gamma \nabla$. So its value on the $(s, t) \in [a, b] \times [a, b]$ is an arrow in Φ

$$\gamma(s) \xrightarrow{(\int_\gamma \nabla)(s,t)} \gamma(t)$$

In particular, if $s = a$ and $t = b$ (the endpoints), $(\int_\gamma \nabla)(a, b)$ is termed the *holonomy of* ∇ *along* γ and is denoted $\mathrm{hol}_\nabla(\gamma)$ or $\int_\gamma \nabla$ (the information of a and b being part of γ). Note that $\mathrm{hol}_\nabla(\gamma)$ is an arrow $\gamma(a) \to \gamma(b)$.

For $a \leq s \leq t \leq b$, $(\int_\gamma \nabla)(s, t)$ equals $\int_{\gamma(s,t)} \nabla$, where $\gamma(s, t)$ denotes the retsriction of γ to the subinterval $[s, t] \subseteq [a, b]$. (But note that $(\int_\gamma \nabla)(s, t)$, unlike $\gamma(s, t)$, is defined whether or not $s \leq t$.) This follows because the restriction of the complete integral for a connection on a manifold, say $[a, b]$, restricts to a complete integral on any submanifold, say $[s, t]$.

Note that if $(\Phi \rightrightarrows M, \nabla)$ admits path integration, ∇ is "flat along any path", i.e. for any path γ in M, $\gamma^*(\nabla)$ is flat (= curvature free), i.e. satisfies (4).

We have for $s \sim t$ in $[a, b]$ that

$$(\int_\gamma \nabla)(s, t) = \nabla(\gamma(s), \gamma(t)); \tag{10}$$

this equation expresses that the complete integral of $\gamma^*(\nabla)$ agrees with $\gamma^*(\nabla)$ on pairs og neighbour points.

Remark 1. There is a condition related to (10), but stronger, namely that this equation holds not just under the assumption that $s \sim t$, but under the weaker assumption that $\gamma(s) \sim \gamma(t)$. However, this would not be realistic for paths with self-intersection, say.

If $(\Phi \rightrightarrows M, \nabla)$ admits path integration, we have defined the *holonomy* $\mathrm{hol}_\nabla(\gamma)$ of ∇ along a path $\gamma : [a, b] \to M$, namely it is the arrow $(\int_{\gamma\nabla})(a, b)$ in Φ, with domain $\gamma(a)$ and codomain $\gamma(b)$. From the assumed uniqueness of complete integrals, it follows that holonomy is invariant under reparametrization (we shall be more explicit below). We thus have a map a map

$$\mathrm{hol}_\nabla : P(M) \to \Phi$$

where $P(M)$ is the space of (smooth Moore-) paths in M, ($P(M)$ as in §2, Example 3.)

Remark 2. More generally, given a path $\gamma : [a, b] \to M$, we have a path $\mathrm{Hol}_\nabla(\gamma) : [a, b] \to \Phi$ in Φ, whose endpoint (value at b) is $\mathrm{hol}_\nabla(\gamma)$. It is defined by $\mathrm{Hol}_\nabla(\gamma)(t) := (\int_\gamma \nabla)(a, t)$. We note that the the domain of the arrow $\mathrm{Hol}_\nabla(\gamma)(t)$ (i.e. the arrow $\int_a^t \gamma^*(\nabla)$) is $\gamma(a)$, the codomain is $\gamma(t)$. – Note that $\mathrm{Hol}_\nabla(\gamma)$ is a path in Φ, whereas $\mathrm{hol}_\nabla(\gamma)$ is just an element (arrow) in Φ.

Let us record the book-keeping conditions which $\mathrm{Hol}_\nabla : P(M) \to P(\Phi)$ satisfies. Recall that P is a functor: if $d : N \to M$ is a map, we get a map $P(d) : P(N) \to$

$P(M)$, by putting $P(d)(\gamma) = d \circ \gamma$ for any path γ in N. In particular, the maps $d_i : \Phi \to M$ $(i = 0, 1)$ give rise to maps $P(d_0)$ and $P(d_1)$ from $P(\Phi)$ to $P(M)$. Let us for any N use α_N and β_N to denote the end points of a path in N, thus e.g. $\alpha(\gamma) = \gamma(a) \in N$ if $\gamma : [a, b] \to N$ is a path; also, for $\gamma : [a, b] \to M$, we let $\kappa(\gamma)$ denote the constant path $[a, b] \to M$ given by $t \mapsto \gamma(a)$ for all $t \in [a, b]$. Then as maps $P(M) \to P(M)$,

$$P(d_0) \circ \mathrm{Hol}_\nabla = \kappa; \tag{11}$$

$$P(d_1) \circ \mathrm{Hol}_\nabla = \text{ identity map of } P(M); \tag{12}$$

and, as maps $P(M) \to \Phi$,

$$\alpha_\Phi \circ \mathrm{Hol}_\nabla = i \circ \alpha_M \tag{13}$$

(where $i : M \to \Phi$ is the map picking out identity arrows), – and

$$\beta_\Phi \circ \mathrm{Hol}_\nabla = \mathrm{hol}_\nabla \,. \tag{14}$$

(These four book-keeping properties, together with reparametriation and subdivision, is a more precise rendering of Virsik's original path-connection notion.)

8 Connections in constant groupoids

We recall some standard notions of SDG, cf. [7], or [6] §I.18. Let M be a manifold and G a group (multiplicatively written, with neutral element e). A *(differential) 1-form with values in* G is a map $\omega : M_{(1)} \to G$ with $\omega(x, x) = e$ for all x, and with $\omega(y, x) = \omega(x, y)^{-1}$ for all $x \sim y$; in the context of SDG, the second condition is a consequence of the first, for many groups G. (For Lie groups, the data of such ω is by [7] equivalent to the classical notion of differential form with values in the Lie algebra $L(G)$ of G, see [7].)

Alternatively, a G-valued 1-form on M may be seen as a connection in $G \rightrightarrows 1$ along the unique map $M \to 1$.

Consider again a connection ∇ in a constant groupoid $M \times G \times M \rightrightarrows M$, and the corresponding G-valued 1-form ω on M. To say that ∇ is flat is equivalent to saying that ω is closed in the sense $\omega(x, y).\omega(y, z) = \omega(x, z)$ for any infinitesimal 2-simplex x, y, z.(Such closed G-valued 1-forms correspond classically to 1-forms with values in $L(G)$ which satisfy the Maurer-Cartan equation, see [7].)

Among the closed G-valued 1-forms are the *exact* ones: a G-valued 1-form ω is called *exact* if it admits a (left) *primitive*; this means a function $f : M \to G$ such that $\omega(x, y) = f(x)^{-1}.f(y)$ for all $x \sim y$ in M.

It is clear that there is a bijective correspondence between G-valued 1-forms on M, on the one side, and connections in the constant groupoid $M \times G \times M \rightrightarrows M$ on the other: to ω corresponds the connection ∇ given by $\nabla(x, y) := (x, \omega(x, y), y)$.

The theory of connections in constant groupoids is therefore identical to the theory of group-valued 1-forms. In this sense, the theory of connections in non-constant groupoids is a generalization of the theory of group-valued 1-forms.

(There is also a bijective correspondence between path connections $\sigma : P(M) \to (M \times G \times M)$ in a constant groupoid, and maps of symmetric reflexive graphs

$s : P(M) \to G$, satisfying a reparametrization and a subdivision law. (Note that a group G may be seen as a groupoid $G \rightrightarrows 1$ with only one object, and that it therefore has an underlying reflexive symmetric graph.) We leave the details to the reader.)

Proposition 2 *Given a constant groupoid $M \times G \times M \rightrightarrows M$. Let ∇ be a connection in it, and ω the corresponding G-valued 1-form. Then ∇ is flat if and only if ω is closed. And ∇ admits a complete integral if and only if ω is exact, i.e. admits a primitive. In fact, if $\overline{\nabla}$ is a complete integral for ∇, then for any $m_0 \in M$, the function $f : M \to G$ given by letting $f(m)$ be the "middle coordinate" in $\overline{\nabla}(m_0, m)$,*

$$\overline{\nabla}(m_0, m) = (m_0, f(m), m)$$

is a primitive of ω; and conversely, given a primitive $f : M \to G$ of ω, a complete integral $\overline{\nabla}$ for ∇ is given by

$$\overline{\nabla}(x, y) = (x, f(x)^{-1}.f(y), y).$$

Complete integrals for connections are unique if and only if primitives are unique modulo left translation by a constant.

Proof. This is safely left to the reader. Let us as a sample prove that uniqueness of complete integrals for ∇ implies uniqueness modulo left translation of primitives for ω. Given two primitives for ω, f_1 and f_2. Then by the middle part of the Proposition

$$\overline{\nabla}(x, y) := (x, f_1(x)^{-1}.f_1(y), y)$$

is a complete integral of ∇, and similarly with f_2 instead of f_1; by assumption these two complete integrals agree, and so we conclude by comparison that for all $x, y \in M$

$$f_1(x)^{-1}.f_1(y) = f_2(x)^{-1}.f_2(y).$$

Now fix x, and multiply this equation on the left by $f_2(x)$. Then we have for all y that

$$[f_2(x).f_1(x)^{-1}].f_1(y) = f_2(y),$$

proving that f_1 and f_2 differ by left translation by the constant appearing in the square bracket.

Not all groupoids admit integration; groupoids of the form described in Example 2 in §1 usually don't, see the argument in §11, Example 3. However, by an essentially classical result one has what in our formulations amount to

Theorem 1 *Any constant groupoid $M \times G \times M \rightrightarrows M$, where G is a Lie group, admits path integration.*

Also any groupoid $\Phi \rightrightarrows M$ which in a suitable sense is *locally* constant, and whose vertex groups are Lie groups, admits path integration; this fact can be reduced to Theorem 1. See [8].

The theory of such groupoids is closely related to (locally constant) principal fibre bundles over Lie groups, as pointed out by Ehresmann in the 1950s; see [13] for a description of this relationship.

9 Holonomy is a path connection

We consider a groupoid $\Phi \rightrightarrows M$, where M is a manifold; let ∇ be a connection in Φ, $\nabla : M_{(1)} \to \Phi$.

Proposition 3 *If* $(\Phi \rightrightarrows M, \nabla)$ *admits path integration, then*

$$\mathrm{hol}_\nabla : P(M) \to \Phi$$

is a path connection.

Proof. We first have to argue that hol_∇ is in fact a morphism of (reflexive symmetric) graphs. First, it is a map of graphs, since the domain (resp. codomain) of $\mathrm{hol}_\nabla(\gamma)$ is $\gamma(a)$, resp. $\gamma(b)$ (where $\gamma : [a, b] \to M$). The "reflexivity structure" is preserved, since for a *constant* path $\gamma : [0, 0] \to M$ (with value $x \in M$, say), the function $(s, t) \mapsto \mathrm{id}_x$ is a complete integral of $\gamma^*(\nabla)$. (Note that in the context of SDG, paths $[0, 0] \to M$ need not be constant.) Finally, for preservation of symmetry: this amounts to proving that the arrows $\mathrm{hol}_\nabla(\gamma)$ and $\mathrm{hol}_\nabla(\tilde{\gamma})$ are inverse of each other in the groupoid $\Phi \rightrightarrows M$, where $\tilde{\gamma} : [a, b] \to M$ is $\gamma \circ \tau$, τ being reflection in the midpoint of $[a, b]$. For simplicity of notation, we shall assume that $[a, b] = [-1, 1]$ so that $\tau(t) = -t$. Let $\gamma : [-1, 1] \to M$ be a path, so $\tilde{\gamma}(t) = \gamma(-t)$. Let $\overline{\nabla}$ be the complete integral of $\gamma^*(\nabla)$. We claim that the complete integral of $\tilde{\gamma}^*(\nabla)$ is given by $\tilde{\overline{\nabla}}(s, t) = \overline{\nabla}(-s, -t)$. The book-keeping is respected, since $\tilde{\overline{\nabla}}(s, t) = \overline{\nabla}(-s, -t)$ is an arrow from $\gamma(-s)$ to $\gamma(-t)$, hence from $\tilde{\gamma}(s)$ to $\tilde{\gamma}(t)$. Also, it is clearly a functor, $\tilde{\overline{\nabla}}(s, t).\tilde{\overline{\nabla}}(t, u) = \tilde{\overline{\nabla}}(s, u)$, since $\overline{\nabla}$ is a functor. Finally, $\tilde{\overline{\nabla}}(s, t)$ extends $\tilde{\gamma}^*(\nabla)$; for, let $s \sim t$ in $[0, 1]$. Then

$$\tilde{\gamma}^*(\nabla)(s, t) = \nabla(\gamma(-s), \gamma(-t)) = \overline{\nabla}(-s, -t) = \tilde{\overline{\nabla}}(s, t).$$

(Note that we did not explicitly use that $\nabla(y, x) = \nabla(x, y)^{-1}$; this property is built into the assumption of existence of a complete integral.)

Both the fact that (7) holds, and the invariance under reparametrizations is an immediate consequence of the fact that connections, as well as their possible complete integrals, pull back (together with uniqueness of complete integrals for connections on intervals). Thus we have a well defined graph map $P(M) \to \Phi$. It remains to prove the subdivision law. So we should prove that if $f : [a, b] \to M$ is a path, and $a \leq c \leq b$

$$h(f) = h(f_1).h(f_2), \tag{15}$$

where, as above, f_1 and f_2 are the restrictions of f to $[a, c]$ and to $[c, b]$, respectively.

Let $H : [a, b] \times [a, b] \to f^*(\Phi)$ be the complete integral of $f^*(\nabla)$. Then the restriction H_1 of H to $[a, c] \times [a, c]$ is the complete integral of $f_1^*(\nabla)$ (identifying $f_1^*(\Phi)$ with a full subgroupoid of $f^*(\Phi)$); similarly for H_2, the restriction of H to $[c, b] \times [c, b]$. The left hand side of (15) is by definition $H(a, b)$, the right hand side is similarly $H_1(a, c).H_2(c, b) = H(a, c).H(c, b)$. But since H is a functor, $H(a, b) = H(a, c).H(c, b)$. This proves (15) and thus the Proposition.

10 Differentiating a Path Connection

Recall from the Appendix that one may form arbitrary affine combinations of mutual neighbour points in a manifold M.

We consider a groupoid $\Phi \rightrightarrows M$, and a path connection $\sigma : P(M) \to \Phi$. We want to differentiate σ into a (first order, infinitesimal) connection σ'. Let $(x, y) \in M_{(1)}$. Since M is a manifold and x and y are neighbours, there is therefore a well defined smooth path $[[x, y]] : [0, 1] \to M$ given by

$$[[x, y]](t) := (1 - t) \cdot x + t \cdot y; \tag{16}$$

we put

$$\sigma'(x, y) := \sigma([[x, y]]). \tag{17}$$

If $x = y$, the path $[[x, y]]$ is constant, and therefore σ returns an identity arrow. This proves that $\sigma'(x, x) = \mathrm{id}_x$. The fact that $\sigma'(x, y)$ and $\sigma'(y, x)$ are inverses follows easily from the "symmetry" assumption (9) on σ.

We now have the two processes: "integration" of a connection ∇ into a path connection hol_∇ (provided the groupoid in question admits path integration), and the process of differentiating a path connection into a connection. We shall prove that these processes are mutually inverse, see Theorems 2 and 3 below. We first note that both these processes are "stable under base change". Consider a groupoid $\Phi \rightrightarrows M$ and a map $g : N \to M$ (where N and M are manifolds). Then we also have a groupoid $g^*(\Phi) \rightrightarrows N$, and to a connection ∇ on $\Phi \rightrightarrows M$, we get a connection $g^*(\nabla)$ on $g^*(\Phi) \rightrightarrows N$; and to a path connection σ on $\Phi \rightrightarrows M$, we get a path connection $g^*(\sigma)$ on $g^*(\Phi) \rightrightarrows N$. Observe that if $\Phi \rightrightarrows M$ admits path integration, then so does $g^*(\Phi) \rightrightarrows N$. In this case, we have for any connection ∇ on $\Phi \rightrightarrows M$ that

$$\mathrm{hol}_{g^*(\nabla)} = g^*(\mathrm{hol}_\nabla). \tag{18}$$

Also, for any path connection σ on $\Phi \rightrightarrows M$, we have

$$(g^*(\sigma))' = g^*(\sigma'). \tag{19}$$

This last fact depends on $g \circ [[x, y]] = [[g(x), g(y)]]$ for $x \sim y \in N$. This follows from the fact that any map between manifolds not only preserves the neighbour relation but also preserves affine combinations of mutual neighbours (this fact is commented on in the Appendix).

We consider a connection ∇ in a groupoid $\Phi \rightrightarrows M$ which admits path integration. So hol_∇ is defined, and is a path connection on $\Phi \rightrightarrows M$. To prove that $(\mathrm{hol}_\nabla)' = \nabla$, we need a further natural property of connections ∇ on $\Phi \rightrightarrows M$, namely that the flatness equation $\nabla(x, y).\nabla(y, z) = \nabla(x, z)$ holds for any "1-dimensional" infinitesimal 2-simplex x, y, z. Recall that to say that $x, y, z \in M$ form an infinitesimal 2-simplex in M is to say that $x \sim y$, $y \sim z$, and $x \sim z$. The notion of 1-dimensionality for such simplices that we shall consider is the following. For any $a \sim b$ in M, we have the set of affine combinations $(1 - t)a + tb$; the set of points in M arising in this way for some $t \in R$, we denote $\mathrm{span}(a, b)$ ("affine span"). It is a property of affine combinations that all points in $\mathrm{span}(a, b)$ are mutual neighbours. An infinitesimal

2-simplex x, y, z is called *1-dimensional* if there exists $a \sim b$ such that x, y, and z all belong to $\mathrm{span}(a, b)$.

Remark. If $a \sim b$, any connection ∇ is flat along the path $[[a, b]] : [0, 1] \to M$, provided the groupoid in question admits path integration. But from this does not follow that ∇ is flat on the *image* of this map (cf. §7 Remark 1). This is why we have to introduce the assumption concerning the "1-dimensionality" property.

So we assume that the flatness equation $\nabla(x, y).\nabla(y, z) = \nabla(x, z)$ holds for any *1-dimensional* infinitesimal 2-simplex x, y, z in M. Under these assumptions, we have that ∇ can be reconstructed from its holonomy:

Theorem 2 *Under these assumptions, we have*

$$(\mathrm{hol}_\nabla)' = \nabla.$$

Proof. Let $a \sim b$ in M. We have the path $[[a, b]] : [0, 1] \to M$, and by definition we have

$$(\mathrm{hol}_\nabla)'(a, b) = \mathrm{hol}_\nabla([[a, b]]) = \Gamma(0, 1), \tag{20}$$

where $\Gamma : [0, 1] \times [0, 1] \to \Phi$ is the trivialization of $[[a, b]]^*(\nabla)$. We are going to describe Γ explicitly. We claim that

$$\Gamma(s, t) = \nabla([[a, b]](s), [[a, b]](t))$$

will do the job, for $s, t \in [0, 1]$. (Note that even though s and t may not be neighbours in $[0, 1]$, $[[a, b]](s)$ and $[[a, b]](t)$ are neighbours in M, so that it makes sense to apply ∇.) First, the Γ thus described is a functor $([0, 1] \times [0, 1]) \to \Phi$, since ∇ is flat on $\mathrm{span}(a, b)$, by the assumption of flatness on 1-dimensional simplices. And clearly for $s \sim t$, it returns the value of $[[a, b]]^*(\nabla)$ on s, t. These two properties characterize the trivialization of $[[a, b]]^*(\nabla)$, so this proves the claim. Substituting $s = 0$ and $t = 1$, we see

$$\Gamma(0, 1) = \nabla([[a, b]](0), [[a, b](1)) = \nabla(a, b),$$

and combining this with (20) proves the desired equality of $(\mathrm{hol}_\nabla)'(a, b)$ and $\nabla(a, b)$, and thus the Theorem.

We next consider a path connection σ in a groupoid $\Phi \rightrightarrows M$ which admits path integration.

Theorem 3 *We have* $\mathrm{hol}_{\sigma'} = \sigma$.

Proof. In view of naturality of differentiation and holonomy formation, it suffices to prove that for any path connection σ on $\Psi \rightrightarrows [a, b]$ $(a \leq b)$, we have

$$\mathrm{hol}_{\sigma'}(\mathrm{id}) = \sigma(\mathrm{id}),$$

where id denotes the identity map $[a, b] \to [a, b]$. For $x \leq y$ in $[a, b]$, we have a path $[x, y] \to [a, b]$ given by the inclusion map of the subset $|[x, y]|$ into the set

$|[a,b]|$, together with the information of the end points x and y; denote this path $[x,y] \hookrightarrow [a,b]$. We construct a complete trivialization h of $\Psi \rightrightarrows [a,b]$ by applying dichotomy: if $x \leq y$

$$h(x,y) = \sigma([x,y] \hookrightarrow [a,b]), \tag{21}$$

and if $y \leq x$,

$$h(x,y) = (\sigma([y,x] \hookrightarrow [a,b]))^{-1}. \tag{22}$$

Note that if $x \leq y$ and also $y \leq x$, $|[x,y]|$ and $|[y,x]|$ agree as subsets of $|[a,b]|$, so the inclusion maps of $[x,y] = [y,x]$ agree; but as *paths*, the information of the endpoints x and y is retained in $[x,y]$). In the case of $x \leq y$ and $y \leq x$, the value of the two expressions (21) and (22) agree, due to the symmetry law for σ (recall that σ is a morphism of reflexive *symmetric* graphs). To see that this is indeed a functor, $h(x,y).h(y,z) = h(x,z)$, we have to consider six cases, by hexachotomy, one for each of the six possible orderings of x,y,z. For the case of $x \leq y \leq z$, this follows from he subdivision law for σ, applied to the inclusion $[x,z] \hookrightarrow [a,b]$, which is a path whose domain may be subdivided by y into the inclusions $[x,y] \hookrightarrow [a,b]$ and $[y,z] \hookrightarrow [a,b]$. The other five cases are similar, using furthermore the symmetry law for σ.

The trivialization of $\Psi \rightrightarrows [a,b]$ thus constructed extends σ'; for, if $x \sim y$, we have $x \leq y$ (and also, by the way, $y \leq x$),

$$h(x,y) = \sigma([x,y] \hookrightarrow [a,b]) = \sigma([[x,y]]),$$

the last equality by the reparametrization property of σ. But $\sigma([[x,y]]) = \sigma'(x,y)$, by definition of differentiation of path connections.

Thus h furnishes the complete integral for σ', and $\mathrm{hol}_{\sigma'}(\mathrm{id})$ is therefore $h(a,b)$, which is $\sigma(\mathrm{id})$. This proves the Theorem.

11 Interpretations in Calculus

We illustrate the meaning of some of the notions presented, by seeing their meaning in "Calculus", meaning the standard "Calculus and Analytic Geometry"-course widely taught.

Example 1. We first discuss the Fundamental Theorem of Calculus. It has two parts: First Part:

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where F is an antiderivative of f, i.e. $F' = f$; and Second Part:

$$\frac{d}{dt} \int_a^t f(x)\, dx = f(t).$$

In the context of SDG, the first part is a definition (since there is in SDG no "integral defined in terms of Riemann sums", say, available). The proof of the Second Part is a 3-line proof (see [14]).

We want to demonstrate how Theorem 2 specializes to this “Second Part”; and also, to analyze to what Theorem 3 specializes.

We are going to specialize the groupoid concepts considered here by considering the “groupoid of 1-variable calculus”, i.e. the constant groupoid $[a, b] \times R \times [a, b]$, where $[a, b]$ $(a \leq b)$ is an interval and $R = (R, +)$ is the (additive group of) the number line.

By the results in §8, a connection ∇ in this groupoid is equivalent to an R-valued 1-form ω on $[a, b]$. Now such an ω may be given in terms of a function $f : [a, b] \to R$ as $f(x)\ dx$. This $f(x)\ dx$ is by definition the 1-form given by

$$(f(x)\ dx)(s, t) = (t - s)f(s)$$

for $s \sim t \in [a, b]$ Also, it is easy to see that a primitive of $f(x)\ dx$ in the sense of group valued 1-forms is the same as an anti-derivative F of f, in the sense of the Calculus Books, i.e. a function $F : [a, b] \to R$ with $F' = f$. So also the complete integral $\overline{\nabla}$ of ∇ can be expressed in terms of such antiderivative F of f: for any $s, t \in [a, b]$,

$$\overline{\nabla}(s, t) = (s, F(t) - F(s), t),$$

or, omitting the “external” coordinates s and t, as we shall henceforth do, $\overline{\nabla}(s, t) = F(t) - F(s)$. Therefore, for $\gamma : [a_1, b_1] \to [a, b]$ a path,

$$\mathrm{hol}_{\nabla}(\gamma) = F(\gamma(b_1)) - F(\gamma(a_1)).$$

in particular, for $s \sim t \in [a, b]$,

$$\begin{aligned}(\mathrm{hol}_{\nabla})'(s, t) = \mathrm{hol}_{\nabla}([[s, t]]) &= F([[s, t]](1)) - F([[s, t]](0)) \\ &= F(t) - F(s) = \int_s^t f(x)\ dx\end{aligned}$$

(the last equality by the First Part of the Fundamental Theorem of Calculus). On the other hand, for $s \sim t$,

$$\nabla(s, t) = (f(x)\ dx)(s, t) = (t - s)f(s).$$

By Theorem 2, the two left hand sides here are equal, whence for all $t \sim s$,

$$\int_s^t f(x)\ dx = (t - s)f(s).$$

Writing $t = s + d$ with $d^2 = 0$ (which is the algebraic way of describing the neighbour relation on the line R), this may be written

$$\int_s^{s+d} f(x)\ dx = d \cdot f(s),$$

or

$$\int_a^{s+d} f(x)\ dx - \int_a^s f(x)\ dx = d \cdot f(s),$$

which is the assertion of the Second Part of the Fundamental Theorem of Calculus (in view of the way, derivatives are defined in SDG).

Example 2. To analyze what Theorem 3 says in terms of Calculus is not so immediate, since the input of the Theorem is a path connection σ , which is such a non-elementary thing as a functional (a function whose input are *functions*, namely *paths*). However, for the "groupoid of 1-variable calculus", each path in $\gamma : [a_1, b_1] \to [a, b]$ may be reparametrized to an "inclusion" path $[\gamma(a_1), \gamma(b_1)] \hookrightarrow [a, b]$, at least if $\gamma(a_1) \leq \gamma(b_1)$. So because of the reparametrization law for σ, the information contained in σ may be encoded in (or at least gives rise to) an R-valued function Σ defined on the set of pairs $x \leq y$ in $[a, b]$, $\Sigma(x, y) := \sigma([x, y] \hookrightarrow [a, b])$. This Σ satisfies $\Sigma(x, y) + \Sigma(y, z) = \Sigma(x, z)$, for $x \leq y \leq z$, by the subdivision law for σ. The Theorem 3 then asserts that there exists a function $f : [a, b] \to R$ such that

$$\Sigma(x, y) = \int_x^y f(t)\, dt$$

(namely the function f such that the 1-form $f(t)\ dt$ corresponds to the connection σ').

Example 3. We discuss first order ordinary differential equations $y' = F(x, y)$, where F is a function $[a, b] \times R \to R$. Such equation represents a connection in the groupoid of the type considered in §1, Example 2: Consider the projection $[a, b] \times R \to [a, b]$. The groupoid $\Phi \rightrightarrows [a, b]$ arising from it by the process of the quoted example has as arrows from s to t ($s, t \in [a, b]$) bijections from one copy of the y-axis (the fibre over s) to another copy of the y-axis (the fibre over t) (so it is (isomorphic to) the constant groupoid $[a, b] \times \mathrm{Diff}(R) \times [a, b] \rightrightarrows [a, b]$, where $\mathrm{Diff}(R)$ is the group of bijections (=diffeormorphisms) $R \to R$). The equation $y' = F(x, y)$ gives rise to a connection ∇ in this groupoid, namely, for $s \sim t$, $\nabla(s, t) = (s, \phi, t)$ where ϕ is the diffeomorphism

$$y \mapsto y + F(s, y)(t - s).$$

(The fact that $\nabla(s, t)$ and $\nabla(t, s)$ are inverse of each other follows from $F(s, y)(t - s) = F(t, y)(t - s)$, which in turn is a consequence of $(t - s)^2 = 0$, and of Taylor expansion of F in the first variable). (Conversely, any connection in this groupoid represents such differential equation $y' = F(x, y)$.) Finding a complete integral $\overline{\nabla}$ for ∇ is tantamount to finding a complete solution for the differential equation. Namely, $\overline{\nabla}(s, t)$ is the map, which associates to $y_0 \in R$ the number $g(t) \in R$, where g is the unique solution of the differential equation $y' = F(x, y)$ with initial value $g(s) = y_0$. (Such g may not exists, e.g. for $F(x, y) = y^2$).) For $F(x, y)$ an *affine* function of y, the $\nabla(s, t)$ belong to the Lie group $\mathrm{Aff}(R) \subseteq \mathrm{Diff}(R)$ of affine maps $R \to R$, and hence complete integrals do exist for ∇ in this case, by Theorem 1. (This is the standard fact from Calculus: "Linear" (i.e. affine) differential equations admit unique complete solutions.)

12 Appendix

12.1 The neighbour relation

Recall that $D(n) \subseteq R^n$ consists of n-dimensional vectors $(d_1, \ldots, d_n)$ with $d_i \cdot d_j = 0$ for all $i, j = 1, \ldots, n$, in particular $d_i^2 = 0$ for all i. If $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{y} = (y_1, \ldots, y_n)$ are two vectors, we say $\underline{x} \sim \underline{y}$ if $\underline{x} - \underline{y} \in D(n)$. This defines the (first order) neighbour relation $\sim$ on R^n. For a general n-dimensional manifold M, the neighbour relation is defined via coordinate charts from R^n. The subset $M_{(1)} \subseteq M \times M$ consisting of pairs of mutual neighbour points in M is called the *first neighbourhood of the diagonal* of M. See [6] §I.17. Also more general objects than manifolds can sometimes be endowed with a good neighbour relation $\sim$, see [2], or [12].

12.2 Affine combinations in manifolds

In [10] Theorem 2.2, we proved that the idea that a manifold M is locally like an affine space has the algebraic consequence that affine combinations of mutual neighbour points $x_0, \ldots, x_n$ in M, defined via affine combinations in an arbitrary coordinate chart, do not depend on the chart chosen; and also, the set of points in M thus obtained (and which may be denoted $\mathrm{span}(x_0, \ldots, x_n)$) has the property that all its points are mutual neighbours.

The "Theorem 2.2" quoted may be supplemented with a "Theorem 2.2+": Any map $f : M \to N$ *preserves* affine combinations of mutual neighbours.

This is proved along the same lines as Theorem 2.2 itself; the detailed proof will be included in a (hopefully) forthcoming sequel to [6].

In the present note, we consider the case $\mathrm{span}(x_0, x_1)$ (where $x_0 \sim x_1$), which, as an affine span of two points, should be thought of as "(at most) 1-dimensional" (we do not need, or attempt, though, to make the dimension notion more precise).

12.3 The ordering of the number line and the notion of interval

In many models of SDG, the basic object ("the line") R carries a pre-order relation $\leq$, compatible with the ring structure on R in the standard way. It is *not* a partial order; the anti-symmetry law does not hold. So we cannot from $x \leq y$ and $y \leq x$ conclude that $x = y$; on the contrary, for all nilpotent elements d in the ring R, $0 \leq d \leq 0$, but this does not imply $d = 0$ (which would destroy much of the synthetic theory, e.g. the neighbour relation). In particular, if $a \leq b$ in R, we have a set $\{x \in R \mid a \leq x \leq b\} \subseteq R$; but a and b cannot be reconstructed from the set. This set, we denote $|[a, b]|$; the set $|[a, b]|$ *together with* the information of the endpoints a and b, we denote $[a, b]$. A "path" in an object M is a map $[a, b] \to M$, meaning precisely: a map $|[a, b]| \to M$, *together with* the information of the endpoints a and b.

We would like the preorder $\leq$ on R to be in a suitable sense a *total* preorder: for each $x, y \in R$, *at least* one of the two alternatives $x \leq y$ or $y \leq x$ hold. This we might call the *strong* dichotomy law for $R, \leq$.

The strong dichotomy law implies a strong "hexachotomy law": for any $x, y, z \in R$, *at least* one of the following six alternatives holds:

$$x \leq y \leq z, \quad x \leq z \leq y, \quad y \leq x \leq z, \ldots, \quad z \leq y \leq x.$$

The strong dichotomy law is consistent with SDG; for instance, one may on R put the 'chaotic' preorder $\leq$ given by: $x \leq y$ for *all* $x, y \in R$.

I don't know at present whether it is consistent to have less chaotic preorders on R satisfying strong dichotomy. But for the purposes here, it is sufficient that certain objects N *perceive* the dicotomy and hexachotomy laws to hold, or that the *weak* dichotomy law holds for N, in the following sense:

We say that the *weak* dichotomy law holds for N if the following is the case: Let $J = |[a, b]|$ be (the underlying set of) an interval. If two maps $f_i : J \times J \to N$ satisfy $f_1(s, t) = f_2(s, t)$ whenever $s \leq t$ and whenever $t \leq s$, then $f_1 = f_2$ $(s, t \in J)$.

(Typically N is Φ, the set of arrows of a groupoid under consideration).

Similarly, there is a weak hexachotomy law for N. We leave the exact formulation to the reader. (I have not been able to prove the weak hexachotomy law for N from the weak dichotomy law for N, which is why the word "hexachotomy" had to be invented; possibly, it is a linguistic misconstruction.) – For simplicity of exposition, we assume the strong laws to hold, having in mind that they may be replaced by weak ones, if needed.

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