CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES
Vol. LI-3 (2010)

ABSTRACT PROJECTIVE LINES

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Abstract. We describe a notion of projective line (over a fixed field k): a groupoid with a certain structure. A *morphism* of projective lines is then a functor preserving the structure. We prove a structure theorem: such projective lines are isomorphic to the coordinate projective line (= set of 1-dimensional subspaces of k^2).

Résumé. Nous décrivons une notion de droite projective (sur un corps fixe k): un groupoïde avec une certaine structure. Un *morphisme* de droites projectives est alors un foncteur préservant la structure. Nous prouvons un théorème de structure: telle droite projective est isomorphe à la droite projective des coordonnées (= l'ensemble des sous-espaces linéaires de dimension 1 dans k^2).

Keywords: Projective line, groupoid, cross ratio.
MSC 2010: 14A25, 51A05.

Introduction

For V a vector space over a field k, one has the Grassmannian manifold $P(V)$ consisting of 1-dimensional linear subspaces of V. If V is $n+1$-dimensional, $P(V)$ is a copy of n-dimensional projective space. For $n \geq 2$, $P(V)$ has a rich combinatorial structure, in terms of incidence relations (essentially: the lattice of linear subspaces), in fact, this structure is so rich that one can essentially reconstruct V from the combinatorial structure.

But for $n = 1$, this combinatorial structure (in the form of a lattice), is trivial; as expressed by R. Baer, "*A line ... has no geometrical structure, if considered as an isolated or absolute phenomenon, since then it is nothing but a set of points with the number of points on the line as the only invariant...*", [1] p. 71.

However, it is our contention that a projective line has another kind of structure, making it possible to talk about a projective line as a set equipped with a certain structure, in such a way that isomorphisms (projectivities) between projective lines are bijective maps which preserve this structure.

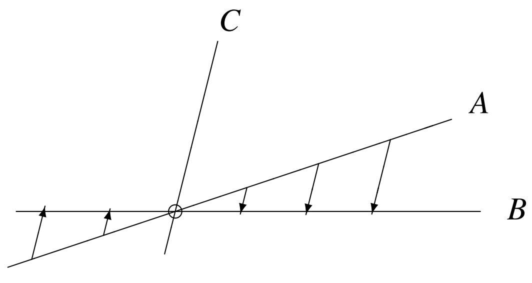
The structure we describe (Section 2) is that of a groupoid (i.e. a category where all arrows are invertible), and with certain properties. The fact that the coordinate projective line $P(k^2)$, more generally, a projective space of the form $P(V)$ (and also the projective plane in the classical synthetic sense) has such groupoid structure, was observed in [3], and further elaborated on in [2]; we shall recall the relevant notions and constructions from [3] in Section 1, and a crucial observation from [2] in Section 3. The present note may be seen as a completion of some of the efforts of these two papers.

1 Groupoid structure on $P(V)$

Let k be a field and let V a 2-dimensional vector space over k. We have a groupoid $\mathbf{L}(V)$, whose set of objects is the set $P(V)$ of 1-dimensional linear subspaces of V, and whose arrows are the linear isomorphisms between these. For $A \in P(V)$, the linear isomorphisms $A \to A$ are in canonical bijective correspondence with the invertible scalars,

$$\mathbf{L}(V)(A,A) = k^*;$$

on the other hand, if A and B are *distinct* 1-dimensional subspaces, then the linear isomorphisms $A \to B$ are all of the form "projection from A to B in a certain unique *direction* C", with $C \in P(V)$ and C distinct from A and B. (This also works in higher dimensions, cf. [3] and [2]; one just has to require that C belongs to the 2-dimensional subspace spanned by A and B.) This is in fact a bijective correspondence, so $\mathbf{L}(V)(A,B)$ is canonically identified with the set $P(V)\backslash\{A,B\}$. Here is a picture (essentially) from [3]:



The linear isomorphism $A \to B$ thus described, we shall denote $(C : A \to B)$. It is clear that the composite of $(C : A \to B)$ with $(C : B \to A)$ gives the identity map of A (projecting forth and back in the same direction). Also it is clear that $(C : A \to B)$ composes with $(C : B \to D)$ to give $(C : A \to D)$. These equations will appear in the axiomatics for abstract projecive lines as the "idempotency laws", (2) and (3) below.

Also, it is clear that two linear isomorphisms from A to B differ by a scalar $\in k^*$; thus, for A and B distinct, and $(C : A \to B)$ and $(D : A \to B)$, there is a unique scalar $\mu \in k^*$ such that

$$(C : A \to B) = \mu \cdot (D : A \to B) = (D : A \to B) \cdot \mu. \tag{1}$$

(We compose from left to right.) This scalar μ is (for A,B,C,D mutually distinct) the classical cross-ratio $(A,B;C,D)$, cf. [3] (3) and [2] Theorem 1.5.3. (For A,B,C distinct, and $D = C$, we have $(A,B;C,C) = 1$.) Permuting the four entries (assumed distinct) will change the cross ratio according to well known formulae (see e.g. [6], [5]) which we shall make explicit and take as axioms.

Thus, the groupoid $\mathbf{L}(V)$, which we in this way have associated to a 2-dimensional vector space V over k, will be an example of an abstract projective line $\mathbf{L}$, in the sense of the next Section.

2 Abstract projective lines: axiomatics

Let k be a field. By a *k-groupoid*, we understand a groupoid $\mathbf{L}$ which is transitive (i.e. the hom set $\mathbf{L}(A,B)$ is non-empty, for any pair of objects A,B in $\mathbf{L}$), and such that all vertex groups $\mathbf{L}(A,A)$ are identified with the (commutative, multiplicative) group k^* of non-zero elements of the field k. We assume that k^* is *central* in $\mathbf{L}$ in the sense that for all $f : A \to B$ and $\lambda \in k^* = \mathbf{L}(A,A) = \mathbf{L}(B,B)$, $\lambda \cdot f = f \cdot \lambda$.

A k-functor between k-groupoids is a functor which preserves k^* in the evident sense.

We now define the notion of *abstract projective line* over k; it is to be a k-groupoid $\mathbf{L}$, equipped with the following kind of structure (L denotes the set of objects of $\mathbf{L}$):

for any two different objects $A,B \in L$, there is given a bijection between the set $\mathbf{L}(A,B)$ and the set $L \setminus \{A,B\}$, and these bijections should satisfy some equational axioms: the *idempotence* laws (2) and (3), and the *permutation* laws (4),...,(7). To state these laws, we use, as in Section 1, the notation:

if $C \in L \setminus \{A,B\}$, then the arrow $A \to B$ corresponding to it (under the assumed bijection) is denoted by $(C : A \to B)$, or just by C, if A and B are clear from the context (say, from a diagram).

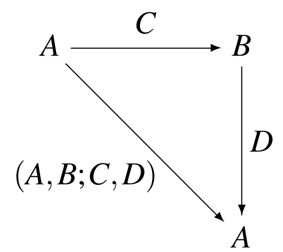
Here is the first set of equations that we assume (the "idempotence equations"): Let A,B,F be mutually distinct, then

$$(F : A \to B) \cdot (F : B \to A) = 1 \in k^* \tag{2}$$

and for A,B,C,F mutually distinct

$$(F : A \to B) \cdot (F : B \to C) = (F : A \to C). \tag{3}$$

The permutation laws which we state next are concerned with the crucial notion of *cross ratio*: If A,B,C,D are four distinct elements of L, we let $(A,B;C,D)$ be the unique scalar (element of k^*) such that



commutes; also, $(A,B;C,D)$ makes sense if $C = D$, and in this case equals $1 \in k^*$, by (2). This scalar is called the *cross ratio* of the 4-tuple A,B,C,D. [1]

Since the elements of L both appear as objects of $\mathbf{L}$ and as labels of arrows of $\mathbf{L}$, the four entries (assumed distinct) in a cross ratio expression can be permuted freely by the 24 possible permutations of four letters. We assume the standard formulas for these permutation instances of a given cross ratio $\mu = (A,B;C,D)$; they give six values,

$$\mu, \mu^{-1}, 1-\mu, (1-\mu)^{-1}, 1-\mu^{-1}, (1-\mu^{-1})^{-1},$$

[1] Convenience, as well as continuity, prompts us to define $(A,B;C,B) = 0$; this is consistent with determinant formulas for cross ratios in $P(k^2)$ to be given later. In fact, one may consistently define $(A,B;C,D)$ whenever $A \neq D$ and $B \neq C$; $(A,A;C,D) = (A,B;C,C) = 1$, and $(A,B;A,D) = (A,B;C,B) = 0$.

see e.g. [6] p. 8 or [5] 0.2. The equations are

$$(A,B;C,D) = (B,A;D,C) = (C,D;A,B) = (D,C;B,A), \qquad (4)$$

and the following equations, where μ denotes $(A,B;C,D)$,

$$(A,B;C,D) = \mu; \qquad (A,B;D,C) = \mu^{-1}; \qquad (5)$$
$$(A,C;B,D) = 1-\mu; \qquad (A,C;D,B) = (1-\mu)^{-1}; \qquad (6)$$
$$(A,D;B,C) = 1-\mu^{-1}; \qquad (A,D;C,B) = (1-\mu^{-1})^{-1}. \qquad (7)$$

(This set of equations is not independent.) We had not needed to be so specific about these "permutation equations", since we shall only need the following consequence: if a map $\Phi : L \to L'$ preserves a cross ratio $(A,B;C,D)$ for some distinct A,B,C,D, then it also preserves any other cross ratio in which the entries are A,B,C,D in some other order.

We have now stated what we mean by an abstract projective line $\mathbf{L}$. For (iso-)*morphisms* ("projectivities") between such: Let $\mathbf{L}$ and $\mathbf{L}'$ be abstract projective lines with object sets (underlying sets) L and L', respectively. By an *isomorphism* $\mathbf{L} \to \mathbf{L}'$ of projective lines, we understand a bijective map $\phi : L \to L'$ with the property that if we put

$$\overline{\phi}(F : A \to B) := (\phi(F) : \phi(A) \to \phi(B)), \qquad (8)$$

(and $\overline{\phi}(\lambda) = \lambda$ for any scalar $\lambda \in k^*$), then $\overline{\phi}$ commutes with composition, i.e. it defines a *functor* $\mathbf{L} \to \mathbf{L}'$ (preserving scalars, i.e. it defines a k-functor). The noticeable aspect of the category $\mathscr{L}$ of abstract projective lines, with (iso)morphisms as just defined, is that the "underlying" functor $\mathbf{L} \mapsto L$ (from $\mathscr{L}$ to the category of sets) is a *faithful* functor, so that it makes sense to say whether a given function $L \to L'$ is a morphism (projectivity) or not.

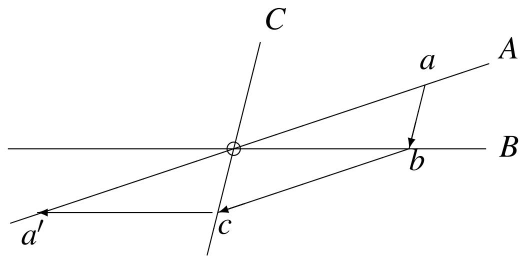
As always in such situations, it is convenient to use the same notation for the object itself, and its underlying set; so we henceforth do not have to distinguish notationally between $\mathbf{L}$ and L.

Cross ratio was defined as a special case of composition; projectivities, in the sense defined here, commute with composition, since projectivities are functors. Hence it is clear that projectivities preserve cross ratios.

In an (abstract) projective line **L**, one may draw some diagrams that are meaningless in more general categories, like the following square (whose commutativity actually can be *proved* on basis of the axiomatics):

$$\begin{array}{ccc} A & \xrightarrow{\;C\;} & B \\ {\scriptstyle -1}\Big\downarrow & & \Big\downarrow{\scriptstyle A} \\ A & \xleftarrow[\;B\;]{} & C \end{array} \tag{9}$$

(where A, B, C are three distinct points in **L**). The commutativity of this diagram, for $\mathbf{L} = P(V)$, expresses an evident geometric fact that one sees by contemplating the figure (essentially from [3], p. 3):



The existence of this diagram (9) shows that "cross ratios do not immediately encode all the geometry" of projective lines; for, no cross ratio (except 1) can be concocted out of just three distinct points; four are needed.

3 Three-transitivity

The "Fundamental Theorem" for projective lines $P(V)$ coming from 2-dimensional vector spaces V is: for any two lists of three distinct points, there is a unique projectivity taking the points of the first list to the points of the second. This theorem, we shall prove holds for abstract projective lines.

Let **L** and $\mathbf{L}'$ be abstract projective lines over the field k.

Theorem 1 (Fundamental Theorem) *Given three distinct points A, B, C in* **L***, and given similarly A', B', C' three distinct distinct points in $\mathbf{L}'$. Then there is a unique projectivity $\phi : \mathbf{L} \to \mathbf{L}'$ taking A to A', B to B' and C to C'.*

Proof. For D distinct from A, B, C, we put $\phi(D) := D'$, where D' is the unique element in $\mathbf{L}'$ with $(A', B'; C', D') = (A, B; C, D)$; equivalently D' is determined by the equation

$$(C' : A' \to B') \cdot (D' : B' \to A') = (A, B; C, D).$$

By construction and the permutation equations, ϕ preserves cross ratios of any distinct 4-tuple, three of whose entries are A, B, C. Next, by the idempotence equations (2) and (3),

$$(A, B; D, E) = (A, B; D, C) \cdot (A, B; C, E),$$

and similarly for the $A', \ldots, E'$. Each of the cross ratios on the right have three entries from the original set A, B, C, and so are preserved, hence so is the cross ratio on the left hand side, $(A, B; D, E)$. So we conclude that any cross ratio, two of whose entries are A and B, is preserved. Next,

$$(A, D; E, F) = (A, D; E, B) \cdot (A, D; B, F),$$

and similarly for the $A', \ldots, F'$, so we conclude that any cross ratio with A as one of its entries is preserved. Finally,

$$(D, E; F, G) = (D, E; F, A) \cdot (D, E; A, G),$$

and similarly for the $A', \ldots, G'$, so we conclude that all cross ratios are preserved.

We have now described the bijection $\phi : \mathbf{L} \to \mathbf{L}'$, and proved that it preserves cross ratio of any four distinct points. To prove that it is a projectivity, in the sense defined, we need to argue that the corresponding $\overline{\phi}$ (as described by (8)) preserves composition of arrows. This is essentially an argument from [2] 2-4-4, which we make explicit:

Proposition 2 *If a bijection $\phi : \mathbf{L} \to \mathbf{L}'$ preserves cross ratio formation, then $\overline{\phi}$ preserves composition.*

It suffices to prove that commutative triangles go to commutative triangles. If the three vertices of the triangle agree, these arrows are scalars $\in k^*$, and $\overline{\phi}$ preserves scalars. If two, but not all three, vertices agree, one arrow is a scalar, and commutativity of the triangle expresses that this scalar is the cross ratio (or its inverse) of the four points that appear as the two vertices and those two labels (likewise points in **L**) that appear on the non-scalar arrows in the triangle; this is a an immediate consequence of the definition (1), possibly combined with the idempotence law (2). We conclude that composites of this form are likewise preserved by $\overline{\phi}$. Finally, we consider the case where the three vertices of the triangle are distinct, so the three arrows in the triangle are of the form $(E:A\to B)$, $(F:B\to C)$, and $(G:A\to C)$ with A,B,C distinct. Consider $(E:A\to B)\cdot(F:B\to C)\cdot(G:C\to A)$, displayed as the top composite in the diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{E} & B & \xrightarrow{F} & C & \xrightarrow{G} & A \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle (A,C;G,F)} \\
A & \xrightarrow{E} & B & \xrightarrow{F} & C & \xrightarrow{F} & A \\
{\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & & & \downarrow{\scriptstyle (A,B;F,E)} \\
A & \xrightarrow[E]{} & B & & \xrightarrow[E]{} & & A.
\end{array}$$

All squares commute; the lower right hand rectangle commutes because of the idempotence law (3) (the two Fs combine into one). The lower composite is 1, because of an idempotence law (2). So we conclude:

$$(A,C;G,F)\cdot(A,B;F,E)=1 \text{ iff } (E:A\to B)\cdot(F:B\to C).\cdot(G:C\to A)=1.$$

Multiplying on the right by $G:A\to C$ (which is inverse to $G:C\to A$), we conclude

$$(A,C;G,F)\cdot(A,B;F,E)=1 \text{ iff } (E:A\to B)\cdot(F:B\to C)=(G:A\to C).$$

Thus commutativity of diagrams can be expressed in terms of cross ratio. Hence since cross ratio are preserved, the composite of $(E:A\to B)$ and $(F:B\to C)$ is preserved by $\overline{\phi}$. This proves the Proposition, and therefore also the existence assertion of the Theorem. The uniqueness is clear, since a projectivity preserves cross ratios, so that we are forced to define $\phi(D)$ as the $D' \in \mathbf{L}'$ with $(A', B'; C', D') = (A, B; C, D)$.

4 $\mathbf{L} = P(k^2)$ as an abstract projective line

The content of the present Section is mostly classical, but it emphasizes the category aspects of $P(k^2)$. Non-zero vectors in k^2 are denoted $a = (a_1, a_2)$, $b = (b_1, b_2)$ etc., and the 1-dimensional linear subspace of k^2 spanned by a is denoted A; similarly, b spans B, etc; A, B, $\ldots$, are the points of the set $\mathbf{L} = P(k^2)$. We now have available the precious tool of *determinants* of 2×2 matrices. We denote the determinant whose rows (or columns) are a, b by the symbol $|a, b|$.

Given distinct A, B, and C, spanned by a, b, and c, respectively. We describe the linear map "projection from A to B in the direction of C" by describing its value on $a \in A$; this value, being in B, is of the form $\lambda \cdot b$ for some unique scalar $\lambda \in k^*$, and an elementary calculation with linear equation systems (say, using Cramer's rule) gives that $\lambda = |c, a| / |c, b|$. Thus

$$(C : A \to B)(a) = \frac{|c, a|}{|c, b|} \cdot b \tag{10}$$

is the basic formula. We can calculate the value of the composite $(C : A \to B) \cdot (D : B \to E)$ on $a \in A$; it takes $a \in A$ into

$$\frac{|c, a|}{|c, b|} \cdot \frac{|d, b|}{|d, e|} \cdot e \in E. \tag{11}$$

In particular, if $E = A$, $a \in A$ goes into $(A, B; C, D) \cdot a$, where

$$(A, B; C, D) := \frac{|c, a|}{|c, b|} \cdot \frac{|d, b|}{|d, a|}$$

$$= \frac{|a, c| \cdot |b, d|}{|a, d| \cdot |b, c|}$$

(using $|c, a| = -|a, c|$, and similarly for the other factors). This is the standard cross ratio $(A, B; C, D)$, and the standard permutation rules follow by known determinant calculations, as do the idempotency laws. So $P(k^2)$ is indeed an abstract projective line, in our sense.

In $\mathbf{L} = P(k^2)$, we describe the points $A \in \mathbf{L}$ by homogeneous coordinates $[a_1 : a_2]$, where a is any vector spanning A. It is convenient to select three particular points in $P(k^2)$, called V, H, and D (for "vertical", "horizontal", and "diagonal", respectively):

$$V = [0:1], \quad H = [1:0], \quad D = [1:1].$$

For any point X distinct from V, there exists a unique $x \in k$ so that $X = [1:x]$. Thus, the $x \in k$ corresponding to H and D are 0 and 1, respectively. For X distinct from V, the corresponding $x \in k$ may be calculated in terms of a cross ratio,

$$x = (V,H;D,X),$$

again by an easy calculation with determinants. Thus $\mathbf{L}\backslash\{V\}$ has, by the chosen conventions, been put in 1-1 correspondence with the affine line k, so

$$\mathbf{L} = \{V\} + k;$$

V is the "point at infinity" of the ("vertical") copy $\{(1,x) \mid x \in k\}$ of the affine line k inside k^2.

The Fundamental Theorem then has the following

Corollary 3 *For every abstract projective line* $\mathbf{L}$ *over k, there exists an isomorphism (= "projective equivalence") with the projective line* $P(k^2)$.

Proof. Pick three distinct points A, B, C in $\mathbf{L}$, and let $\phi = \phi_{A,B,C}$ be the unique projectivity (as asserted by the Theorem) $\mathbf{L} \to P(k^2)$ sending A to $[1:0]$, B to $[0:1]$, and $[C]$ to $[1:1]$.

The isomorphism ϕ described is not canonical, since it depends on choice of three distinct points A, B, C. However, it gives rise to certain canonical *bundles*; this will be exploited in Section 6.

The isomorphism/projectivity ϕ described in this Corollary, although not canonical, allows us to perform calculations in $\mathbf{L}$ using coordinates, in the form of such projective equivalence $\mathbf{L} \cong P(k^2)$.

Let us for instance prove commutativity of (9). It suffices to prove that it holds in $\mathbf{L} = P(k^2)$. For, then it follows from the Fundamental Theorem that it also holds for three distinct points in an abstract projective line $\mathbf{L}$.

So consider points A, B, C in $P(k^2)$, and pick non-zero vectors $a \in A$, $b \in B$, and $c \in C$. Using (11), we see that the composite $(C : A \to B) \cdot (A : B \to C)$ takes $a \in A$ into

$$\frac{|c,a|}{|c,b|} \cdot \frac{|a,b|}{|a,c|} \cdot c,$$

and since $|c,a| = -|a,c|$, two factors cancel except for the sign, and we are left with

$$-\frac{|a,b|}{|c,b|} \cdot c = -\frac{|b,a|}{|b,c|} \cdot c;$$

an easy calculation shows that this by $B : C \to A$ goes to $-a$. (See [2] 1-4-2 for a more coordinate free proof.)

To complete the comparison with the classical "coordinate-" projective line $P(k^2)$, we need to compare projectivities in our sense (functors) with classical projectivities, meaning maps $P(k^2) \to P(k^2)$ that are "tracked" by linear automorphisms $k^2 \to k^2$.

Let $f : k^2 \to k^2$ be such linear automorphism. Then it defines a map $P(f) : P(k^2) \to P(k^2)$ by $[a_1 : a_2] \mapsto [f(a_1) : f(a_2)]$. We shall see that this map preserves composition of arrows, hence is a functor; for, by (10), $f(C : A \to B))$ takes $f(a) \in P(f)(A)$ to

$$\frac{|f(c), f(a)|}{|f(c), f(b)|} \cdot f(b) = \frac{|c,a|}{|c,b|} \cdot f(b) \in P(f)(B)$$

(using the product rule for determinants and then cancelling the occurrences of the determinant of f that appear). The fact that composition is preserved is then a consequence of the formula (11).

On the other hand, every projectivity $\phi : P(k^2) \to P(k^2)$ (in our sense) is of the form $P(f)$ for some linear automorphism $f : k^2 \to k^2$ (which is in fact unique modulo k^*). Let $\phi(H) = A$, $\phi(V) = B$ and $\phi(D) = C$. Pick non-zero vectors $a \in A$, $b \in B$ and $c \in C$. Any linear automorphism $f : k^2 \to k^2$ with matrix

$$f = \begin{bmatrix} a_1 & \lambda b_1 \\ a_2 & \lambda b_2 \end{bmatrix},$$

with $\lambda \neq 0$ has the property that it takes $(1,0)$ to a and $(0,1)$ to λb, hence $P(f)$ takes H to A and V to B. Also f takes $(1,1)$ to $a+\lambda b$; so to ensure $P(f)(D) = C$, we must ensure $a+\lambda b \in C$, i.e. we must ensure linear dependence of the pair consisting of $a+\lambda b$ and c. This means that we should pick λ so that the determinant $|a+\lambda b, c|$ is 0; there is a unique λ solving this, namely $-|a,c|/|b,c|$. With this λ, the maps ϕ and $P(f)$ agree on H, V, and D, and since they both are projectivities, they agree everywhere, by the Fundamental Theorem. This proves that every projectivity $\phi : P(k^2) \to P(k^2)$ (functor) is indeed tracked by a linear automorphism $k^2 \to k^2$.

Remark. The projectivity $\phi : P(k^2) \to P(k^2)$ tracked by a linear automorphism $f : k^2 \to k^2$ with matrix $[\alpha_{ij}]$ is also classically described as the *fractional linear transformation*

$$x \mapsto \frac{\alpha_{21} + \alpha_{22}x}{\alpha_{11} + \alpha_{12}x}.$$

This refers to the identification of $x \in k$ with $[1 : x] \in P(k^2)$.

5 Structures on punctured projective lines

We identify k with the subset $P(k^2)\backslash\{V\} \subseteq P(k^2)$ via $x \mapsto [1 : x]$ (recall that V denotes $[0 : 1]$). The group $PGL(2,k)$ of auto-projectivities of $P(k^2)$ contains a subgroup of those auto-projectivities which are tracked by matrices of the form $\begin{bmatrix} 1 & 0 \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$ (with $\alpha_{22} \neq 0$); they are those $\phi \in PGL(2,k)$ which satisfy $\phi(V) = V$, or equivalently which map $k \subseteq P(k^2)$ to itself. Such ϕ map k to itself by an *affine* bijection, $x \mapsto \alpha_{21} + \alpha_{22}x$. Thus, the subgroup of auto-projectivities of $P(k^2)$ which fix V, is identified with the group $\mathrm{Aff}(k)$ of affine bijections $k \to k$.

We shall prove

Proposition 4 *Given an abstract projective line* $\mathbf{L}$*. 1) For any point* A *in* $\mathbf{L}$*, the set* $\mathbf{L}\backslash\{A\}$ *carries a canonical structure of affine line. 2) For any two distinct points* A, B *in* $\mathbf{L}$*, the set* $\mathbf{L}\backslash\{A\}$ *carries a canonical structure of vector line, with* B *as* 0*. 3) For any three distinct points* A, B, C *in* $\mathbf{L}$*, the set* $\mathbf{L}\backslash\{A\}$ *carries a canonical structure of vector line with a chosen basis, with* B *as* 0 *and* C *as the chosen basis vector.*

Proof. For 1): Given $A \in \mathbf{L}$. Pick distinct B and C in $\mathbf{L}\backslash\{A\}$. Consider the unique projectivity $\phi_{A,B,C} : \mathbf{L} \to P(k^2)$ with $A \mapsto V$, $B \mapsto H$ and $C \mapsto D$, as in the Fundamental Theorem. Then since $\phi_{A,B,C}$ maps A to V, it maps $\mathbf{L}\backslash\{A\}$ bijectively to $P(k^2)\backslash\{V\} = k$. We import the affine structure which k has back to $\mathbf{L}\backslash\{A\}$ via this bijection. This affine structure on $\mathbf{L}\backslash\{A\}$ does not depend on the choice of B and C; for, $\phi_{A,B,C}$ and $\phi_{A,B',C'}$ (same A!) will differ by a projectivity $\phi_{A,B,C} \circ \phi^{-1}_{A,B',C'} : P(k^2) \to P(k^2)$ which fixes V, and such projectivity preserves the affine structure on $k \subseteq P(k^2)$, as we observed. This proves 1). For 2) and 3): It is a well known fact that an affine line with a chosen point B carries a canonical structure of vector line with B as 0. Also, a vector line with a chosen point $C \neq 0$ carries a canonical structure of vector line with a chosen basis vector C, i.e. it is *canonically* isomorphic to the vector space k. Then it is clear that 2) and 3) are consequences of 1).

Remark. Instead of deriving 2) from 1) using the "well known fact", we might prove 2) directly along the same lines as used for proving 1), namely by observing that a projectivity $P(k^2) \to P(k^2)$, which fixes V as well as H, is tracked by a diagonal matrix, and therefore restricts along $k \subseteq P(k^2)$ to a linear $k \to k$. Also, 3) may be seen as an immediate consequence of the Fundamental Theorem.

The affine, resp. vector line, structures described in this Proposition depend on the choice of the point A, resp. on the choice of the points A, B. We can record the dependence using some notions from fibre bundle theory. This is the content of the following Section.

6 The canonical fibre bundles

We consider a pull-back diagram of sets, having the following form

$$\begin{array}{ccc} U \times F & \xrightarrow{h} & E \\ {\scriptstyle proj}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow[\gamma]{} & B \end{array}$$

with γ a surjection. Thus for each $u \in U$, h provides a bijection $h(u,-)$ from F to the fibre $E_{\gamma(u)}$. One may say that such pull-back diagram provides $E \to B$ with structure of *fibre* bundle with *fibre* F (or *modelled on* F). The situation gives rise to a "cocycle" $\overline{h} : U \times_B U \to Aut(F)$, where $Aut(F)$ is the group of all bijections $F \to F$, namely

$$\overline{h}(u_1, u_2) := h(u_1, -)^{-1} \circ h(u_2, -)$$

(composing from right to left).

If F carries some structure T, say structure of vector space, affine space, or projective line, the structure defines a subgroup $Aut_T(F)$ of $Aut(F)$, namely the subgroup of bijections $F \to F$ which preseve the structure in question. For vector space structure T, the customary notation for $Aut_T(F)$ is $GL(F)$, and similarly $\mathrm{Aff}(F)$ for affine space structure.

If now F carries T-structure, and if the cocycle $\overline{h}$ factors through $Aut_T(F)$, the fibres E_x can canonically be provided with T-structure as well. Namely, pick a $u \in U$ with $\gamma(u) = x$, and transport the T-structure from F to E_x via the bijection $h(u,-) : F \to E_x$. The structure thus defined does not depend on the choice of u; any two choices u_1 and u_2 will give the same T-structure on E_x since $h(u_1,-)^{-1} \circ h(u_2,-)$ was assumed to be a T-automorphism.

Thus if F is a vector space, and if the cocycle $\overline{h}$ takes values in $GL(F)$, $E \to X$ acquires structure of a vector bundle; similarly for affine-space bundles or projective-line bundles.

We consider now a fixed (abstract) projective line $\mathbf{L}$. The isomorphisms $\mathbf{L} \cong P(k^2)$, as given by the Fundamental Theorem, are not canonical, but depend on the choice of three distinct points in $\mathbf{L}$. Out of these isomorphisms grow, however, certain canonical fibre bundles:

Let $\mathbf{L}^{(2)}$, resp. $\mathbf{L}^{(3)}$, denote the set of pairs, resp. triples, of distinct points in $\mathbf{L}$. We also have the set $\mathbf{L}^{(2)} \times_{\mathbf{L}} \mathbf{L}^{(2)}$ of quadruples $((A,B),(A,B'))$ with A distinct from B and from B'. These sets appear in pull-back diagrams

$$\begin{array}{ccc} \mathbf{L}^{(3)} \times k & \xrightarrow{h_a} & \mathbf{L}^{(2)} \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{L}^{(3)} & \longrightarrow & \mathbf{L} \end{array} \qquad \begin{array}{ccc} \mathbf{L}^{(3)} \times k & \xrightarrow{h_l} & \mathbf{L}^{(2)} \times_{\mathbf{L}} \mathbf{L}^{(2)} \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{L}^{(3)} & \longrightarrow & \mathbf{L}^{(2)} \end{array} \tag{12}$$

where the displayed vertical maps are projection onto the first factor, or factors, and likewise for the horizontal maps in the bottom row. The maps in the top row are essentially given by the ϕs of the Fundamental Theorem; thus $h_a(A,B,C,\mu)$ is (A,X), where $X \in \mathbf{L}\backslash\{A\}$ is the unique point with $\phi_{A,B,C}(X) = \mu$; and $\phi_l(A,B,C,\mu)$ is $((A,B),(A,X))$ where X, as before, is the unique point with $\phi_{A,B,C}(X) = \mu$.

To the left of these pull-back diagrams, we can, if we want, adjoin yet another one, namely the following pull-back over $\mathbf{L}^{(0)} = 1$ (so a cartesian product diagram), i.e.

$$\begin{array}{ccc} \mathbf{L}^{(3)} \times P(k^2) & \xrightarrow{h_p} & \mathbf{L} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbf{L}^{(3)} & \longrightarrow & 1 \end{array} \tag{13}$$

with $h_p(A,B,C,\mu)$ the unique X such that $\phi_{A,B,C}(X) = \mu$. Here $\phi_{A,B,C} : \mathbf{L} \to P(k^2)$ is the isomorphism of projective lines provided by the Fundamental Theorem, cf. Corollary 3 (and its proof). The cocycle $\mathbf{L}^{(3)} \times \mathbf{L}^{(3)} \to Aut(P(k^2))$ takes values in the subgroup of projectivities of $P(k^2)$, i.e. in $PGL(2,k)$. – Also, to the right of the pull-back diagrams in (12), we may adjoin another “extreme” one (which we shall not use), likewise deriving from the Fundamental Theorem.

$$\begin{array}{ccc} \mathbf{L}^{(3)} \times P(k^2) & \xrightarrow{\cong} & \mathbf{L}^{(3)} \times \mathbf{L} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbf{L}^{(3)} & \xrightarrow[=]{} & \mathbf{L}^{(3)}. \end{array}$$

The cocycles associated to the two pull-back diagrams in (12) take values in the subgroup $\mathrm{Aff}(k) \subseteq PGL(2,k)$, resp. in $GL(k) = k^* \subseteq PGL(2,k)$. For the first of these diagrams, note that the right hand vertical map $\mathbf{L}^{(2)} \to \mathbf{L}$ has for its fibre over $A \in \mathbf{L}$ (a set which may canonically be identified with) $\mathbf{L}\backslash\{A\}$; and for given $(A,B,C) \in \mathbf{L}^{(3)}$, the map h_a maps (A,B,C,μ)

according to the recipe in terms of $\phi_{A,B,C}$, given in the proof of Proposition 4; and, as we observed there, for $((A,B,C),(A,B',C')) \in \mathbf{L}^{(3)} \times_{\mathbf{L}} \mathbf{L}^{(3)}$, (same A!) the value of the "difference" (cocycle) $\phi_{A,B,C} \circ \phi^{-1}_{A,B',C'}$ belongs to Aff(k).

For the second of the diagrams, the right hand vertical map $\mathbf{L}^{(2)} \times_{\mathbf{L}} \mathbf{L}^{(2)} \to \mathbf{L}^{(2)}$ has for its fibre over $(A,B) \in \mathbf{L}^{(2)}$ a set which again may be canonically identified with $\mathbf{L}\backslash\{A\}$ (by identifying $((A,B),(A,C))$ with C). And for given $(A,B,C) \in \mathbf{L}^{(3)}$, the map h_l maps (A,B,C,μ) according to the same recipe as the one for h_a. The cocycle now takes for its input tuples $((A,B,C),(A,B,C'))$ (same A and B!), and, as we observed in the Remark, $\phi_{A,B,C} \circ \phi^{-1}_{A,B,C'}$ takes values in k^*.

Thus, the first diagram in (12) exhibits the $\mathbf{L}\backslash\{A\}$s as the fibres of an affine line bundle over $\mathbf{L}$; and the second diagram in (12) exhibits the $\mathbf{L}\backslash\{A\}$s as the fibres of a vector line bundle over $\mathbf{L}^{(2)}$, both these bundles modelled on the fibre k, viewed as, respectively, an affine line or a vector line.

One may similarly consider the diagram (13) as exhibiting the (abstract) projective line $\mathbf{L}$ as a projective line bundle over 1, modelled on the coordinate projective line $P(k^2)$.

Stacks of projective lines

The notion of projective line, and of morphism (= isomorphism = projectivity) between such, as described here, is a (1-sorted) first order theory. This immediately implies that the notion of a *bundle* of projective lines over a space M makes sense, and in fact, such bundles pull back along maps, and descend along surjections, so projective line bundles form canonically a stack over the base category of sets, or, with suitable modifications, over the base category of spaces, say. Continuity, or other forms of cohesion, will usually follow by the the fact that the constructions employed are canonical, as in [4], Section A.5. The study of bundles of projective lines in the category of schemes, from [5], was the input challenge for the present work, and I hope to push further into loc. cit. using the abstract-projective-line concepts.

Example. Let k denote the field of three elements $\mathbb{Z}_3$. Every 4-element set carries a *unique* structure of abstract projective line over this k. We invite the reader to construct this structure (a groupoid with 4 objects, and each hom-set a 2-element set); the composition laws follow from the idempotence equations; the cross ratio of the four distinct points (in any order) is -1.

(Another argument: the group $PGL(2;\mathbb{Z}_3)$ has 24 elements, which is also the number of permutations of a 4-element set, hence every permutation of a 4-element set is a projectivity.)

It follows that for any space M, and for any 4-fold covering $E \to M$, the bundle $E \to M$ is uniquely a bundle of projective lines over k. Clearly, such $E \to M$ need not have a section $M \to E$, so does not come about from a bundle of affine lines over M, by completing the fibres by points at infinity (the fibrewise infinity points would provide a cross section).

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