SOME CATEGORY THEORETIC NOTIONS ARISING IN SYNTHETIC DIFFERENTIAL GEOMETRY: ATOM, ETALE, DISCRETE

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In [KR 77], Reyes and I came across objects J in certain toposes E, having the property that the functor $(-)^J: E \to E$ preserves colimits; in particular, the basic infinitesimal objects $D, D_2, D\times D, \ldots$ in synthetic differential geometry (SDG, for short) were proved to have this property in some models for SDG, including the Zariski topos. Dubuc [D 79] subsequently included this property of D_1, D_2, etc., as part of the notion of "well-adapted model for SDG", cf. also Axiom 3 of [K 81a] and [K 81b]. In [KR 77], we called such objects J internally projective, [D 79] calls them petite, [K 81b] calls them atoms, and Lawvere and Freyd have recently studied them under the name "tiny objects". In the present note, we stick to the word atom. (Recall that Bunge [Bu 69] calls an object $J \in E$ an atom if $\hom(J,-): E \to \underline{Set}$ commutes with colimits. Only when the global section functor $\Gamma: E \to \underline{Set}$ preserves colimits is there an obvious relationship between the two notions: atoms in our sense are atoms in that of Bunge's. The well-adapted models for SDG usually have this property for Γ. - On the other hand, there is no relation between our atom-notion and the lattice theoretic one of Barr and Diaconescu [BD 80].)

In [KR 77] we utilized atoms J that were furthermore equipped with a point $1 \to J$. To a given class $\underline{D}$ of pointed atoms, we considered the correpsonding notion of $\underline{D}$-etale map; this is a

map $f: N \to M$ such that for any $D \in \underline{D}$, the commutative square

$$\begin{array}{ccc} N^D & \xrightarrow{f^D} & M^D \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & M \end{array}$$

is a pull- back (the vertical maps being induced by the given point of D).

In [K 80], we considered, for given object $M \in E$, and for certain specific $\underline{D}$, the category Et/M, the full sub-category of E/M whose objects are the ($\underline{D}$-) etale maps to M. Likewise, we considered the category $sh(Et/M)$ of sheaves on Et/M for the canonical topology (because this, in the specific circumstances of loc. cit., is the category where the tangent bundle $TM \to M$, when coreflected into it, becomes a Lie algebra object). However, by the following theorem, $sh(Et/M) = Et/M$:

Theorem 1. The inclusion functor $i: Et/M \hookrightarrow E/M$ is left exact (in fact left continuous) and has a right adjoint; in particular, Et/M is a topos, and i is the inverse image functor of a geometric surjection $E/M \to Et/M$.

Proof. The proof technique is the wellknown one of putting two and two together: in [K 80] we state (p. 239) that the subcategory Et/M is closed under finite limits in E/M; and in [KR 77] Lemma 4.6 we prove that it is closed under all (finite) coproducts; the proof for arbitrary colimits is essentially the same, and the existence of a right adjoint for i then follows rom the Special Adjoint Functor Theorem. We include both parts of the proof for completeness:

Stability under inverse limits (this does not depend on the assumptions that the objects of $\underline{D}$ are atoms). We first prove a lemma:

Lemma 2. Suppose $g \circ f$ and g are etale. Then f is etale.

(Call this (v') in [SDG] p. 121)

Proof. Since the notion of etale here is derived from the notion of pull-back, the Lemma follows immediately from the well-known fact that if a double commutative square

$$\begin{array}{ccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \downarrow & & \downarrow & & \downarrow \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \end{array}$$

is a pull-back, and the right hand square is a pull-back, then so is the left hand square.

Remark 3. The stability property expressed in the Lemma should probably have been included in the concept of "abstract etaleness notion" in [K 81b] I.19 (and probably, Joyal did include it), and is included in Benabou's concept of calibrage, cf. [Be 75] and [CM 81].

We now prove that Et/M is stable under formation of pull-backs in E/M. Since pull-backs (like all connected limits) in

E/M are formed in E, we consider a pull-back in E

$$\begin{array}{ccc} X_3 & \xrightarrow{g_2} & X_1 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle f_1} \\ X_2 & \xrightarrow[f_2]{} & X_0 \end{array},$$

each X_i equipped with a structural map ξ_i to M, and all relevant triangles commutative. Assume ξ_0, ξ_1 and ξ_2 are etale. From the Lemma follows that f_1 and f_2 are etale, and since the pull-back of an etale map is etale, g_1 is etale. So the composite $\xi_2 \circ g_1$ is etale, but it equals ξ_3.

Since the terminal object in E/M, namely $1_M: M \to M$ is clearly etale, we have the stability of Et/M under finite limits, as asserted in the Theorem. It is not hard to prove that Et/M is also closed under arbitrary products in E/M, hence under all limits.

Stability under colimits. Since colimits in E/M are calculated in E, we write such colimit in form

$$\begin{array}{c} \varinjlim X_j \\ {\scriptstyle \xi_\infty}\downarrow \\ M \end{array}$$

with the map ξ_∞ being induced by the structural maps $\xi_j: X_j \to M$. Assume each ξ_j etale, and assume $D \in \underline{D}$. We must prove

$$(\varinjlim X_j)^D \cong (\varinjlim X_j) \times_M M^D .$$

We have

$$\begin{aligned} (\varinjlim X_j)^D &\cong \varinjlim (X_j^D) && D \text{ being an atom} \\ &\cong \varinjlim (X_j \times_M M^D) && \xi_j \text{ being etale } \forall j \\ &\sim (\varinjlim X_j) \times_M M^D , \end{aligned}$$

the last isomorphism because pulling back in a topos preserves colimits.

This proves the Theorem.

Question 4. Does the inclusion have a left adjoint? There is no obvious "yes"-answer, since Et/M is not closed under formation of subobjects in E/M (see below), so the "old theorem" stated on p. 87 in [F 64] is not applicable. Neither do I know whether E/M has a cogenerator, making Special Adjoint Functor Theorem applicable.

In 1979, Lawvere studied further categorical properties of atoms (from the viewpoint of elementary toposes, see Remark 8). In particular, in his talk in Cambridge in 1981, "A possible base for SDG", he studied the full subcategory $\underline{T}(\underline{D})$ of discrete objects relative to a given class $\underline{D}$ of atoms, an object S being discrete if the canonical diagonal map $S \to S^D$ is an iso $\forall D \in \underline{D}$. And he indicated why $\underline{T}(\underline{D})$ should be a topos. A missing assumption was added in 1983 by Freyd: the objects of $\underline{D}$ should have full support; this of course is implied by our assumtion that they be pointed.

Now it is obvious that for $\underline{D}$ consisting of pointed atoms, to say S discrete is equivalent to saying that the unique map $S \to 1$ is etale, so that, for this case the Lawvere theorem follows from ours.

However, ours is stronger: Freyd has noted that if D is a fully supported atom in E, then

$$D_M := (D \times M \to M) \quad \in E/M$$

is a fully supported atom in E/M. So a class $\underline{D}$ of fully supported atoms in E gives rise to a class $\underline{D}_M$ of such in E/M. In case the objects of $\underline{D}$ are pointed, we may ask for the relationship between the toposes $\underline{T}(\underline{D}_M)$ and Et/M. We have

Proposition 5. We have an inclusion $Et/M \hookrightarrow \underline{T}(\underline{D}_M)$ of subcategories of E/M (in general, it will be a proper inclusion).

Proof. Suppose $\xi: X \to M$ is etale. Let $D \in \underline{D}$. Then the right hand square in the following diagram is a pull-back, and the left hand square is a pull-back by the construction of exponential objects in E/M,

$$* \qquad \begin{array}{ccccc} \xi^{D_M} & \longrightarrow & X^D & \xrightarrow{\pi} & X \\ \downarrow & \text{p.b.} & \downarrow & & \downarrow \xi \\ M & \xrightarrow{\Delta} & M^D & \xrightarrow{\pi} & M \end{array},$$

(the maps π being induced by the given point of D). So the total diagram is a pull-back, and since the bottom composite is 1_M, the top composite is an isomorphism.

The inclusion is generally proper. For, any $\underline{T}(\underline{D})$ is closed under subobjects, as has been observed by Lawvere and Freyd, whereas Et/M in general does not have this property. For instance, any subobject U of M may be considered a subobject $\overline{U}$ of the terminal object 1_M of E/M. Now $1_M \in Et/M$, but $\overline{U}$ need bot belong here, since not all monic maps to M are etale, in general.

However, if $M \in \underline{T}(\underline{D})$, then it is easy to prove that $Et/M = \underline{T}(\underline{D}_M)$; for, in *, $\pi: M^D \to M$ is then iso, and hence is Δ; furthermore, if $\xi \in \underline{T}(\underline{D}_M)$, the top composite is iso, so $\pi: X^D \to X$ is iso, so the right hand square is a (trivial) pull-back, so ξ is etale.

Question 6. Does $Et/M = \underline{T}(\underline{D}_M)$ imply that $M \in \underline{T}(\underline{D})$? The fact that $\underline{T}(\underline{D}_M) \subset E/M$ is closed under subobjects can also be expressed: the geometric morphism $E/M \to \underline{T}(\underline{D}_M)$ is hypercon-nected, [J 81]. Also, we have, by Proposition 5, a factoriza-tion of the geometric morphism $E/M \to Et/M$ through $\underline{T}(\underline{D}_M)$. John-stone raised the question whether it is the hyperconnected/localic factorization, i.e.

Question 7. Is $\underline{T}(\underline{D}_M) \to Et/M$ localic ?

Remark 8. The arguments of Lawvere and Freyd are made using explicitly the right adjoints $(-)_D$ of $(-)^D$, for $D \in \underline{D}$ ($\underline{D}$ now assumed to be in a suitable sense internally indexed), and do not depend on adjoint-functor-theorems. Possibly, the right adjoints of

$$E/M \xrightarrow{(-)^D} E/M^D \xrightarrow{\Sigma_\pi} E/M,$$

can be used in an analogous way in proving Theorem 1, , these right adjoints being not special cases of $(-)_D$'s.

We should finally remark that the difference between "fully supported" and "pointed" atom is not so drastic: most of the to-poses where atoms, etaleness etc. have been studied have the pro-perty that Γ preserves colimits (has a right adjoint), so any fully supported object has a point.

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