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Anders Kock

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COMBINATORICS OF NON-HOLONOMOUS JETS

ANDERS KOCK, Aarhus

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The notion of non-holonomous and semi-holonomous r-jet as considered by Ehresmann [1], was defined by induction in r. In the present paper, we point out that when the notion of 1-jet becomes representable (as is the case in synthetic differential geometry [4], [5]), then the inductive definition becomes equivalent to an explicit definition. In fact, just as the notion of "1-jet at x" becomes representable in the sense that it becomes equivalent to "map defined on the 1-monad $\mathcal{M}_1(x)$ around x", so the notion of "non-holonomous r-jet" becomes representable by a certain set $\mathcal{M}_r^{\sim}(x)$, the "non-holonomous r-monad around x". We show how to compose non-holonomous jets, and that the subclasses of holonomous and semi-holonomous jets are stable under this composition. To this end, we utilize (the combinatorial paraphrasing of) the classical inductive definition, in conjunction with the notion of 1-prolongation of a differentiable category (also due to Ehresmann).

In the final §, we describe the applications of the theory of non- and semi-holonomous jets to the theory of connections. This is a combinatorial paraphrasing of parts of [8] and [9]. I want to thank I. Kolář for valuable discussions, which became the germ of the present investigations.

1. NON-HOLONOMOUS r-MONADS AND JETS

In the context of synthetic differential geometry (see e.g. [5]), every object M (say a manifold) comes equipped with a "first neighbourhood of the diagonal" $M_{(1)} \subset M \times M$ (see in particular [1]). Using set theoretic language, we have on M a binary relation $\sim$ given by

$$x \sim y \quad \text{iff} \quad (x, y) \in M_{(1)} .$$

This relation is reflexive and symmetric:

$$x \sim x\,; \quad x \sim y \Rightarrow y \sim x\,,$$

but not transitive. We call it the *1-neigbour relation*. Any map $f: M \to N$ preserves this relation. Furthermore, let for $x \in M$,

$$\mathcal{M}_1(x) := [\![y \in M \mid y \sim x]\!]\,;$$

then, at least for suitably nice M, the inclusion $\mathcal{M}_1(x) \subset M$ reflects $\sim$, meaning that if y and $z \in \mathcal{M}_1(x)$ are 1-neighbours in M, then they are 1-neigbours in $\mathcal{M}_1(x)$. (This property is only used in the proof of Proposition 2.1.)

We generalize $M_{(1)}$ as follows. Let $r \geqq 0$ be an integer. Define

$$\tilde{M}_{(r)} := [\![(x_0, \ldots, x_r) \in M^{r+1} \mid x_{i-1} \sim x_i \ \forall i = 1, \ldots, r]\!] .$$

(Note $M_{(1)} = \tilde{M}_{(1)}$.) An element $(x_0, \ldots, x_r)$ of $\tilde{M}_{(r)}$ will be called an *r-chain* with x_0 as its *center*, and x_r as its *extremity*; so we have maps

$$\tilde{M}_{(r)} \overset{c}{\underset{e}{\rightrightarrows}} M$$

associating to each r-chain its center and extremity, respectively. There is some further combinatorial structure on the family of $\tilde{M}_{(r)}$'s namely "degeneracy" maps

$$\sigma_i\colon \tilde{M}_{(r)} \to \tilde{M}_{(r+1)} \quad (i = 0, \ldots, r)$$

given by

$$\sigma_i(x_0, \ldots, x_r) = (x_0, \ldots, x_i, \ x_i, \ldots, x_r)$$

(i'th entry repeated). (We don't have "face" maps $\partial_i\colon \tilde{M}_{(r)} \to \tilde{M}_{(r-1)}$ since the relation $\sim$ is not transitive.)

Composing r degeneracy maps gives the diagonal map

$$\Delta\colon M \to \tilde{M}_{(r)}$$

given by $\Delta(x) = (x, \ldots, x)$; also composing $r-1$ degeneracy maps suitably gives r maps

$$M_{(1)} \underset{s_{r-1}}{\overset{s_0}{\vdots}} \tilde{M}_{(r)}$$

given by

$$s_i(x, y) = (x, \ldots, x, \ y, \ldots, y)$$

(x written $i+1$ times, y written $r-i$ times). The σ_i's and hence Δ and the s_i's preserve c and e: $e \circ \sigma_i = e$, etc.

For $x \in M$, we let $\tilde{M}_r(x)$ denote the fibre of $\tilde{M}_{(r)}$ over x by the center map $c\colon \tilde{M}_{(r)} \to M$; so

$$\tilde{\mathcal{M}}_r(x) = [\![(x_0, \ldots, x_r) \mid x_{i-1} \sim x_i \ \forall i; \ x_0 = x]\!] ;$$

we call it the *non-holonomous r-monad around* x, and x is its *center*. The restriction of e to $\tilde{\mathcal{M}}_r(x)$ defines a map $e\colon \tilde{\mathcal{M}}_r(x) \to M$, which is not monic in general, unless $r = 0$ or 1. A map $f\colon \tilde{\mathcal{M}}_r(x) \to N$ is called a *non-holonomous* N-valued *r-jet at* x, and x is called the *source* of the jet f. In this sense, the notion of non-holonomous r-jet is represented by an object, namely $\tilde{\mathcal{M}}_r(x)$. – Henceforth, we will often say "r-jet" instead of "non-holonomous r-jet".

Also, if $\pi\colon E \to M$ is an arbitrary map, an *r-jet of a section of* π *at* x is an r-jet at x,

$$f\colon \tilde{\mathcal{M}}_r(x) \to E$$

with $$\pi(f(\boldsymbol{x})) = e(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \mathcal{M}_r^{\sim}(x) .$$

We get an object $\tilde{J}^r(E) \to M$ over M, whose fibre over $x \in M$ is the set of r-jet sections of π at x.

In particular, we may form $\tilde{J}^s(\tilde{J}^r E)$. We have the following comparison which relates our definition of (non-holonomous) r-jet with the inductive definition of Ehresmann [3]:

Proposition 1.1. *For any $\pi\colon E \to M$, there is a canonical bijection over M*

$$\tilde{J}^1(\tilde{J}^r(E)) \to \tilde{J}^{r+1}(E) .$$

Proof/construction. Let $F \in \tilde{J}^1(\tilde{J}^r(E))_{x_0}$; so for any $x_1 \sim x_0$,

$$F(x_0, x_1) \in \tilde{J}^r(E)_{x_1} ,$$

so for any r-chain $(x_1, \ldots, x_{r+1})$ with center x_1

$$F(x_0, x_1)\,(x_1, \ldots, x_{r+1}) \in E_{x_{r+1}} .$$

So for any $(r + 1)$-chain $(x_0, \ldots, x_{r+1})$ with center x_0, put

(1.1) $$f(x_0, x_1, \ldots, x_{r+1}) := F(x_0, x_1)\,(x_1, \ldots, x_{r+1}) .$$

This defines $f \in \tilde{J}^{r+1}(E)_{x_0}$. The passage the other way is similar.

An r-jet $f\colon \mathcal{M}_r^{\sim}(x) \to N$ is called *semi-holonomous* if for any $\boldsymbol{x} \in \mathcal{M}_{r-1}^{\sim}(x)$,

$$f(\sigma_i(\boldsymbol{x})) = f(\sigma_j(\boldsymbol{x})) \quad \forall i, j = 0, \ldots, r - 1 ,$$

and *holonomous* if $f(\boldsymbol{x})$ depends only on $e(\boldsymbol{x})$. Clearly, holonomous implies semi-holonomous. For $\pi\colon E \to M$ as above, we denote by $\bar{J}^r E$, respectively $J^r E$, the subset of $\tilde{J}^r E$ consisting of semi-holonomous, respectively holonomous, r-jet sections of π. Note $J^1 E = \bar{J}^1 E = \tilde{J}^1 E$.

Clearly, if $F \subset E$, $\pi\colon E \to M$, we get $\tilde{J}^s(F) \subset \tilde{J}^s(E)$ canonically, so in particular, we may view $J^1(\bar{J}^r(E))$ as a subset of $J^1(\tilde{J}^r(E))$. We have the following comparison which relates our definition of "semi-holonomous" with the inductive definition of Ehresmann [2]. Consider the diagram

(1.2) $$\begin{array}{ccc} \bar{J}^{r+1} & \hookrightarrow & \tilde{J}^{r+1} \\ \downarrow & & \downarrow \cong \\ J^1(\bar{J}^r E) & \hookrightarrow & J^1(\tilde{J}^r E) \end{array}$$

where the indicated bijection is that of Proposition 1.1.

Proposition 1.2. This bijection identifies $\bar{J}^{r+1}E$ with that subset of $J^1(\bar{J}^r E)$ consisting of F's which satisfy

$$(1.3) \qquad F(x_0, x_1)(x_1, x_1, x_2, \ldots, x_r) = F(x_0, x_0)(x_0, x_1, \ldots, x_r)$$

for all $\boldsymbol{x} = (x_0, \ldots, x_r) \in \tilde{\mathcal{M}}_r(x_0)$ (x_0 the source of F).

Proof. Let F be a 1-jet section of $\bar{J}^r E$ at x_0, satisfying (1.3), and let f correspond to F under the bijection. Then, for $j, i \geqq 1$ and $\boldsymbol{x} \in \tilde{\mathcal{M}}_r(x_0)$,

$$\begin{aligned} f(\sigma_i(\boldsymbol{x})) &= F(x_0, x_1)(x_1, \ldots, x_i, x_i, \ldots, x_r) \\ &= F(x_0, x_1)(x_1, \ldots, x_j, x_j, \ldots, x_r) = f(\sigma_j(\boldsymbol{x})), \end{aligned}$$

since $F(x_0, x_1) \in \bar{J}^r E$, whereas for $j = 0$, $i \geqq 1$,

$$\begin{aligned} f(\sigma_i(\boldsymbol{x})) &= F(x_0, x_1)(x_1, \ldots, x_i, x_i, \ldots, x_r) \\ &= F(x_0, x_1)(x_1, x_1, \ldots, x_i, \ldots, x_r) \end{aligned}$$

(since $F(x_0, x_1) \in \bar{J}^r E$),

$$= F(x_0, x_0)(x_0, x_1, \ldots, x_i, \ldots, x_r) = f(\sigma_0(\boldsymbol{x})).$$

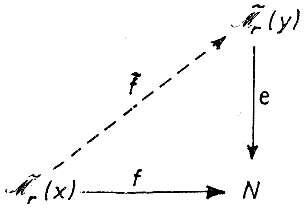
The calculation for the other implication is similar.

2. THE COMPOSITION OF NON-HOLONOMOUS JETS

If $f: \tilde{\mathcal{M}}_r(x) \to N$ is an r-jet at x, we define its *target* y to be

$$y = f(x, x, \ldots, x) = f(\Delta(x)) \in N.$$

Proposition 2.1. *Let f be an r-jet at x with target y. There is a canonical lifting $\tilde{f}$ of f across e: $\tilde{\mathcal{M}}_r(y) \to N$,*



Proof/construction. Let x_0 denote x. Given $\boldsymbol{x} = (x_0, \ldots, x_r) \in \tilde{\mathcal{M}}_r(x)$ we define $\tilde{f}(\boldsymbol{x})$ to be the $(r + 1)$-tuple

$$(f(x_0, x_0, \ldots, x_0), f(x_0, x_1, \ldots, x_1), \ldots, f(x_0, x_1, \ldots, x_r));$$

we have to argue that this is an r-chain, i.e. that consecutive entries are 1-neighbours. So we should prove

$$(2.1) \qquad f(x_0, \ldots, x_i, x_i, \ldots, x_i) \sim f(x_0, \ldots, x_i, x_{i+1}, \ldots, x_{i+1}).$$

For fixed $(x_0, \ldots, x_i)$, we consider the right hand side as a function $g(x_{i+1})$ of $x_{i+1} \in \in \mathcal{M}_1(x_i)$. Since $x_i \sim x_{i+1}$ in M, it follows that $x_i \sim x_{i+1}$ in $\mathcal{M}_1(x_i)$, and hence that $g(x_i) \sim g(x_{i+1})$ in N, which is (2.1).

From the Proposition follows that (non-holonomous) r-jets may be composed: Let f be as above, and let g: $\mathcal{M}_r^{\sim}(y) \to P$ be an r-jet at y. We then put $g \circ f$ equal to the composite map

$$\mathcal{M}_r^{\sim}(x) \xrightarrow{\tilde{f}} \mathcal{M}_r^{\sim}(y) \xrightarrow{g} P .$$

Also, for each $x \in M$, there is a canonical r-jet e_x at x, with source and target x, namely e: $\mathcal{M}_r^{\sim}(x) \to M$. Note that the lifting $\tilde{e}$ of this e is the identity map of $\mathcal{M}_r^{\sim}(x)$.

It is possible to verify directly that jet composition thus defined is associative and has the e_x's as units, so that r-jets form a category; and also that the semi-holonomous (respectively holonomous) jets form a subcategory. However, these proofs seem more transparent using the inductive definition of non-holonomous jets, in conjunction with a general theory of jet-prolongation of categories, as expounded in the next §, notably Theorem 3.1. So for the time being, let us note that the structure $\circ$ and the e_x's equip the set of r-jets (with source and target in M) as an *oriented multiplicative graph* $\tilde{\Pi}^{(r)}M$ with M as object- (or vertex-) set (the notion of oriented multiplicative graph is like the notion of category, except that we do not assume the associative law for 0, nor that the e_x's are neutral elements for 0).

3. THE PROLONGATION OF CATEGORIES

Let C be a category with M as its set of objects. So we have source and target maps α, β: $C \to M$. The composition in C is denoted $^\circ$ and written from right to left; the unit at $x \in M$ is denoted u_x or $u(x)$.

The classical definition (Ehresmann) of the prolonged category $C^{(1)}$ is now paraphrased as follows: $C^{(1)}$ has M as object set; an arrow from x to y is a 1-jet of a section of α

$$f: \mathcal{M}_1(x) \to C$$

with $\beta(f(x)) = y$. If g is an arrow in $C^{(1)}$ from y to z, then the composite $g \circ f$ is the 1-jet at x which sends $x_1 \sim x$ to the arrow

$$g(\beta(f(x_1)) \circ f(x_1)$$

in C (cf. [4], Remark 6.5, with $r = 1$). The unit arrow $u_x^{(1)}$ at x is the 1-jet $\mathcal{M}_1(x) \to C$ which sends $x_1 \sim x$ to $u(x_1)$. The fact that these data do determine a category $C^{(1)}$ is straightforward (and omitted in loc. cit.). The construction works even when C, is only an oriented multiplicative graph; in this case, $C^{(1)}$ is again only an oriented multiplicative graph.

More generally, for an oriented multiplicative graph C as above, we define its r'th non-holonomous prolongation $\tilde{C}^{(r)}$ to be the following oriented multiplicative graph: its object set is M; an arrow from x to y is an r-jet of a section of α,

$$f\colon \mathcal{M}_r^{\sim}(x) \to C$$

with $\beta(f(\Delta(x))) = y$. Note that the target of f as an arrow in $\tilde{C}^{(r)}$ is $\beta(f(\Delta(x)))$, whereas the target of f as an abstract jet is $f(\Delta(x))$. We shall write β^r for the target formation in $\tilde{C}^{(r)}$. If g is an arrow in $\tilde{C}^{(r)}$ from y to z, then the composite $g \circ f$ is defined to be the r-jet at $x = x_0$, which sends $\boldsymbol{x} = (x_0, \ldots, x_r)$ to the arrow

$$g((\beta \circ f)^{\sim}(\boldsymbol{x})) \circ f(\boldsymbol{x})$$

in C, or more detailed,

$$(g \circ f)(\boldsymbol{x}) = g(\beta(f(x_0, \ldots, x_0)), \beta(f(x_0, \ldots, x_1)), \ldots, \beta(f(x_0, \ldots, x_r))) \circ f(x_0, \ldots, x_r)\,.$$

The displayed element does make sense as a composite in C since

$$\alpha(g(\beta \circ f)^{\sim}(\boldsymbol{x})) = e((\beta \circ f)^{\sim}(\boldsymbol{x})) = (\beta \circ f)(\boldsymbol{x}) = \beta(f(\boldsymbol{x}))\,.$$

The unit arrow at x is taken to be the map $u^{(r)}\colon \mathcal{M}_r^{\sim}(x) \to C$ which sends $\boldsymbol{x}$ to $u_{e(\boldsymbol{x})}$.

Theorem 3.1. *Let C be an oriented multiplicative graph. For each $r = 0, 1, 2, \ldots$, there is a canonical isomorphism of oriented multiplicative graphs*

$$(\tilde{C}^{(r)})^{(1)} \xrightarrow{\cong} \tilde{C}^{(r+1)}\,. \tag{3.1}$$

If C is a category, then so is each $\tilde{C}^{(r)}$, and (3.1) is an isomorphism of categories.

Proof. For $x_0 \in M$, we identify $\mathcal{M}_1(x_0)$ with $\mathcal{M}_1^{\sim}(x_0)$ via $x_1 \mapsto (x_0, x_1)$; now the isomorphism (3.1) is a special case of that of Proposition 1.1, with $\pi\colon E \to M$ replaced by $\alpha\colon C \to M$. We only have to verify that (3.1) thus defined preserves source, target, composition, and units. For source, this is part of Proposition 1.1 (and obvious). For the remaining things to be checked, let f correspond to F, g to G etc., under the correspondence (1.1). Identifying $\mathcal{M}_1^{\sim}(x_0)$ with $\mathcal{M}_1(x_0)$, (1.1) simplifies (for f and F with source x_0) to

$$f(x_0, x_1, \ldots, x_{r+1}) = F(x_1)(x_1, \ldots, x_{r+1})\,. \tag{3.2}$$

To see that this correspondence preserves target, let $F \in (\tilde{C}^{(r)})^{(1)}$. The target $\beta^r(F(x_1))$ of the arrow $F(x_1) \in \tilde{C}^{(r)}$ (whose source is x_1) is

$$\beta(F(x_1)(x_1, \ldots, x_1)) = \beta(f(x_0, x_1, \ldots, x_1))\,,$$

so in particular, the target $\beta^1(F)$ of $F \in (\tilde{C}^{(r)})^{(1)}$ (whose source is x_0) is

$$\beta^r(F(x_0)) = \beta(F(x_0)(x_0, \ldots, x_0)) = \beta(f(x_0, \ldots, x_0))$$

which is $\beta^{r+1}(f)$.

To prove that (3.2) preserves composition, let f and g be composable in $\tilde{C}^{(r+1)}$. Let $\boldsymbol{x} = (x_0, \ldots, x_{r+1})$ and let

$$(y_0, \ldots, y_{r+1}) = \boldsymbol{y} = (\beta \circ f)^{\sim}(\boldsymbol{x})\,.$$

So $\beta^{r+1}f = y_0$. Let $h = g \circ f$ in $\tilde{C}^{(r+1)}$ correspond to H in $(\tilde{C}^{(r)})^{(1)}$. So

$$(3.3) \qquad H(x_1)(x_1, \ldots, x_{r+1}) = (g \circ f)(x_0, x_1, \ldots, x_{r+1}) = g(y_0, \ldots, y_{r+1}) \circ f(x_0, \ldots, x_{r+1}) .$$

On the other hand (letting $\mathbf{x}' = (x_1, \ldots, x_{r+1})$, and similarly for $\mathbf{y}'$)

$$(3.4) \qquad (G \circ F)(x_1)(\mathbf{x}') = (G(\beta^r(F(x_1))) \circ F(x_1))(\mathbf{x}') = (G(y_1) \circ F(x_1))(\mathbf{x}') = G(y_1)((\beta \circ F(x_1))^{\sim}(\mathbf{x}')) \circ F(x_1)(\mathbf{x}') .$$

But

$$(\beta \circ F(x_1))^{\sim}(\mathbf{x}') = (\beta(F(x_1))(x_1, \ldots, x_1), \ldots, \beta(F(x_1))(x_1, \ldots, x_{r+1})) = (\beta(f(x_0, x_1, \ldots, x_1), \ldots, \beta(f(x_0, x_1, \ldots, x_{r+1})) = (y_1, \ldots, y_{r+1}) = \mathbf{y}' ,$$

so the equation (3.4) continues

$$= G(y_1)(\mathbf{y}') \circ F(x_1)(\mathbf{x}') = g(y_0, \ldots, y_{r+1}) \circ f(x_0, \ldots, x_{r+1}) ,$$

which, when compared with (3.3), gives the result about composition.

Finally, the unit $u^{(1)}(x)$ at $x = x_0$ in $(\tilde{C}^{(r)})^{(1)}$ is given by

$$u^{(1)}(x_0)(x_1)(x_1, \ldots, x_{r+1}) = u^{(r)}(x_1)(x_1, \ldots, x_{r+1}) = u(x_{r+1}) = u^{(r+1)}(x_0)(x_0, \ldots, x_{r+1})$$

so that $u^{(1)}(x_0)$ and $u^{(r+1)}(x_0)$ correspond under (3.2). This proves the first claim of the theorem.

The second claim is now an obvious corollary of the first: use induction in r and the fact that if D is a category, then so is its first prolongation $D^{(1)}$.

Consider the oriented multiplicative graph $\tilde{\Pi}^{(r)}M$ of (non-holonomous) r-jets with source and target in M, and let $\Pi_0 M$ denote the "codiscrete" category on M (i.e. the category with arrow set $M \times M$, and the two projections as source and target map). Then one immediately sees that

$$(\Pi_0 M)^{\sim(r)} = \tilde{\Pi}^{(r)}M ,$$

so the theorem has the following

Corollary 3.2. *Non holonomous r-jets with source and target in M form a category $\tilde{\Pi}^{(r)}M$ under the composition given in § 2.*

Let C be a category, as above. We define $\bar{C}^{(r)} \subseteq \tilde{C}^{(r)}$ to be the set of those r-jet sections of $\alpha: C \to M$ which are semi-holonomous.

Proposition 3.3. *The subset $\bar{C}^{(r)} \subseteq \tilde{C}^{(r)}$ is a subcategory.*

Proof. This again uses Theorem 3.1, and induction on r, but now in conjunction with Proposition 1.2. Let us by induction assume $\bar{C}^{(r)}$ to be a subcategory of $\tilde{C}^{(r)}$.

Let f and g be composable in $\tilde{C}^{(r+1)}$ and both $\in \bar{C}^{(r+1)}$. Then $g \circ f$ corresponds to H as given by (3.3), and $H \in (\bar{C}^{(r)})^{(1)}$ since $\bar{C}^{(r)}$ is stable under composition. So we just have to prove (1.3) for H. Now with notation as in the proof of the theorem

$$\begin{aligned} H(x_1)(x_1, x_1, x_2, \ldots, x_r) &= (g \circ f)(x_0, x_1, x_1, x_2, \ldots, x_r) \\ &= g(y_0, y_1, y_1, y_2, \ldots, y_r) \circ f(x_0, x_1, x_1, x_2, \ldots, x_r) \\ &= g(y_0, y_0, y_1, \ldots, y_r) \circ f(x_0, x_0, x_1, \ldots, x_r) \\ &= H(x_0)(x_0, x_1, \ldots, x_r)\,. \end{aligned}$$

This proves inductively that $\bar{C}^{(r+1)}$ is stable under composition. Also $u_x^{(r+1)}$ is semi-holonomous, in fact holonomous. So $\bar{C}^{(r+1)}$ is a subcategory.

Proposition 3.4. *The subset $C^{(r)} \subseteq \bar{C}^{(r)}$ is a subcategory.*

Proof. Using the characterization of holonomous jets as those whose value on a chain only depend on the extremity, this becomes easy. We omit details.

4. COMPOSITE CONNECTIONS

The maps $c, e\colon \tilde{M}^{(r)} \to M$, together with $\Delta\colon M \to \tilde{M}^{(r)}$, equip $\tilde{M}^{(r)}$ as an oriented graph with M as its object- (or vertex-) set. (It even has an involution

$$(x_0, \ldots, x_r) \mapsto (x_r, \ldots, x_0)\,.)$$

Likewise, a category C with M as object set may be viewed as an oriented graph in the same sense. (If C is a groupoid it even has an involution $f \mapsto f^{-1}$.)

An r'th *order (non-holonomous) connection R on C* is now defined to be a morphism

$$R\colon \tilde{M}^{(r)} \to C$$

of oriented graphs over M. (If C is a groupoid, R may or may not preserve involution; if it does, one probably should express this by saying: R is hysteresis-free; first order connections on nice objects in models of synthetic differential geometry automatically have this property.)

We say R is semi-holonomous (respectively holonomous) if $R \circ \sigma_i = R \circ \sigma_j \ \forall i, j$ (respectively depends only on endpoints of chains).

Following [9] and [8], we define the composite $R * S$ of two connections R and S on C, of order r and s, respectively, to be the $r+s$ order connection given by

$$(R * S)(x_0, \ldots, x_{r+s}) = S(x_r, \ldots, x_{r+s}) \circ R(x_0, \ldots, x_r)\,.$$

Likewise, following [9], we construct out of R an r-tuple of 1st order connections $\zeta_i(R)$ $(i = 0, \ldots, r-1)$, by taking $\zeta_i(R)$ to be composite

$$M_{(1)} \xrightarrow{s_i} \tilde{M}_{(r)} \xrightarrow{R} C\,.$$

The (paraphrasing of) some of the results of [8] and [9] now get very easy proofs; we give some of these.

Theorem 4.1. *If R is semi-holonomous, then all $\zeta_i(R)$ are equal.*

Proof. We must prove $R \circ s_i = R \circ s_{i+1}$ $(i = 0, \dots, r-2)$. But

$$s_i = \sigma_{i+1} \circ s_i$$

$$s_{i+1} = \sigma_i \circ s_i ,$$

and since $R \circ \sigma_i = R \circ \sigma_{i+1}$ by assumption, the result follows.

Theorem 4.2. *If $R = S_0 * \dots * S_{r-1}$ (with the S_i's 1st order connections), then $\zeta_i(R) = S_i$ $(i = 0, \dots, r-1)$.*

Proof. We have, for any $x \sim y$

$$\begin{aligned} \zeta_i(R) &= R(x, \dots, x, y, \dots, y) \qquad (x \text{ written } i+1 \text{ times}) \\ &= S_{r-1}(y, y) \circ \dots \circ S_{i+1}(y, y) \circ S_i(x, y) \circ S_{i-1}(x, x) \circ \dots \circ S_0(x, x) \\ &= S_i(x, y) \end{aligned}$$

since $S_j(z, z) = u_z \ \forall z, j$.

Theorem 4.3. *If $R = S_0 * \dots * S_{r-1}$ as above, then R is semi-holonomous iff $S_i = S_j \ \forall i, j$.*

Proof. The implication "$\Rightarrow$" follows combining Theorems 4.1 and 4.2. On the other hand, if $S_i = S_j \ \forall i, j$ then

$$R(\sigma_i(x_0, \dots, x_{r-1})) = S(x_{r-2}, x_{r-1}) \circ \dots \circ S(x_i, x_i) \circ \dots \circ S(x_0, x_1) .$$

Here $S(x_i, x_i) = u(x_i)$, so may be omitted, so we get $S(x_{r-2}, x_{r-1}) \circ \dots \circ S(x_0, x_1)$ which is independent of i.

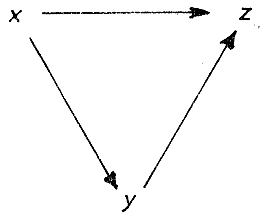
A 1st order connection with values in a groupoid C over M is *curvature free* [7] if $x \sim y$, $y \sim z$ and $z \sim x$ implies

$$\nabla(z, x) \circ \nabla(y, z) \circ \nabla(x, y) = u_x ,$$

or equivalently (assuming $\nabla(z, x) = \nabla(x, z)^{-1}$)

$$\nabla(x, z) = \nabla(y, z) \circ \nabla(x, y) ,$$

which may be expressed as: "taking path lift of ∇ along the two possible paths from x to z in



yields the same result". Now Theorem 7 in [9] says that when $R = \nabla * \ldots * \nabla$ (∇ a 1st order connection), and ∇ is curvature free then R is holonomous. In our context, the conclusion "R holonomous" can be expressed: "the path lift of ∇ along any two paths (r-chains) from x_0 to x_r yields the same result". We can actually prove this form of Virsik's Theorem, using the technique of [6], notably Proposition 4.1 (ii), but this involves considerations of not purely combinatorial nature and so will not be included here.

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Author's address: Matematisk Institut, Aarhus University Ny Munkegade — Bygning 530, 8000 Aarhus C, Danmark.