CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 23, nº 4 (1982), p. 347-362.

<http://www.numdam.org/item?id=CTGDC_1982__23_4_347_0>

CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE *Vol. XXIII-4 (1982)*

THE ALGEBRAIC THEORY OF MOVING FRAMES *

by Anders KOCK

In this note we aim at describing the algebraic structure which the set of frames, i.e. the set of positively oriented orthonormal coordinate-systems in physical space, has (under the assumptions: an orientation has been chosen; a unit of measure has been chosen, say «meter»; and the idealizing assumption that space is Euclidean). Of course, this set has a well known structure: it can be identified with the group $SO(3) \times_a \mathrm{R}^3$ (semidirect product). Our point is to analyze *how* canonical this identification is, or alternatively, to free our algebraic thinking of space from resorting to «an arbitrary but fixed reference frame».

So the aim is to axiomatize an objectively given reality as it is, not as it *becomes* after an arbitrary choice. (Of course, the choice of the orientation and the «meter» as unit is an arbitrary choice, which means that we have not gone all the way in the program.)

The motivation was to get a more natural understanding of the Maurer-Cartan form on the set of frames. This we present in detail elsewhere in the context of synthetic differential geometry.

What canonical structure, then, does the set of frames have? Our answer is that it is a «pregroup»: a set, equipped with a trinary operation $\lambda(P,Q,R)$ which behaves as if it were the formation of $Q.P^{-1}.R$ in a group. This trinary operation has an objective significance: $\lambda(P,Q,R)$ is that unique frame which R is carried to by that rigid motion of space which takes the frame P to the frame Q. It thus can be realized physically by connecting R to P by a rigid system of rods, and then moving P to Q. Then R becomes $\lambda(P,Q,R)$.

* This research was partially supported by the Australian Research Grants Committee.

Another natural model for the axiomatics is elliptic 3-space, and this model provides a way of thinking geometrically on pregroups, namely by thinking of P, Q, R and $\lambda(P, Q, R) = S$ as the four corners of a (Clifford) parallelogram: PQ «left» parallel to RS, PR «right» parallel to QS.

Our notion of pregroup may be said to be related to the notion of group in the same way as the notion of affine space is related to the notion of vector space. Note that physical space is canonically an affine space, but only after an arbitrary choice of a base point, it becomes a vector space. A pure algebraic axiomatics of affine space has been given in [11], page 425.

Professor A. Ehresmann has kindly pointed out to me that already in 1951, the Soviet mathematician V.V. Vagner [12] considered the notion of «coordinate structure» of which ours is a special case. He found the same ternary operation (on the «set of coordinate systems») as we consider here, except that he explains this operation in terms of coordinate changes, rather than, as we do, in terms of motions.

Furthermore, Vagner identifies the equational theory of this ternary operation as the theory of the notion of «Schar» (H. Prüfer, and R. Baer [1]) or «abstract coset» (J. Certaine [4]). Of course, then, Schar = asstract coset = pregroup, but our equational presentation of this notion is different from that of Prüfer, Baer and Certaine, who also use axioms with five variables, whereas our axioms have only four variables. Their axiomatization is the most natural when one wants to axiomatize the ternary operation $ab^{-1}c$ in the theory of groups, the typical axiom then being

$$(ab^{-1}c)\,d^{-1}e = ab^{-1}(cd^{-1}e)\ ;$$

- whereas our axiomatization grows out of the geometric interpretation. (J. Certaine, in [4], also gives a notion of «system of free vectors» which, in essence, is the same as our description of pregroups in terms of double categories, Section 6.)

1. PREGROUPS: AXIOMS.

A *pregroup* is a set A equipped with a trinary operation λ satisfying the Axioms 1-3 below. On basis of the axioms, it can be proved (see Section 3) that any pregroup can be embedded into some group G in such a way that

$$\lambda(P,Q,R) = Q.P^{-1}.R,$$

which, once this fact is proved, makes it unnecessary to remember the axioms. Anyway, the axioms for pregroups, even though they don't look very nice algebraically, can be *drawn* nicely, by the method of parallelograms, as mentioned in the introduction. We do this for the case of Axiom 3. Axiom 2 is redundant, and only included for future reference.

Finally, observe that the axioms are self-dual in the following sense: if A, λ is a pregroup, we can give A another pregroup structure ρ, by putting

$$\rho(P,Q,R) := \lambda(P,R,Q).$$

The structure ρ is called the *dual* or *opposite* of λ. The equational consequences of the axioms therefore also come in pairs.

Axiom 1. $\lambda(P,Q,P) = Q, \quad \lambda(P,P,Q) = Q.$

Axiom 2. $\lambda(R,\lambda(P,Q,R),P) = Q, \quad \lambda(Q,P,\lambda(P,Q,R)) = R.$

Axiom 3. $\lambda(R,\lambda(P,Q,R),T) = \lambda(P,Q,T),$

$\lambda(Q,T,\lambda(P,Q,R)) = \lambda(P,T,R).$

We define two relations on the set $A \times A$, called the *geometric* and the *analytic* relation, respectively; namely write

(1.1) $$(P,Q) \sim_g (R,S) \quad \text{if} \quad \lambda(P,Q,R) = S$$

and

(1.2) $$(P,R) \sim_a (Q,S) \quad \text{if} \quad \lambda(P,Q,R) = S.$$

PROPOSITION 1. *The two relations* $\sim_g$ *and* $\sim_a$ *are equivalence relations.*

PROOF. By the self duality of the theory, it suffices to prove this for the case of $\sim_g$. Now

$$(P,Q) \sim_g (P,Q) \iff \lambda(P,Q,P) = Q,$$

which is Axiom 1, proving reflexivity. Next assume $(P,Q) \sim_g (R,S)$, i.e. $\lambda(P,Q,R) = S$ and prove $(R,S) \sim_g (P,Q)$, i.e. $\lambda(R,S,P) = Q$. But this holds by substituting $\lambda(P,Q,R)$ for S and using Axiom 2. Finally assume

$$(P,Q) \sim_g (R,S) \quad \text{and} \quad (R,S) \sim_g (T,U),$$
$$\text{i.e., } \lambda(P,Q,R) = S \quad \text{and} \quad \lambda(R,S,T) = U.$$

Substitute the former in the latter to obtain $\lambda(R, \lambda(P,Q,R), T) = U$. By Axiom 3, the left hand side here is $\lambda(P,Q,T)$. So $\lambda(P,Q,T) = U$, which means $(P,Q) \sim_g (T,U)$, as desired. Note that for the three proofs, we used the first halves of the axioms. The proof that $\sim_a$ is an equivalence uses the three second halves of the axioms.

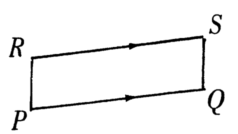
We write $\vec{PQ}$ for the $\sim_g$ equivalence class of (P,Q), and $\underset{\to}{PQ}$ for the $\sim_a$ equivalence class.

(The intention is: $\vec{PQ}$ is the *motion* which carries P into Q; $\underset{\to}{PQ}$ is the *substitution* which transforms coordinates with respect to P into coordinates with respect to Q; hence the names «geometric» and «analytic» which we borrowed from E. Cartan [3], Section 62. $\underset{\to}{PQ}$ can also be thought of as: the coordinate expression of P in the coordinate system Q.)

Given four points P, Q, R, S, then the following conditions are, by the very definition, equivalent:

$$(1.3) \qquad S = \lambda(P,Q;R), \quad \vec{PQ} = \vec{RS}, \quad \underset{\to}{PR} = \underset{\to}{QS}.$$

We indicate this state of affairs by saying that (P,Q,R,S) form a *parallelogram*, and indicate it by a diagram

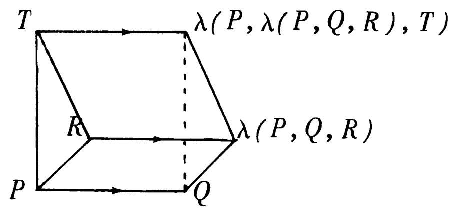


The arrowheads indicate those two pairs (P,Q) and (R,S) which are *geometrically* equivalent. They may be reversed:

$$(1.4) \; (\vec{PQ} = \vec{RS}) \Rightarrow (\underset{\to}{PR} = \underset{\to}{QS}) \Rightarrow (\underset{\to}{QS} = \underset{\to}{PR}) \Rightarrow (\vec{QP} = \vec{SR})$$

but they may not in general be transferred to the other two edges of the parallelogram since, unlike parallelograms in affine geometry, we do not have $\vec{PQ} = \vec{RS} \not\Rightarrow \vec{PR} = \vec{QS}$.

The first statement in Axiom 3 can be stated diagrammatically by saying that the third face of the prism



is also a parallelogram. There is a similar picture for the second half of Axiom 3, a «horizontal» concatenation of the two parallelograms, which the reader may draw now, or when proving Proposition 2 below.

If G is a group, we can equip the underlying set of G as a pregroup G^b, by putting $\lambda(p, q, r) = q.p^{-1}.r$. It is trivial to verify the axioms.

As a second example of a pregroup, let A be the set of points in elliptic 3-space (see e.g. [2] Section 3-5, or [9] VIII.5). Let $\lambda(P, Q, R)$ be the point where the line through R, left-(Clifford-)parallel to the line $[P, Q]$ through P and Q, intersects the line through Q, right-parallel to $[P, R]$ (assuming P, Q, R distinct; for non-distinct points, the value $\lambda(P, Q, R)$ is forced by Axiom 1).

The third example, which is the main one has been mentioned in the introduction: orthonormal frames in physical space.

2. THE TWO GROUPS ASSOCIATED TO A PREGROUP.

We let $A^\#$ and $A_\#$, respectivley, denote the set of equivalence classes for $\sim_g$, respectively $\sim_a$. Note that, given $a \in A^\#$ and $R \in A$, there is a unique $S \in A$ so that $a = \vec{RS}$; for, let $a = \vec{PQ}$; and take S to be $\lambda(P, Q, R)$. A similar thing holds for $A_\#$.

Let a and β be elements in $A^\#$. Represent a as $\vec{PQ}$ and then β

as $\vec{QT}$, and define

$$\beta \circ \alpha = \vec{QT} \circ \vec{PQ} := \vec{PT}.$$

PROPOSITION 2. *The structure* $\circ$ *on* $A^{\#}$ *is well defined and makes* $A^{\#}$ *into a group.*

PROOF. To see that this is well defined, assume

$$\vec{PQ} = \vec{P'Q'} \text{ and } \vec{QT} = \vec{Q'T'}.$$

Then we have two parallellograms in a «horizontal» prism, hence also the third: $\vec{PT} = \vec{P'T'}$. - Associativity of the operation is obvious. $\vec{PP}$ is a neutral element, and $\vec{QP}$ is inverse of $\vec{PQ}$.

The group structure on $A_{\#}$ is defined dually: $\underset{\to}{PQ} . \underset{\to}{QR} := \underset{\to}{PR}$.

The groups $A^{\#}$ and $A_{\#}$ act on A from left and from right respectively:

Let $\alpha \in A^{\#}$ and $P \in A$. Find that Q so that $\alpha = \vec{PQ}$ and put $\alpha . P = Q$; so

(3.1) $$\vec{PQ} . P = Q.$$

Also let $\beta \in A_{\#}$ and $P \in A$. Find that R so that $\beta = \underset{\to}{PR}$ and put $P . \beta = R$; so

(3.2) $$P . \underset{\to}{PR} = R.$$

It is evident that these actions are unitary and associative; for associativity of the left $A^{\#}$-action, for instance, we have

$$(\vec{QR} \circ \vec{PQ}) . P = \vec{PR} . P = R = \vec{QR} . Q = \vec{QR} . (\vec{PQ} . P).$$

Also, the left and the right actions commute: if $\lambda(P, Q, R) = S$, we have $\vec{PQ} = \vec{RS}$ and $\underset{\to}{PR} = \underset{\to}{QS}$, so that

$$\vec{PQ} . (P . \underset{\to}{PR}) = \vec{PQ} . R = \vec{RS} . R = S$$

and

$$(\vec{PQ} . P) . \underset{\to}{PR} = Q . \underset{\to}{PR} = Q . \underset{\to}{QS} = S.$$

We claim that both the left and the right actions are pregroup automorphisms. To see this for the left action, say, assume $\alpha = \vec{PP'} \in A^{\#}$ and that $\lambda(P, Q, R) = S$. We must prove $\lambda(\alpha P, \alpha Q, \alpha R) = \alpha S$. We have:

$aP = P'$, and also

$$\overrightarrow{PQ} = \overrightarrow{RS}, \quad \overrightarrow{PaP} = \overrightarrow{QaQ} = \overrightarrow{RaR} = \overrightarrow{SaS}.$$

From $\overrightarrow{PaP} = \overrightarrow{RaR}$, we conclude by «reversing arrows, (1.4)» that $\overrightarrow{aPP} = \overrightarrow{aRR}$, so that

$$\overrightarrow{aPaQ} = \overrightarrow{QaQ} \circ \overrightarrow{PQ} \circ \overrightarrow{aPP} = \overrightarrow{SaS} \circ \overrightarrow{RS} \circ \overrightarrow{aRR} = \overrightarrow{aRaS},$$

which is the desired conclusion.

Finally the action is «principal homogeneous»: to a pair P, Q, there exists exactly one $a \in A^{\#}$ (namely $\overrightarrow{PQ}$) that takes P to Q.

The proofs of the similar facts for the right action of $A_{\#}$ are dual. So we have

PROPOSITION 3. *The formulae (3.1) and (3.2) define commuting left and right- actions of the groups $A^{\#}$ and $A_{\#}$, respectively, on A, by pregroup automorphisms, and the actions make A a left and right principal homogeneous object (or a « bi-torsor»).*

If G is a group, there are canonical group isomorphisms

$$G^{b\#} \to G \quad \text{and} \quad G^{b}_{\#} \to G \quad \text{given by} \quad \overrightarrow{PQ} \mapsto QP^{-1} \quad \text{and} \quad \underrightarrow{PR} \mapsto P^{-1}R.$$

They are well defined. For if $\overrightarrow{PQ} = \overrightarrow{RS}$, we have $\lambda(P, Q, R) = S$, i.e. $S = QP^{-1}R$ so that $SR^{-1} = QP^{-1}$. Similarly for the other isomorphism. It is trivial to verify that they are group isomorphisms. Under the identification $G^{b\#} \to G$, the left action of $G^{b\#}$ on $G^{b} = G$ is just multiplication by G on G from the left. Dually for the right action of $G^{b}_{\#}$ on $G^{b} = G$.

3. NON-CANONICAL ISOMORPHISMS.

If we *choose* an element $O \in A$ (in the frame situation, this would be called «a frame of reference»), then we get bijections

$$\phi_O : A \to A^{\#} \quad \text{and} \quad \psi_O : A \to A_{\#}$$

given by $\phi_O(P) = \overrightarrow{OP}$ and $\psi_O(P) = \underrightarrow{OP}$, respectively.

PROPOSITION 4. *ϕ_O and ψ_O are isomorphisms of pregroups:*

$$A \xrightarrow{\approx} A^{\#b} \quad and \quad A \xrightarrow{\approx} A^b_{\#}.$$

PROOF. Let $\lambda(P,Q,R) = S$. To prove the statement for ϕ_O means to prove that this implies

$$\vec{OQ} \circ (\vec{OP})^{-1} \circ \vec{OR} = \vec{OS}.$$

But $(\vec{OP})^{-1} = \vec{PO}$, and $\vec{PQ} = \vec{RS}$. The result is now immediate from the definition of the group structure on $A^{\#}$:

$$\vec{OQ} \circ \vec{PO} \circ \vec{OR} = \vec{PQ} \circ \vec{OR} = \vec{RS} \circ \vec{OR} = \vec{OS}.$$

The proof for ψ_O is similar. We note that the two group structures imposed on A by ϕ_O and ψ_O are identical.

In view of the interpretation

$$\underset{\to}{PQ} = \text{« } P \text{ expressed in the coordinate system } Q \text{ »}$$

it is more natural to consider

$$\bar{\psi}_O : A \to A_{\#} \quad \text{given by} \quad \bar{\psi}_O(P) = \underset{\to}{PO}.$$

It gives an isomorphism from A to the opposite pregroup of $A^b_{\#}$.

COROLLARY. *Every pregroup A admits an injective pregroup homomorphism $A \to G^b$ into some group G.*

Combining the two non-canonical isomorphisms of Proposition 4 we obtain a non-canonical isomorphism

$$\epsilon_O = \psi_O \circ \phi_O^{-1} : A^{\#} \to A_{\#} : \vec{OP} \mapsto \underset{\to}{OP}.$$

We analyze how non-canonical it is:

PROPOSITION 5. *Let O, O' be points in A. Then $a_{OO'} \circ \epsilon_{O'} = \epsilon_O$, where $a_{OO'}$ is the inner automorphism of $A_{\#}$ given by the element $\underset{\to}{OO'}$.*

PROOF. Let $a \in A^{\#}$, $a = \vec{OP} = \vec{O'P'}$, say. Then $\underset{\to}{OO'} = \underset{\to}{PP'}$. So

$$\epsilon_O(a) = \psi_O \phi_O^{-1}(\vec{OP}) = \psi_O(P) = \underset{\to}{OP} = \underset{\to}{OO'} . \underset{\to}{O'P'} . \underset{\to}{P'P} =$$
$$= \underset{\to}{OO'} . \underset{\to}{O'P'} . (\underset{\to}{OO'})^{-1} = a_{OO'}(\epsilon_{O'}(a)).$$

It is possible to reformulate and strengthen the proposition into a statement about the map $A \to Iso(A^{\#}, A_{\#})$ being compatible with actions of $A^{\#}$ and $A_{\#}$ ($A_{\#}$, say, acts on the left on $Iso(A^{\#}, A_{\#})$ via

$$A_{\#} \xrightarrow{in} Iso(A_{\#}, A_{\#}),$$

« formation of inner automorphism », followed by composition).

4. COORDINATES.

In the pregroup A of orthonormal frames in space, $A^{\#}$ has a *canonical* geometric interpretation as the group of Euclidean motions, whereas $A_{\#}$ *canonically* becomes identified with a definite algebraically described group, namely the semidirect product $SO(3) \times_{a} \mathrm{R}^3$. To see this latter fact, let $\underset{\to}{PQ} \in A_{\#}$. Let $(\underline{\underline{B}}, \underline{v}) \in SO(3) \times \mathrm{R}^3$ be given as follows: $\underline{v}$ is the coordinates of the base point of P, relative to the coordinate system Q; next make a parallel translation of P so that its base point agrees with that of Q and let the 3×3 matrix $\underline{\underline{B}}$ have for its rows the coordinates (with respect to Q) of the three rods («vectors») that constitute P. In short, $(\underline{\underline{B}}, \underline{v})$ is the coordinate expression of P relative to Q.

To $\underset{\to}{PQ}$, associate $(\underline{\underline{B}}, \underline{v}) \in SO(3) \times \mathrm{R}^3$. It is well defined, for $\underset{\to}{PQ} = \underset{\to}{RS}$ iff $\vec{PR} = \vec{QS}$. The latter means that the rigid motion that carries Q to S takes P to R, so that the coordinates of P relative to Q equal the coordinates of R relative to S.

PROPOSITION 6. *The canonical map $A_{\#} \to SO(3) \times \mathrm{R}^3$ becomes an isomorphism of groups if we define the semidirect-product group structure on the latter set by*

$$(4.1) \qquad (\underline{\underline{B}}, \underline{v}) . (\underline{\underline{C}}, \underline{w}) := (\underline{\underline{B}} . \underline{\underline{C}},\ \underline{v} . \underline{\underline{C}} + \underline{w}).$$

PROOF. Let the frame P consist of the four *points* P_0, P_1, P_2, P_3 and similarly for Q and R. Let us calculate the coordinate expression of P_0 in the coordinate system R. We apply usual vector calculus in physical space F: so $\vec{P_0 P_1}$ is a vector in the vector space $F^{\#}$ canonically associated to F, since F is an affine space (see e.g. [11]). We have

$$\vec{R_0 P_0} = \vec{R_0 Q_0} + \vec{Q_0 P_0} = \vec{R_0 Q_0} + \sum_i v_i . \vec{Q_0 Q_i} =$$
$$= \vec{R_0 Q_0} + \sum_{ij} v_i . C_{ij} . \vec{R_0 R_j} .$$

This says precisely that the coordinates of P_0 in terms of the frame R is $\underline{w} + \underline{v}.\underline{\underline{C}}$. - The calculation of $\underline{\underline{B}}.\underline{\underline{C}}$ is similar.

The index a in $SO(3)\times_a R^3$ refers to the group structure defined by (4.1).

The canonical right action of $A_\#$ on A can, under the canonical identification $A_\# = SO(3)\times_a R^3$ be described as:

given $P \in A$ and $(\underline{\underline{B}}, \underline{v}) \in SO(3)\times_a R^3$, find that unique frame Q such that P has coordinates $(\underline{\underline{B}}, \underline{v})$ with respect to it.

The associative law of the action

$$(P.(\underline{\underline{B}},\underline{v})).(\underline{\underline{C}},\underline{w}) = P.((\underline{\underline{B}},\underline{v}).(\underline{\underline{C}},\underline{w}))$$

can be expressed by saying: if Q has coordinates $(\underline{\underline{C}}, \underline{w})$ with respect to a frame R, then right multiplication (in $SO(3)\times_a R^3$) by $(\underline{\underline{C}}, \underline{w})$ converts coordinate expressions (for frames) in terms of Q into coordinate expressions in terms of R. Also, if a point P_0 in space has coordinates $\underline{v}$ with respect to Q, then it has coordinates $\underline{v}.\underline{\underline{C}} + \underline{w}$ with respect to R; thus $(\underline{\underline{C}}, \underline{w})$ is a «coordinate transformation matrix».

The Maurer-Cartan form on the set of frames A associates to an infinitesimal motion $P(t)$ of a frame P (i.e. a tangent vector to A at P) the derivative at $t = 0$ of $P(t) \rightarrow P \in A_\# = SO(3)\times_a R^3$, which is an element in the Lie algebra of the latter, and equals the derivative at 0 of the coordinate expression of $P(t)$ in terms of P.

More generally, if A is a pregroup with a differentiable structure compatible with the algebraic, we can canonically describe a 1-form on A with values in the Lie algebra $L(A_\#)$ ($A_\#$ being in this case a Lie group) namely as the derivative of $t \mapsto P(t) \rightarrow P \in A_\#$. The Lie algebra $L(A_\#)$ can also be described as the set of vector fields on A which are *left* invariant (= invariant under the left action of $A^\#$). In geometric terms these vector fields are those whose field vectors form parallelograms under motions.

5. ALGEBRA OF FRAME BUNDLES : PREGROUPOIDS.

The notion of pregroup considered above can be weakened to a notion of pregroupoid, which is an algebraic structure which (for instance) the set A of orthonormal frames of an arbitrary Riemannian manifold M has.

Explicitly, a *pregroupoid* over a set M is a set A equipped with a surjective map $\pi : A \to M$ and with a partially defined trinary operation λ, satisfying the six equations of Axioms 1-3 in Section 1 and Condition (5.2) below. The condition for when $\lambda(P, Q, R)$ is defined is

$$\pi(P) = \pi(R) ; \tag{5.1}$$

and we assume that when this is the case

$$\pi(\lambda(P, Q, R)) = \pi(Q). \tag{5.2}$$

Note that the axioms are no longer self-dual.

For the case of orthonormal frames on an n-dimensional Riemannian manifold, $\lambda(P, Q, R) = S$ is taken to mean:

1) P and R are frames with same base point, whence it makes sense to talk about the coordinates of P in terms of R (this being an orthogonal $n \times n$ matrix) and

2) this matrix equals the coordinate matrix of Q in terms of S (the two frames Q, S being again frames at the same point).

We define two relations $\sim_g$ and $\sim_a$ as in Section 1; $\sim_g$ is an equivalence relation on $A \times A$ defined by

$$(P, Q) \sim_g (R, S) \quad \text{if} \quad \lambda(P, Q, R) = S.$$

So if $\vec{PQ} = \vec{RS}$ (in the notation of Section 1), then

$$\pi(P) = \pi(R) \wedge \pi(Q) = \pi(S).$$

Also, $\sim_a$ is an equivalence relation on the set

$$A \times_M A = \{ (P, R) \mid \pi(P) = \pi(R) \})$$

with

$$(P, R) \sim_a (Q, S) \quad \text{if} \quad \lambda(P, Q, R) = S.$$

The proof that these two relations are actually equivalence relations is as in Section 1. The $\sim_a$-equivalence class of (P, R) is denoted $\underset{\rightarrow}{PR}$. The sets of $\sim_g$- (respectively $\sim_a$-) equivalence classes are denoted $A^{\#}$ and

$A_\#$, respectively.

Generalizing the group structures on $A^\#$ and $A_\#$ and their actions on A, we have instead:

PROPOSITION 7. *The set* $A_\#$ *carries a natural group structure and acts on the right on* A, *making* A *into a right* $A_\#$*-torsor (= principal fibre bundle) over* M. *The set* $A^\#$ *carries the structure of (the set of arrows of) a groupoid with* M *as its set of objects, and* $A \to M$ *has the structure of a discrete op-fibration (or internal diagram, e.g.* [7]*) over* $A^\#$. *The two actions of* $A^\#$ *and* $A_\#$ *on* A *commute with each other.*

PROOF/CONSTRUCTION. The construction of the group structure on $A_\#$, and of its action on A, is as in Section 2, since we, as there, for any $\beta \in A_\#$ and $P \in A$, can find a unique R with $\underset{\to}{PR} = \beta$. This R is in the same π-fibre as P, and from this follows that the action is fibrewise over M. From the uniqueness of such R, given β, also follows that $A \to M$ is in fact a torsor over $A_\#$.

To make $A^\#$ into a groupoid, we construct two maps $\partial_0, \partial_1 : A^\# \to M$ by

$$\partial_0(\vec{PQ}) = \pi(P), \quad \partial_1(\vec{PQ}) = \pi(Q).$$

This is well defined, since if $\vec{PQ} = \vec{RS}$, then $\lambda(P,Q,R) = S$, so that $\pi(P) = \pi(R)$ by (5.1) and $\pi(Q) = \pi(S)$ by (5.2).

If $\partial_1(\vec{PQ}) = \partial_0(\vec{Q'T'})$ we have $\pi(Q) = \pi(Q')$ so that we can form $\lambda(Q', T', Q) = T$, and we then put

$$\vec{Q'T'} \circ \vec{PQ} = \vec{PT}.$$

This defines the composition of the groupoid. The unit over $m \in M$ is $\vec{PP}$ for any P with $\pi(P) = m$.

If $\alpha \in A^\#$ has $\partial_0(\alpha) = m \in M$, and P is in the fibre over m, we can represent α in the form $\vec{PQ}$ and define $\alpha . P = Q$. Now $\pi(Q) = \partial_1(\alpha)$. So each arrow α of $A^\#$ defines a map $\tilde{\alpha}$ from the $\partial_0(\alpha)$-fibre of A to the $\partial_1(\alpha)$-fibre of A. This is the structure of «discrete op-fibration», or «left action» of the groupoid $A^\#$ on $A \to M$. The proof that $\tilde{\alpha}$ commutes with the right fibrewise action of $A_\#$ is as in Section 2. This proves the proposition.

We finally give a partial analysis of the remaining relations between the notions: pregroupoid, torsor, groupoid.

Given a right G-torsor $A \to M$ over M, we can make A into a pregroupoid by putting, for $\pi(P) = \pi(R)$:

$$\lambda(P, Q, R) := Q.\, g,$$

where $g \in G$ is the unique group element with $P.\, g = R$.

Then $A^{\#}$ is canonically isomorphic to the groupoid $A A^{-1}$ constructed in [5] page 34 or [10] page 25, and $A_{\#}$ is canonically isomorphic to the group G.

Given a groupoid Γ (where we compose from right to left) with M as its set of objects, we get, by choosing an $m_0 \in M$ (thus non-canonically), a pregroupoid $A = \Gamma^b(m_0)$ over M by letting the fibre over $m \in M$ be $hom_{\Gamma}(m_0, m)$. Then we put

$$\lambda(P, Q, R) = Q \circ P^{-1} \circ R,$$

which makes sense if $\pi(R) = \pi(P)$, i.e. if $\partial_1(P) = \partial_1(R)$. Then $A^{\#}$ is canonically isomorphic to Γ, by the well defined map $\overrightarrow{PQ} \mapsto Q \circ P^{-1}$, and $A_{\#}$ is canonically anti-isomorphic to $hom_{\Gamma}(m_0, m_0)$ via $\underrightarrow{PR} \mapsto R^{-1} \circ P$.

6. PREGROUPS AND PREGROUPOIDS AS DOUBLE CATEGORIES.

We remind the reader of the notion of (small) *category*, and the special cases *groupoid* (all arrows are invertible), and *preorder* (all diagrams commute). An *equivalence relation* is a groupoid which is also a preorder (there exists an arrow $P \to Q$ iff P and Q are equivalent).

We also recall the notion of *double category* (cf. [6], II.4 or [8], 1.1). A *double groupoid* is a double category in which every square is invertible, for the horizontal as well as for the vertical compositions. It follows that the horizontal arrows form a groupoid and so do the vertical ones. Similarly for the notion of *double preorder*.

Finally, a *double equivalence relation* is a double groupoid which is a double preorder. It induces two ordinary equivalence relations $\sim_h$ and $\sim_v$ on its sets of objects. Conversely, given a set A with two equivalence relations $\sim_h$ and $\sim_v$, then these arise in this way, provided it is possible to declare certain quadruples (P, Q, R, S) (where

$$P \sim_h Q, \quad R \sim_h S, \quad P \sim_v R, \quad Q \sim_v S\,)$$

to be *squares*, in such a way that certain stability conditions hold; thus, for $P \sim_h Q$,

$(6.1)_1$ $\quad (P, Q, P, Q)$ is a square

(expressing existence of vertically neutral squares) ; also

$(6.2)_1$ $\quad (P, Q, R, S)$ square $\Rightarrow (R, S, P, Q)$ square

(expressing that squares can be vertically inverted) ; and

$(6.3)_1$ $\quad (P, Q, R, S)$ square and (R, S, T, U) square
$\Rightarrow (P, Q, T, U)$ square

expressing that squares can be vertically composed.

Similarly $(6.1)_2$, $(6.2)_2$, $(6.3)_2$ for the horizontal composition.

A double category is said to have *unique fillers* if for any pair consisting of a vertical arrow and a horizontal arrow with common domain, there exists a unique square with the two given arrows as its domains.

PROPOSITION 8. *To equip a set A with a pregroup structure is equivalent to making A the set of objects of a double equivalence relation with unique fillers, for which both $\sim_h$ and $\sim_v$ are codiscrete (i.e. have just one equivalence class).*

PROOF. Let A have a pregroup structure λ. We declare a quadruple (P, Q, R, S) to be a square if $S = \lambda(P, Q, R)$. Then Axiom 1 yields (6.1), Axiom 2 yields (6.2), and Axiom 3 yields (6.3). So we have a double equivalence relation, and by the very construction it has unique fillers. Also, for any P, Q, $R \in A$,

(6.4) $\quad P \sim_h Q$ and $P \sim_v R$.

Conversely, given a double equivalence relation with unique fillers, with $\sim_h$ and $\sim_v$ both codiscrete, for any P, Q and R, we have (6.4), so there is a unique square with (P, Q) and (P, R) as its vertical, respectively horizontal domain. The fourth corner of this square is declared to be $\lambda(P, Q, R)$. The Axioms 1, 2 and 3 now follow by the uniqueness of fillers, and the fact that we have neutral, inverse and composite squares in a double groupoid.

We remark that the correspondence given in the proposition identifies the category of pregroups with a certain full subcategory of the category of double categories.

Let us also remark that the notion of «pregroupoid over a set M» as developed in Section 5, can be expressed in terms of double equivalence relations, namely: a double equivalence relation with unique fillers, in which $\sim_h$ is codiscrete, but where the set of equivalence classes for $\sim_v$ is M.

Note that, when we view a double category as a category object in *Cat*, then we may express the «unique filler» condition as follows: the functor «domain» from the category of morphisms (i.e. the category of squares for horizontal composition) to the category of objects (i.e. the category of horizontal arrows) is a discrete op-fibration. (I am indebted to M. Adelman for this observation.)

February 1980

Revised March 1982

REFERENCES.

1. R. BAER, Zur Einführung des Scharbegriffs, *J. Reine Angew. Math.* 160 (1929), 199-207.
2. W. BLASCHKE, *Kinematik und Quaternionen*, D. V. W., Berlin 1960.
3. E. CARTAN, *Groupes finis et continus et la Géométrie différentielle*, Gauthier-Villars, Paris, 1937.
4. J. CERTAINE, The ternary operation $(abc) = ab^{-1}c$ of a group, *Bull. A. M.S.* 49 (1943), 869-877.
5. C. EHRESMANN, Les connexions infinitésimales dans un espace fibré différentiable, *Coll. Topologie Bruxelles*, C.B.R.M. 1950.
6. C. EHRESMANN, Catégories structurées, *Ann. Sci. Ec. Norm. Sup.* 80 (1963), 349-426.
7. P. JOHNSTONE, *Topos Theory*, Academic Press, London 1977.
8. M. KELLY & R. STREET, Review of the elements of 2-categories, Cat. Sem. Sydney 1972/73, *Lecture Notes in Math.* 420, Springer (1974).
9. F. KLEIN, *Vorlesungen über nicht-euklidische Geometrie*, Springer, 1928.
10. P. LIBERMANN, Sur les prolongements des fibrés principaux..., *Analyse Globale*, Les Presses de l'Université de Montréal, 1971.
11. S. MAC LANE & G. BIRKHOFF, *Algebra*, MacMillan, New York, 1967.
12. V.V. VAGNER, Ternary algebraic operations in the theory of coordinate structures, *Doklady Akad. Nauk SSSR* (N.S.) 81 (1951), 981-984. (In Russian).

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