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COMBINATORIAL NOTIONS RELATING TO PRINCIPAL FIBRE BUNDLES

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This article aims at clarifying the relationship between principal fibre bundles and groupoids, and along with it, the relationship between connections in the bundle or groupoid, and the associated connection forms. These notions are essentially due to E. Cartan [2] and to Ehresmann [4], [5], who in them saw some of the fundamental aspects of differential geometry, cf. also [10].

To this end, we introduce the notion of *pregroupoid* over a base 'space' B ('space' may mean either 'topological space', 'smooth manifold', or 'object in a topos $\mathscr{E}$', and accordingly for 'map' (or 'operation' or 'law')). We use the word 'set' synonymously with 'space', and give some standard comments for this abuse below. Formally, a pregroupoid over B is a set $E \to B$ over B equipped with a partially defined ternary operation λ, satisfying certain equations. In essence, a pregroupoid over B is the same as a principal fibre bundle, or torsor, over B, but whereas for a torsor, a group has to be given in advance, a pregroupoid canonically creates its own group. Also, by a dual construction, a pregroupoid creates a groupoid over B. Identifying the pregroupoid with a principal bundle H, this groupoid is Ehresmann's HH^{-1}, [4].

In the context of differential geometry, a typical example of a pregroupoid is the bundle E of orthonormal frames on a Riemannian manifold B; for x, y, z such frames, with x and z being frames at the same point of B, $\lambda(x, y, z)$ is the frame (at the same point as y) which has the same coordinates in terms of y as z does in terms of x.

To describe the relationship between connections and connection forms, we need to assume that the base 'space' B comes equipped with a reflexive symmetric 'neighbour' relation. Except for the two trivial extreme cases, topological spaces do not carry any natural relation of this kind, nor do smooth manifolds. However, for the latter, the method of synthetic differential geometry (cf. e.g. [7]) becomes available: essentially, it consists in embedding the category Mf of smooth manifolds into a suitable 'well-adapted' topos $\mathscr{E}$. When viewed in $\mathscr{E}$, any smooth manifold does acquire canonically such a reflexive symmetric relation, ‘the first neighbourhood of the diagonal’. It is a subobject $B_{(1)}$ of $B \times B$ in $\mathscr{E}$; its construction and use is well established also prior to synthetic differential geometry, cf. e.g. [11]. (One may think of $B_{(1)}$ as the manifold B but equipped with an ‘enlarged’ structure sheaf, whose stalks have nilpotent elements.) One effect of working in the topos $\mathscr{E}$ is that to a certain extent one can talk and reason about objects in a topos *as if they were* sets, a by now well established technique which finds its justification in categorical logic, cf. e.g. [7].

We now present the notion of pregroupoid (cf. also [9, §5]). A pregroupoid is a set E (‘total space’), a surjective map $\pi: E \to B$ onto another set B (‘base space’), and a law λ which to x, y, z with $\pi(x) = \pi(z)$ associates an element $\lambda(x, y, z)$ with $\pi(\lambda(x, y, z)) = \pi(y)$,

$$\begin{tikzcd} z \arrow[r] \arrow[d, equal] & \lambda(x,y,z) \arrow[d, equal] \\ x \arrow[r] & y \end{tikzcd}$$

and satisfying those equational conditions which $\lambda(x, y, z) := y \cdot x^{-1} \cdot z$ satisfies in any group; it suffices to require

$$\begin{cases} \lambda(x, y, x) = y, \\ \lambda(x, x, z) = z, \end{cases}$$
$$\begin{cases} \lambda(z, \lambda(x, y, z), t) = \lambda(x, y, t), \\ \lambda(y, t, \lambda(x, y, z)) = \lambda(x, t, z). \end{cases}$$

Some of the purely equational theory deriving from this was developed in [9], and we begin by summarizing the main points of this in Section 1. In Section 2, we further assume that B is equipped with a symmetric reflexive relation, the ‘neighbour’-relation $\sim$, and in terms of that, we describe the notion of (Ehresmann-) *connection* and connection-*form*. This purely combinatorial theory does in fact include the differential-geometric theory which the terminology suggests, namely via synthetic differential geometry (cf. e.g. [7]), with the relation $\sim$ being ‘the first neighbourhood of the diagonal’, ([3] or [6]); but we shall not here go into that aspect. In Sections 3 and 4, we deal with parallel transport, and integration, of connections and forms, respectively.

1. Groupoid and group associated canonically to a pregroupoid

Let $\pi: E \to B$, λ be a pregroupoid. We construct a groupoid E^* with B as its set of objects; an arrow $\alpha: a \to b$ $(a, b \in B)$ is an equivalence class of pairs $x, y \in E$ with $\pi(x) = a$, $\pi(y) = b$, under the equivalence relation

$$(x, y) \equiv (z, \lambda(x, y, z)).$$

The arrow $a \to b$ defined by the pair (x, y) is denoted $\overline{xy}$ or $\overline{x, y}$. Identity arrow over $a \in B$ is $\overline{xx}$ for any x over a, and $\overline{yt} \circ \overline{xy} = \overline{xt}$ defines the composition.

Also, we construct a group E_*; it has for its elements equivalence classes of pairs (x, z) with $\pi(x) = \pi(z)$, under the equivalence relation

$$(x, z) \equiv (y, \lambda(x, y, z)).$$

The element defined by (x, z) is denoted $\underline{xz}$ or $\underline{x, z}$. Group multiplication is defined by $\underline{xz} \cdot \underline{zt} = \underline{xt}$.

The groupoid E^* acts on E on the left, and the group E_* acts on E on the right. These actions commute with each other. The E_*-action is principal homogeneous, and makes $\pi: E \to B$ into a principal E_*-bundle. If $\alpha: a \to b$ is an arrow in E^*, left multiplication by α defines a map $E_a \to E_b$ (where E_a denotes the fibre of E over a, etc.). The action of E^* is given by $\alpha \cdot z = \lambda(x, y, z)$ where $\alpha = \overline{xy}$, and the action of E_* is given by $y \cdot \sigma = \lambda(x, y, z)$ where $\sigma = \underline{xz}$. These actions are unitary, associative, and both are principal, but in two different senses: for any $x, y \in E$, there is a unique arrow $\alpha: \pi(x) \to \pi(y)$ in E^* with $\alpha \cdot x = y$ (namely $\overline{xy}$); for each x, z in the same fibre, there is a unique $\sigma \in E_*$ with $x \cdot \sigma = z$, (namely $\underline{xz}$).

We collect for convenience several basic equations and bi-implications concerning the operations introduced. Note that $\overline{xy}$ 'behaves like $y \cdot x^{-1}$ does in a group', whereas $\underline{xz}$ 'behaves like $x^{-1} \cdot z$ does'; we also put these 'rules of thumb' in our reference list, but in quotes. All the rules are to be read with the proviso that the conditions for the expressions to be meaningful must be satisfied, e.g. in (1.4): $\partial_0(\alpha) = \pi(x)$.

$$\text{``}\lambda(x, y, z) = y \cdot x^{-1} \cdot z\text{''}. \tag{1.0}$$

$$\overline{xy} = \overline{zu} \quad \text{iff} \quad \underline{xz} = \underline{yu} \quad \text{iff} \quad \lambda(x, y, z) = u. \tag{1.1}$$

$$\begin{cases} \overline{yt} \circ \overline{xy} = \overline{xt}, \\ \underline{xz} \cdot \underline{zt} = \underline{xt}. \end{cases} \tag{1.2}$$

$$\begin{cases} \overline{ab} \circ \overline{cd} = \overline{c, \lambda(a, b, d)}, \\ \underline{ab} \cdot \underline{cd} = \underline{a, \lambda(c, b, d)}. \end{cases} \tag{1.3}$$

$$\begin{cases} \alpha \cdot x = y \quad \text{iff} \quad \overline{xy} = \alpha & (\text{``}\overline{xy} = y \cdot x^{-1}\text{''}), \\ x \cdot \sigma = z \quad \text{iff} \quad \underline{xz} = \sigma & (\text{``}\underline{xz} = x^{-1} \cdot z\text{''}). \end{cases} \tag{1.4}$$

$$\begin{cases} \overline{xy} \cdot x = y, \\ x \cdot \underline{xz} = z. \end{cases} \tag{1.5}$$

$$\overline{xy} \cdot z = y \cdot \underline{xz} = \lambda(x, y, z). \tag{1.6}$$

$$\begin{cases} \overline{x, \alpha \cdot y} = \alpha \circ \overline{xy}, \\ \overline{\alpha \cdot x, y} = \overline{xy} \circ \alpha^{-1}, \\ \underline{x, z \cdot \sigma} = \underline{xz} \cdot \sigma, \\ \underline{x \cdot \sigma, z} = \sigma^{-1} \cdot \underline{xz}. \end{cases} \tag{1.7}$$

$$\begin{cases} \overline{x\cdot\sigma, y\cdot\sigma} = \overline{xy}, \\ \underline{\alpha\cdot x, \alpha\cdot z} = \underline{xz}. \end{cases} \tag{1.8}$$

Also $\overline{xx}$ and $\underline{xx}$ act as neutral elements for the relevant compositions.

If G is a group and B a set, $B\times G$ becomes a pregroupoid over B by putting

$$\lambda((b_1, g_1), (b_2, g_2), (b_1, g_3)) := (b_2, g_2\cdot g_1^{-1}\cdot g_3).$$

We then have canonically, by identifying $\overline{(b_1, g_1), (b_2, g_2)}$ with $(b_1, g_2\cdot g_1^{-1}, b_2)$,

$$E^* = B\times G\times B,$$

and composition in $B\times G\times B$ is given by

$$(b_2, h, b_3)\circ(b_1, g, b_2) = (b_1, h\cdot g^{-1}, b_3).$$

Also, we have canonically, by identifying $\underline{(b, g), (b, h)}$ with $g^{-1}\cdot h$,

$$E_* = G.$$

The actions of E^* and E_* are then defined through the evident formulae:

$$(b_1, g, b_2)\cdot(b_1, h) = (b_2, g\cdot h),$$

$$(b, h)\cdot g = (b, h\cdot g).$$

2. Connections and forms

Both reflexive-symmetric relations and groupoids are *graphs*, meaning: a collection of 'vertices' (or 'objects'), and a collection of 'edges' (or 'arrows'); each row has a domain and a codomain (which are objects); to each arrow is given another arrow (the 'inverse' arrow) with domain and codomain interchanged, and to each object is given an identity arrow. No composition of arrows is assumed.

A *graph map* maps objects to objects, arrows to arrows, and commutes with inversion and identity formation. If the two graphs have the same object set B, and the graph map maps each object to itself, we call it a graph map *over B*.

In the following, B is a set with a reflexive symmetric relation $\sim$ on it; the graph thus given is denoted $B_{(1)}$.

The first to envisage a graph viewpoint for connections and forms was Joyal; but the relation $\sim$ and its utilization for connections appear already in [6].

If Φ is a groupoid with B as its object set, a graph map over B

$$\nabla : B_{(1)} \to \Phi$$

is called a *connection* on Φ. If G is a group (considered as a groupoid with one object, hence as a graph with one vertex), a graph map

$$\Omega : B_{(1)} \to G$$

is called a 1-*form on B* (with values in G); cf. [8] for the differential geometric motivation for this notion and its equivalence with the classical notion of differential form with values in the Lie algebra of G.

We now consider groupoids Φ of the form E^* with E a pregroupoid over B. We provide E with a reflexive symmetric relation $\sim$ by setting $x \sim y$ iff $\pi(x) \sim \pi(y)$ in B.

Proposition 2.1. *There is a natural* 1–1 *correspondence between connections* ∇ *on* E^*, *and* E_**-valued* 1*-forms* Ω *on* E *which are right* E_**-equivariant in the sense that*

$$\Omega(x, z \cdot \sigma) = \Omega(x, z) \cdot \sigma \quad \forall \sigma \in E_*,\ \forall x, z \text{ with } \pi(x) = \pi(z); \tag{2.1}$$

Ω *is called the* connection-form *of the connection* ∇.

Proof/construction. Given ∇, define Ω by

$$\Omega(x, y) := \underline{\nabla(\pi(x), \pi(y)) \cdot x, y}. \tag{2.2}$$

Given Ω, define ∇ by

$$\nabla(a, b) := \overline{x, y \cdot \Omega(y, x)}, \tag{2.3}$$

where x and y are arbitrary elements in E over respectively a and b. This well-defines $\nabla(a, b)$, for, if we had chosen $x' = x \cdot \sigma$ instead of x, we would get

$$\overline{x \cdot \sigma, y \cdot \Omega(y, x \cdot \sigma)} = \overline{x \cdot \sigma, y \cdot \Omega(y, x) \cdot \sigma} = \overline{x, y \cdot \Omega(y, x)}$$

by (2.1) and (1.8). Similarly, if we had chosen $y' = y \cdot \sigma$ instead of y; then we would use $\Omega(y \cdot \sigma, x) = \sigma^{-1} \cdot \Omega(y, x)$, which follows from (2.1) and $\Omega(r, s) = \Omega(s, r)^{-1}$. We leave the further checking to the reader. It is of course similarly trivial/equational. □

3. Integrable connections and exact forms

We first summarize some notions from [8]. Let throughout this section B be a set with a symmetric reflexive relation $\sim$. If G is a group, a (G-valued) 0-form on B is just a map $B \to G$, a (G-valued) 1-form is as defined in Section 2, and a (G-valued) 2-form is a law which to any '2-simplex' in B (meaning a triple x, y, z with $x \sim y \sim z \sim x$) associates an element of G, and which satisfies some equational conditions which we shall not need here.

To a 0-form f on B, we associate two 1-form $\bar{\mathrm{d}}f$ and $\underline{\mathrm{d}}f$, given by, respectively

$$(\bar{\mathrm{d}}f)(a, b) = f(b) \cdot f(a)^{-1}, \tag{3.1}$$

$$(\underline{\mathrm{d}}f)(a, b) = f(a)^{-1} \cdot f(b), \tag{3.2}$$

for $a \sim b$.

Also, to a 1-form ω, we associate two 2-forms $\bar{\mathrm{d}}\omega$ and $\underline{\mathrm{d}}\omega$, given by, respectively

$$(\bar{\mathrm{d}}\omega)(a, b, c) = \omega(c, a) \cdot \omega(b, c) \cdot \omega(a, b), \tag{3.3}$$

$$(\underline{d}\omega)(a,b,c) = \omega(a,b)\cdot\omega(b,c)\cdot\omega(c,a). \qquad (3.4)$$

for any 2-simplex a, b, c.

Clearly $\bar{d}(\bar{d}f)$ and $\underline{d}(\underline{d}f)$ are both the zero 2-form, meaning it takes value $e \in G$ on any 2-simplex (e = neutral element of G). In the following, we shall give preference to the lower-bar case (3.2)–(3.4), and write $\mathrm{d}f$ for $\underline{d}f$, $\mathrm{d}\omega$ for $\underline{d}\omega$. Also, we say that a 1-form ω is *closed* if $\mathrm{d}\omega = 0$.

We say that the pair $(B, \sim)$, G *admits integration* if "closed 1-forms are exact", meaning precisely: to any closed 1-form ω, there is a 0-form $f: B \to G$ with $\mathrm{d}f = \omega$ ('a *primitive* of ω'), and any two such primitives differ by left multiplication by a fixed $g \in G$,

$$f_1(b) = g \cdot f_2(b) \quad \forall b \in B.$$

Remark 3.1. One of the fundamental theorems of differential geometry may be expressed: if G is a Lie group, and B is a connected and simply connected manifold, then B, G admits integration. This requires the use of the context of synthetic differential geometry, and in particular, taking $\sim$ to be the first neighbourhood of the diagonal of B. For a proof, see [8].

Closely connected with coboundary of 1-forms is curvature of connections. If Φ is a groupoid over B, and ∇ is a connection in Φ, the *curvature* of ∇ is the law which to any 2-simplex (a, b, c) in B associates the composible arrow

$$\nabla(c,a) \circ \nabla(b,c) \circ \nabla(a,b) \in \Phi(a,a),$$

and we say that ∇ is *curvature-free* if for all 2-simplices, this arrow is an identity arrow.

Also, closely connected with primitives of 1-forms are integrals of connections: if ∇ is a connection on Φ, an *integral* for it is a map $\tilde{\nabla}: B \times B \to \Phi$ with $\tilde{\nabla}(a,b): a \to b$ for all $a, b \in B$, and with

$$\tilde{\nabla}(a,b) = \nabla(a,b) \quad \text{for } a \sim b$$

$$\tilde{\nabla}(b,c) \circ \tilde{\nabla}(a,b) = \tilde{\nabla}(a,c) \quad \forall a,b,c.$$

Equivalently: $\tilde{\nabla}$ is a functor over B from the codiscrete groupoid $\pi_0 B$ on B, extending ∇. Or, equivalently again, it is a curvature free connection $\tilde{\nabla}$ in Φ, but now with B viewed as having the codiscrete $\sim$ on it ($a \sim b$ for all a, b), and with $\tilde{\nabla}$ extending ∇.

Clearly, if ∇ has an integral, then it is a curvature free.

We now consider the case where the groupoid Φ is of the form E^* for some pregroupoid E over B. Then

Proposition 3.2. *Under the correspondence of Proposition* 2.1, *curvature free connections correspond to closed forms.*

Proof. Suppose ∇ is curvature free, and let x, y, z be a 2-simplex in E. Write a, b, c for $\pi(x)$, $\pi(y)$, and $\pi(z)$, respectively. Then

$$\Omega(x,y)\cdot\Omega(y,z)\cdot\Omega(z,x)=\underline{\nabla(a,b)\cdot x,y}\cdot\underline{\nabla(b,c)\cdot y,z}\cdot\underline{\nabla(c,a)\cdot z,x}.$$

Now multiply the two entries in the first factor on the left by $\alpha=\nabla(c,a)\circ\nabla(b,c)$, which we may by (1.8), and similarly, multiply the two entries in the second factor by $\alpha=\nabla(c,a)$. By (1.2), the expression then reduces to

$$\underline{\nabla(c,a)\circ\nabla(b,c)\circ\nabla(a,b)\cdot x,x}; \tag{3.5}$$

the left hand entry is x, by the assumption on ∇ being curvature free, so we get $\underline{xx}=e$. Conversely, if Ω is closed, and a, b, c is a 2-simplex in B, we may choose x, y, and z above a, b, and c, respectively. The same calculation as before now leads from the assumption on Ω to the conclusion that (3.5) is e, whence, by (1.4), $\nabla(c,a)\circ\nabla(b,c)\circ\nabla(a,b)$ is an identity arrow in E^*. This proves the proposition. □

Corollary 3.3. *Let ∇ be a connection on E^* with connection from Ω. Then there is a natural* 1–1 *correspondence between integrals $\tilde{\nabla}$ of ∇, and extensions $\tilde{\Omega}$ of Ω*

$$\tilde{\Omega}: E\times E\to E_*$$

which are right E_*-equivariant, and satisfy

$$\tilde{\Omega}(x,y)\cdot\tilde{\Omega}(y,z)\cdot\tilde{\Omega}(z,x)=e \quad \forall x,y,z\in E. \tag{3.6}$$

Proof. Apply the correspondence of Proposition 2.1, but now with B equipped with the codiscrete $\sim$, to get the 1–1 correspondence between $\tilde{\nabla}$'s and $\tilde{\Omega}$'s. Then, by Proposition 3.2, $\tilde{\nabla}$ is an integral (i.e. is curvature free) iff $\tilde{\Omega}$ is closed, i.e. satisfies (3.6). Since the correspondence $\tilde{\nabla}\Leftrightarrow\tilde{\Omega}$ is given by (2.2) and (2.3), as is the correspondence between ∇ and Ω, it follows that $\tilde{\nabla}$ extends ∇ iff $\tilde{\Omega}$ extends Ω. □

It is easy to see that there is a bijective correspondence between, on the one side, maps $\tilde{\Omega}: E\times E\to E_*$ which are right E_*-equivariant, and are closed 1-forms with respect to the codiscrete $\sim$ on E (i.e. they satisfy (3.6)), and, on the other side, orbits of right E_*-equivariant maps $f: E\to E_*$ under the left (valuewise) action of E_*, namely, to the orbit of f, associate $\mathrm{d}f$ given by $(\mathrm{d}f)(x,y) := f(x)^{-1}\cdot f(y)$, and to $\tilde{\Omega}$, associate the orbit of $\tilde{\Omega}(x_0,-): E\to E_*$, for some (any) fixed $x_0\in E$.

We can now prove

Theorem 3.4. *Assume the pair $(B,\sim)$, E_* admits integration. Then, any curvature-free connection ∇ on E^* has a unique integral $\tilde{\nabla}: B\times B\to E^*$.*

Proof. We shall use a useful and well-known fact of categorical logic: if a construction can be performed using a hypothetical splitting s of an epic π, but the result of the construction is independent of the chosen splitting, then the construction can be performed, even though no splitting of π exists. (Proof of this fact: construct locally; glue together by uniqueness.) So suppose $s: B \to E$ is a hypothetical splitting of π. The connection form Ω of ∇ is closed (Proposition 3.2), hence so is the 1-form $s^*\Omega$ on B ($s^*\Omega(a,b) = \Omega(s(a), s(b))$). By the integration assumption, there exists a map $h: B \to E_*$ with $\mathrm{d}h = s^*\Omega$, so for $a \sim b$ in B

$$h(a)^{-1} \cdot h(b) = \Omega(s(a), s(b)) = \underline{\nabla(a,b) \cdot s(a), s(b)} \tag{3.7}$$

by (2.2). We construct $\tilde{\nabla}$ essentially by (2.3):

$$\tilde{\nabla}(a,b) := \overline{s(a), s(b) \cdot h(b)^{-1} \cdot h(a)} = \overline{s(a) \cdot h(a)^{-1}, s(b) \cdot h(b)^{-1}} \tag{3.8}$$

(the last equality by (1.8)). Then it is immediate from (1.2) that $\tilde{\nabla}$ is curvature free. Also, it extends ∇, for, if $a \sim b$ in B, we have by (3.8), (3.7), and (2.3)

$$\tilde{\nabla}(a,b) = \overline{s(a), s(b) \cdot h(b)^{-1} \cdot h(a)} = \overline{s(a), s(b) \cdot \Omega(s(b), s(a))} = \nabla(a,b).$$

To prove uniqueness of such $\tilde{\nabla}$ (and, as explained, the uniqueness justifies the use of the splitting s), let $\tilde{\nabla}_1, \tilde{\nabla}_2$ be two integrals of ∇. By Corollary 3.3, they correspond to right-equivariant closed 1-forms $\tilde{\Omega}_1, \tilde{\Omega}_2$ on E (with codiscrete $\sim$), extending Ω, and hence, as we remarked, to two (orbits of) right E_*-equivariant maps $f_1, f_2: E \to E_*$. Both of these are right-invariant primitives of Ω, hence factor across π and belong to the same orbit under the left E_*-action, whence $\tilde{\Omega}_1 = \tilde{\Omega}_2$, whence $\tilde{\nabla}_1 = \tilde{\nabla}_2$. This proves the uniqueness and thus the theorem. □

Recall that a groupoid Φ with object set B is called *transitive* if for any $a, b \in B$, there is at least one arrow $a \to b$. 'The' *isotropy group* of a transitive Φ is the group $\Phi(a,a)$ for any $a \in B$; it is only determined up to (non-canonical) isomorphism.

Corollary 3.5. *Let Φ be a transitive groupoid on B, with isotropy group G. Assume the pair $(B, \sim)$, G admits integration. Then a connection ∇ in Φ admits an integral iff it is curvature free. Such an integral is unique.*

Proof. Any transitive Φ is of the form E^* for some pregroupoid E on B (cf. [9]), namely with $E_b = \Phi(a,b)$ (fixed a), and $\lambda(x,y,z) = y \circ x^{-1} \circ z$. Then E_* is isomorphic to the isotropy group of Φ. Now apply the theorem to E. □

4. Lift along curves, and holonomy

Let G be a group, and let I be a set with a symmetric reflexive relation $\sim$ on it, such that $(I, \sim)$, G admits integration and such that any G-valued 1-form on I is closed (hence exact, with unique primitive, modulo left multiplication). In the application, I will be the unit interval.

Let us also consider a groupoid Φ over B, transitive and with isotropy group $\cong G$. We assume B is equipped with a symmetric reflexive relation $\sim$. By a *path* in B we mean any map $f: I \to B$ which preserves the relation $\sim$. If f is a path, the groupoid Φ over B pulls back to a groupoid Φ_f over I, with hom sets given by

$$\Phi_f(t_1, t_2) := \Phi(f(t_1), f(t_2)).$$

Also, given a connection $\nabla: B_{(1)} \to \Phi$, it pulls back to a connection $\nabla_f: I_{(1)} \to \Phi_f$: for $t_1 \sim t_2$ in I

$$\nabla_f(t_1, t_2) := \nabla(f(t_1), f(t_2)) \in \Phi(f(t_1), f(t_2)) = \Phi_f(t_1, t_2).$$

Applying Corollary 3.5 for the groupoids Φ_f and the connection ∇_f (which is curvature free, since its connection form, when Φ_f is written in form E^* for a pre-groupoid E over I, is closed, by assumption on I), we get

Theorem 4.1. *To any path $f: I \to B$, there exists a unique functor $\bar{\nabla}_f: \pi_0 I \to \Phi$ which is f on objects and satisfies, for all t_1, t_2, s in I with $t_1 \sim t_2$,*

$$\nabla(f(t_1), f(t_2)) \circ \bar{\nabla}_f(s, t_1) = \bar{\nabla}_f(s, t_2).$$

("*Lift* of f by ∇, or *lift* of ∇ *along* f".)

Assume now further that I is equipped with two specified elements 0 and 1. By the *holonomy groupoid* $\mathscr{H}(\nabla) \subseteq \Phi$ of the connection ∇, we understand the smallest subgroupoid containing $\bar{\nabla}_f(0, 1)$ for all paths $f: I \to B$. It is a transitive subgroupoid of Φ is we assume B to be path-connected: to any $a, b \in B$, there exists a path f with $f(0) = a$, $f(1) = b$. 'The' holonomy group $\underline{\mathscr{H}}(\nabla)$ of ∇ is defined to be 'the' isotropy group of $\mathscr{H}(\nabla)$.

Corollary 4.2. *Assume B path connected, and such that the pair $(B, \sim)$, E_* admits integration, and suppose ∇ is a curvature free connection on E^*. Then the holonomy groupoid $\mathscr{H}(\nabla)$ is trivial (i.e. codiscrete: for any pair a, b in B, there is exactly one arrow).*

Proof. Let $\tilde{\nabla}: B \times B \to E^*$ be the integral of ∇ (Corollary 3.5). For any path $f: I \to B$ from a to b, the function $I \times I \to E^*$ given by

$$(s, t) \mapsto \tilde{\nabla}(f(s), f(t))$$

will be an integral of ∇_f, so

$$\bar{\nabla}_f(0, 1) = \tilde{\nabla}(f(0), f(1)) = \tilde{\nabla}(a, b),$$

which is independent of the path f. □

A transitive groupoid Φ on B with isotropy group G is said to be *reducible to* the subgroup $G' \subset G$ if there is a subgroupoid $\Phi' \subset \Phi$ with isotropy group G' and still transitive on B. (The notion really applies not to subgroups but to conjugacy classes of subgroups.) We get in particular, under the standing hypotheses of this section, the following corollary of Theorem 4.1:

Corollary 4.3. *If ∇ is a connection on the transitive groupoid Φ over B, and B is path connected, then Φ is reducible to the holonomy subgroup of ∇.*

From the assumptions made so far, it does not follow that $\nabla(a,b) \in \bar{\mathscr{H}}(\nabla)$. We have to assume that among the paths f from a to b, there exists one with $\bar{\nabla}_f(a,b) = \nabla(a,b)$; in the applications, such a path will be the unique *affine* path $t \mapsto (1-t)\cdot a + t\cdot b$, which, for $a \sim b$, can be proved to be independent of the choice of coordinate chart employed to make sense to this affine combination.

Let E be a pregroupoid. As in [9, §3], we may (for any $a \in B$), by choosing a base point 0 in E_a, construct an isomorphism of groups $\varepsilon_0 : E^*(a,a) \cong E_*$; any two such differ by inner automorphism on E_*. Therefore, if $H \subset E_*$ is a *normal* subgroup, and $\mathscr{H} \subset E^*$ a subgroupoid, it makes sense to say that $\mathscr{H}(a,a) \subseteqq H$ (meaning: $\varepsilon_0(\mathscr{H}(a,a)) \subseteqq H$ for one, hence any, $0 \in E_a$).

Let us in what follows suppress mentioning of $\sim$ on B and I, and let us accumulate the hypotheses made so far: $(E \to B, \lambda)$ is pregroupoid, (B, E_*) admits integration; B is path connected, and (I, E_*) admits integration. Under these hypotheses, we have the following version of Ambrose–Singer's Theorem:

Theorem. *Let $H \lhd E_*$ be a normal subgroup such that $B, E_*/H$ admits integration. If ∇ is a connection on E^* with connection form Ω, such that all values of $\mathrm{d}\Omega$ belong to H, then the holonomy groups $\mathscr{H}(a,a)$ of ∇ are contained in H.*

Proof. The set E/H of orbits of the right action of $H \subseteq E_*$ can be equipped with structure of pregroupoid λ_H (over B), with $p: E \to E/H$ a pregroupoid homomorphism in the evident sense. (This follows from the easily proved equations

$$\lambda(x\cdot g, y, z) = \lambda(x,y,z)\cdot (g^{-1})^{\underline{xz}},$$

$$\lambda(x, y\cdot g, z) = \lambda(x,y,z)\cdot g^{\underline{xz}},$$

$$\lambda(x,y,z\cdot g) = \lambda(x,y,z)\cdot g$$

where $a^b := (b^{-1}\cdot a\cdot b)$. Then $(E/H)_*$ is canonically identified with E_*/H. Also, there is a groupoid homomorphism $E^* \to (E/H)^*$, and using this, we get a connection ∇_H on $(E/H)^*$, whose curvature form Ω_H is related to the curvature form Ω of ∇ by

$$\Omega_H(p(x), p(y)) = \Omega(x,y)\cdot H \in E_*/H.$$

The assumption on $\mathrm{d}\Omega$ now says by Proposition 3.2 that ∇_H is curvature free. By Corollary 4.2, its holonomy groupoid $\bar{\mathscr{H}}(\nabla_H) \subseteqq (E/H)^*$, is codiscrete, or

$$\bar{\mathscr{H}}(\nabla_H) \subseteqq \{H\},$$

H viewed as an element of E_*/H. Since lifts of paths by ∇ by $p: E \to E/H$ go to their lifts by ∇_H, we conclude

$$\mathscr{H}(\nabla) \subseteq H.$$

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