Fairness in allocation problems

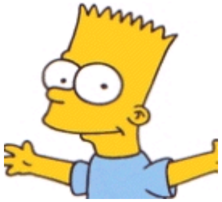
Ioannis Caragiannis
University of Patras

Advanced Course on AI
Chania, July 2019

An ancient problem

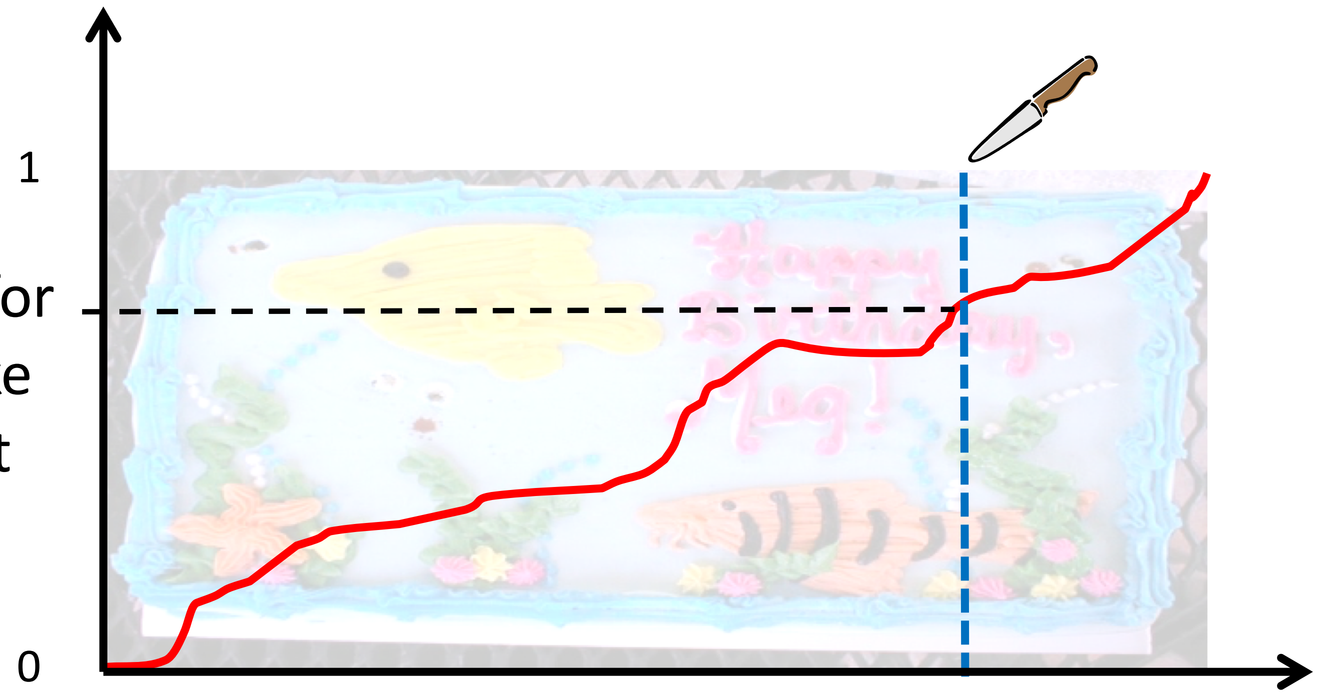
- **Cake cutting**
  - Input: **agents** with different **preferences** for parts of the cake
  - Goal: **divide** the cake in a **fair** manner
- Mathematical formulations initiated by Steinhaus, Banach, & Knaster (1948)
- Basic algorithm/protocol: **cut-and-choose**

Cake cutting



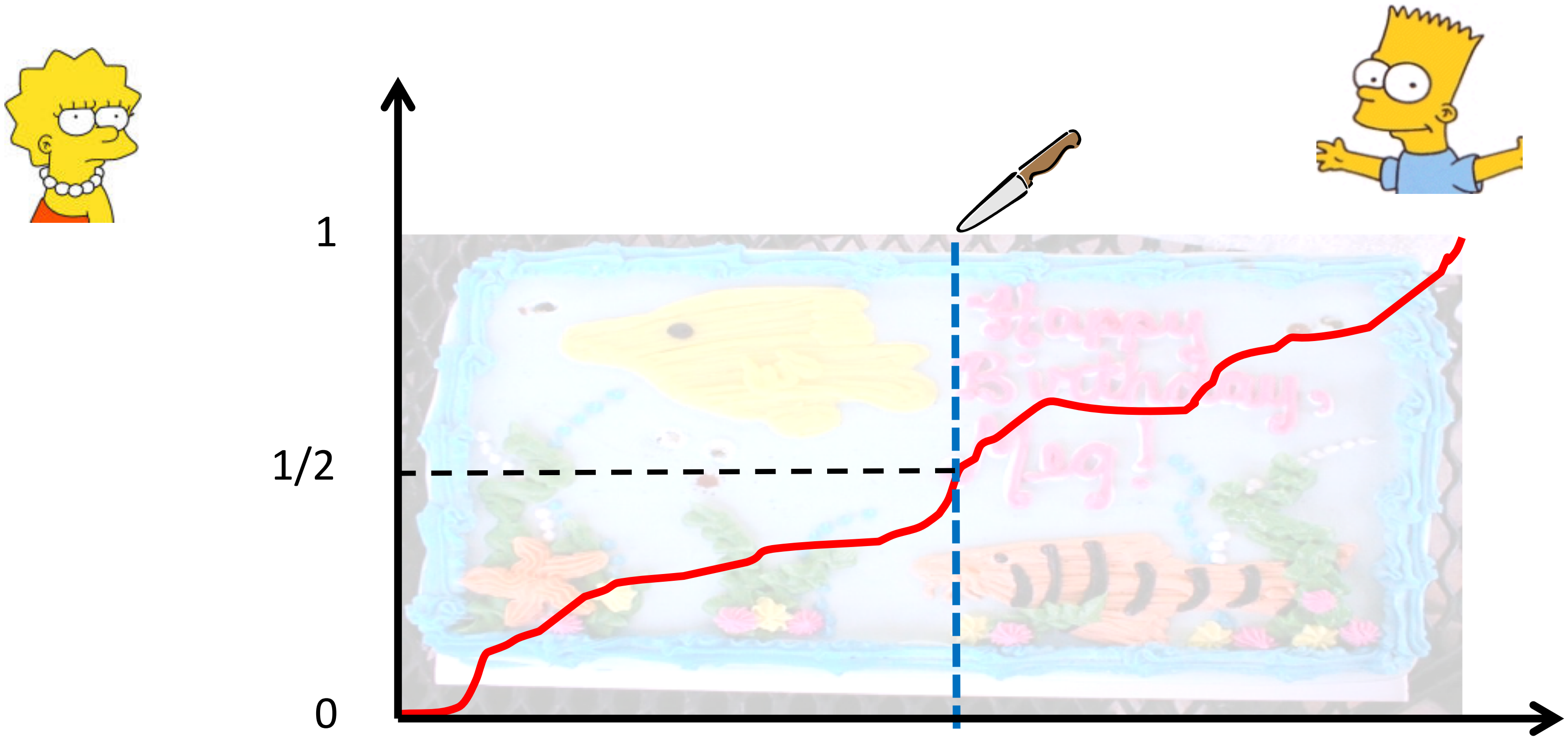
Cake cutting

Value of the agent for the piece of the cake at the left of the cut



Cake cutting

- Cut-and-choose: Lisa **cuts**, Bart **chooses** first



Allocations of goods

- **Indivisible** goods

- **Agents** with additive **valuations** for goods

- Goal: **divide** the goods **fairly**

An allocation problem

			ROLLS ROYCE		
	$1000	$200	$600	$100	$100
	$700	$500	$100	$400	$300
	$500	$700	$400	$200	$200

An allocation problem

	$1000	$200	$600	**$100**	$100
	**$700**	**$500**	**$100**	$400	$300
	$500	$700	$400	$200	**$200**

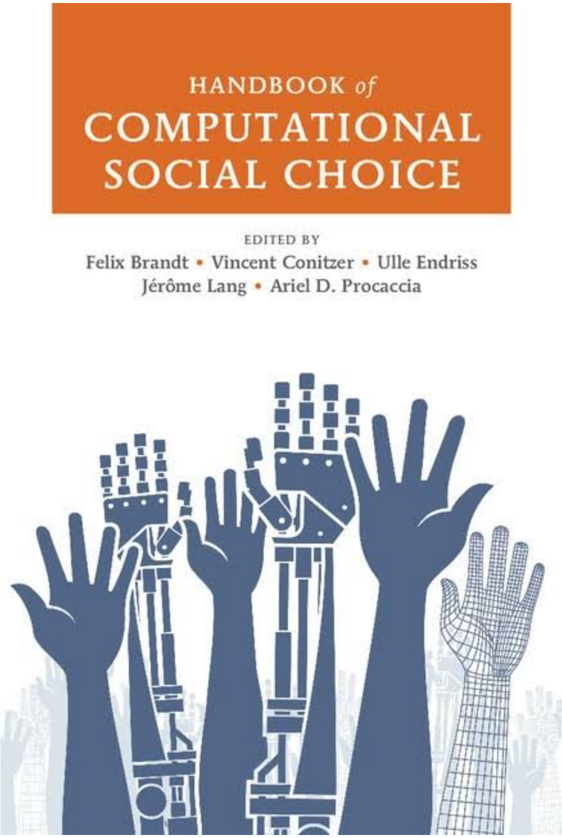
Allocation problems: some history

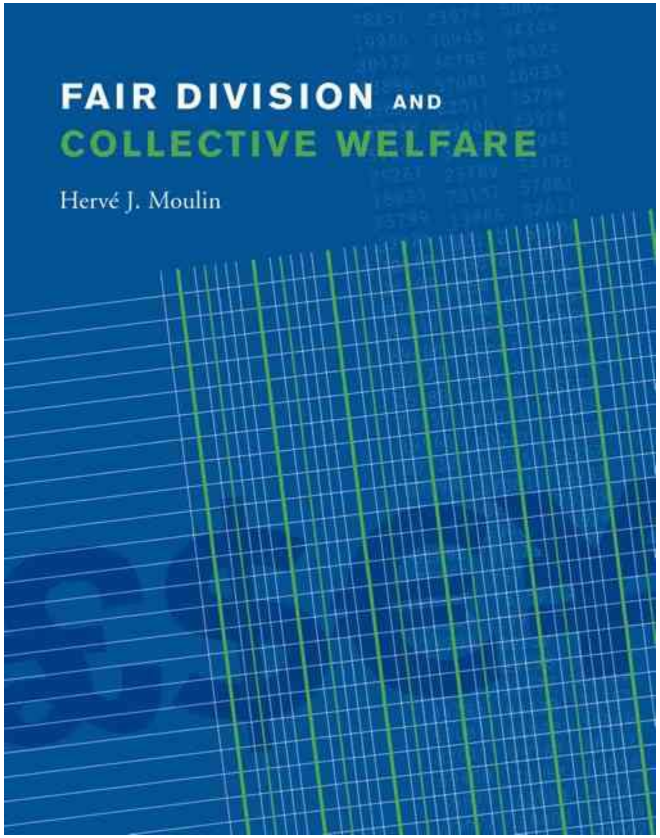
- Ancient Egypt:
  - Land division around Nile (i.e., of the most fertile land)
- Ancient Greece:
  - Sponsorships in theatrical performances
- First references to cut-and-choose protocol
  - Theogony (Hesiod, 8th century B.C.): run between Prometheus and Zeus
  - Bible: run between Abraham and Lot

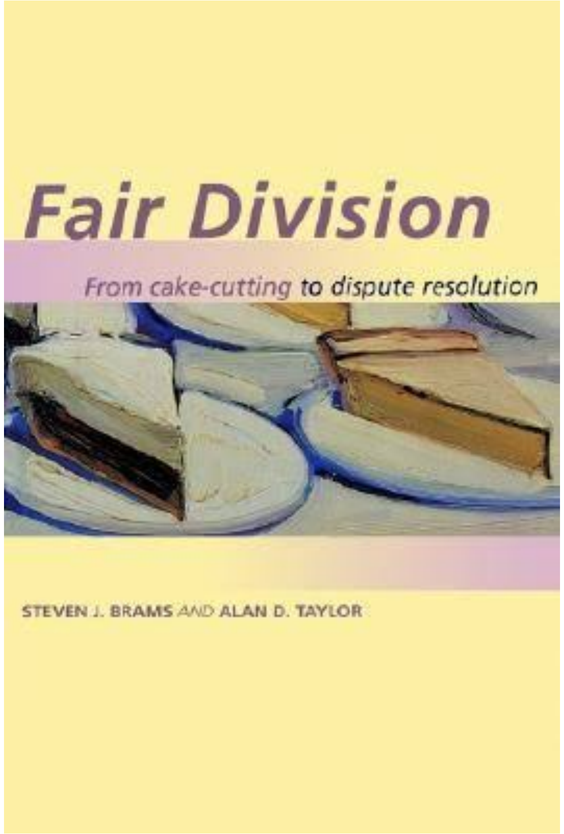
Related implementations/tools

- http://www.spliddit.org
  - Algorithms for various classes of problems (allocations of goods, rent division, etc.)
  - Ariel Procaccia
- http://www.nyu.edu/projects/adjustedwinner/
  - Implementation of the "Adjusted Winner" algorithm for two agents
  - Steven Brams & Alan Taylor
- http://www.math.hmc.edu/~su/fairdivision/calc/
  - Algorithms for allocating goods
  - Francis Su

Further reading







Structure of the lecture

- Basic notions
- Fairness vs. efficiency
- EF1: a relaxed version of envy-freeness
- More fairness notions
- Fairness, knowledge, and social constraints

Basic notions

Formally …

- n **agents**
- A set of **goods** G
- Agent i has **valuation** $v_i(g)$ for good g
- Valuations are additive, i.e.,

$$v_i(S) = \sum_{g \in S} v_i(g)$$

- **Allocation**: a partition $A=(A_1, \ldots, A_n)$ of the goods in G

What does “**fairly**” mean?

- **Fairness notions**
  - Envy freeness
  - Proportionality

What does “**fairly**” mean?

- **Fairness notions**
  - **Envy freeness**: every agent prefers her own bundle to the bundle of any other agent

$$\forall j, i, v_i(A_i) \geq v_i(A_j)$$

EF: an example

			ROLLS ROYCE		
	$1000	$200	$600	$100	$100
	$700	$500	$100	$400	$300
	$500	$700	$400	$200	$200

EF: an example

					
	**$1000**	$200	$600	$100	$100
	$700	$500	$100	$400	$300
	$500	$700	$400	$200	$200

EF: an example

$1000	$200	$600	$100	$100
$700	$500	$100	**$400**	**$300**
$500	**$700**	**$400**	$200	$200

What does "**fairly**" mean?

- **Fairness notions**
  - **Envy freeness**: every agent prefers her own bundle to the bundle of any other agent
  - **Proportionality**: every agent feels that she gets at least 1/n-th of the goods

$$\forall i, v_i(A_i) \geq \frac{1}{n} v_i(G)$$

Proportionality: an example

			ROLLS ROYCE		
	$1200	$200	$300	$200	$100
	$800	$500	$200	$300	$200
	$800	$400	$400	$300	$100

Proportionality: an example

$1200	$200	$300	$200	$100
$800	**$500**	$200	$300	**$200**
$800	$400	**$400**	**$300**	$100

What does “fairly” mean?

- **Fairness notions**
  - **Envy freeness**: every agent prefers her own bundle to the bundle of any other agent
  - **Proportionality**: every agent feels that she gets at least 1/n-th of the goods

Properties

- **Theorem**: EF implies Proportionality

Properties

- **Theorem**: EF implies Proportionality
- **Proof**: Since agent i does not envy any other agent, $\forall j \neq i, v_i(A_i) \geq v_i(A_j)$

Properties

- **Theorem**: EF implies Proportionality
- **Proof**: Since agent i does not envy any other agent, $\forall j \neq i, v_i(A_i) \geq v_i(A_j)$

  Trivially, $v_i(A_i) \geq v_i(A_i)$

Properties

- **Theorem**: EF implies Proportionality
- **Proof**: Since agent i does not envy any other agent, $\forall j \neq i, v_i(A_i) \geq v_i(A_j)$

  Trivially, $v_i(A_i) \geq v_i(A_i)$

  Summing all these n inequalities, we get

$$n \cdot v_i(A_i) \geq \sum_{j=1}^{n} v_i(A_j) = v_i(G)$$

Properties

- **Theorem**: EF implies Proportionality
- **Proof**: Since agent i does not envy any other agent, $\forall j \neq i, v_i(A_i) \geq v_i(A_j)$

  Trivially, $v_i(A_i) \geq v_i(A_i)$

  Summing all these n inequalities, we get

  $$n \cdot v_i(A_i) \geq \sum_{j=1}^{n} v_i(A_j) = v_i(G)$$

  and, equivalently, $\boxed{v_i(A_i) \geq \frac{1}{n} v_i(G)}$

Properties

- **Theorem**: For 2 agents, Proportionality is equivalent to EF

Properties

- **Theorem**: For 2 agents, Proportionality is equivalent to EF
- **Proof**: Since $v_1(A_1) \geq v_1(G)/2$, it must also be $v_1(A_2) \leq v_1(G)/2$, i.e., $v_1(A_1) \geq v_1(A_2)$.

Proportionality may not imply EF for more than two agents

	\$800	\$300	\$300	\$300	\$300
	\$800	\$500	\$200	\$300	\$200
	\$800	\$400	\$400	\$300	\$100

Proportionality may not imply EF for more than two agents

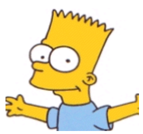
	**$800**	$300	$300	$300	$300
	$800	**$500**	$200	$300	**$200**
	$800	$400	**$400**	**$300**	$100

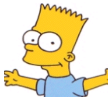
Fairness vs. Efficiency

A motivating example

goods

agents

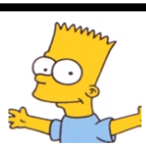
				
	\$3	\$0	\$5	**\$12**
	**\$0**	**\$8**	**\$8**	\$4

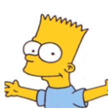
 

allocation ({  }, {    } is EF

A motivating example

goods

agents				
	**$3**	$0	$5	**$12**
	$0	**$8**	**$8**	$4

allocation ({  }, {    } is EF

allocation ({   }, {   } is EF and, in a sense, better!

Efficiency

- **Economic** efficiency
  - Pareto-optimality
  - Social welfare maximization
- **Computational** efficiency
  - Polynomial-time computation
  - Low query complexity

Efficiency

- **Economic** efficiency
  - Pareto-optimality
  - Social welfare maximization
- **Computational** efficiency
  - Polynomial-time computation
  - Low query complexity

a property of allocations

a property of allocation algorithms/protocols

Warming up: Pareto-optimality vs fairness

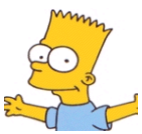
- Definition: an allocation $A = (A_1, A_2, \ldots, A_n)$ is called **Pareto-optimal** if there is no allocation $B = (B_1, B_2, \ldots, B_n)$ such that $v_i(B_i) \geq v_i(A_i)$ for every agent i and $v_{i'}(B_{i'}) > v_{i'}(A_{i'})$ for some agent i'

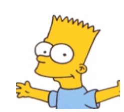
- Informally: there is no allocation in which **all agents are at least as happy** and **some agent is strictly happier**

Envy-freeness vs. Pareto-optimality

goods

agents

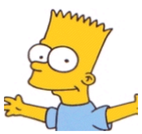
				
	$3	$0	$5	**$12**
	**$0**	**$8**	**$8**	$4

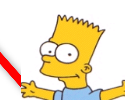
- Observation: In a Pareto-optimal allocation, agent  does not get  and agent  does not get 

Envy-freeness vs. Pareto-optimality

goods

agents

				
	$3	$0	$5	**$12**
	**$0**	**$8**	**$8**	$4

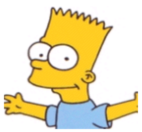
- Observation: In a Pareto-optimal allocation, agent  does not get  and agent  does not get 

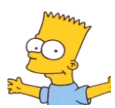
An envy-free allocation that is not Pareto-optimal

Envy-freeness vs. Pareto-optimality

goods

agents

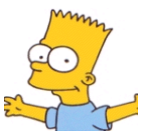
				
	**$3**	$0	**$5**	**$12**
	$0	**$8**	$8	$4

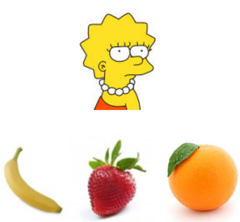
		**PO**	**EF**
		**?**	**?**

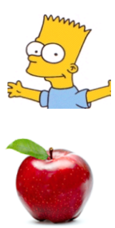
Envy-freeness vs. Pareto-optimality

goods

agents

				
	**$3**	$0	**$5**	**$12**
	$0	**$8**	$8	$4



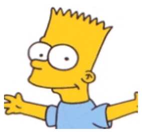


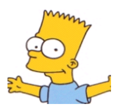
PO	EF
YES	NO

Envy-freeness vs. Pareto-optimality

goods

agents

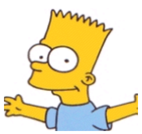
				
	**$3**	$0	**$5**	$12
	$0	**$8**	$8	**$4**

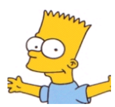
		**PO**	**EF**
		**YES**	**NO**
		**?**	**?**

Envy-freeness vs. Pareto-optimality

goods

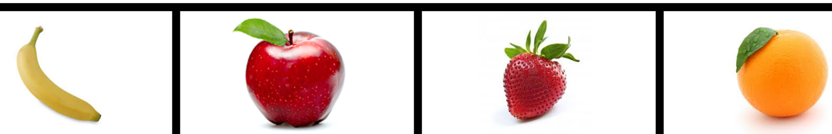
agents

				
	**$3**	$0	**$5**	$12
	$0	**$8**	$8	**$4**

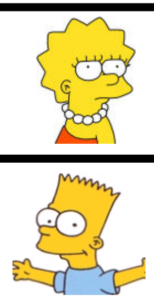
		**PO**	**EF**
		YES	NO
		NO	NO

Envy-freeness vs. Pareto-optimality

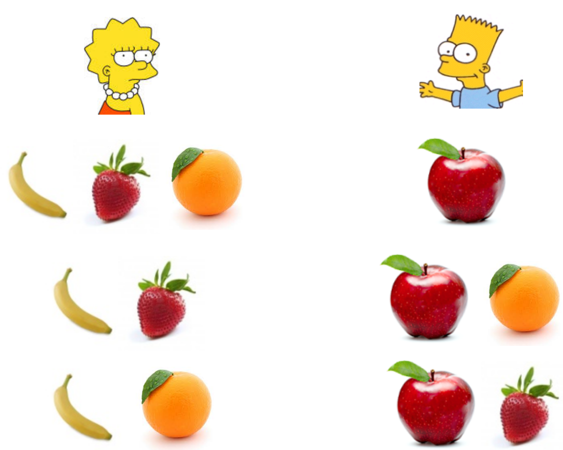
goods



agents



	$3	$0	$5	**$12**
	$0	**$8**	**$8**	$4

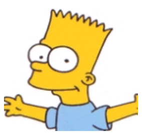


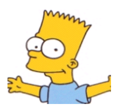
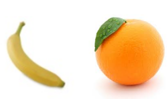
PO	EF
YES	NO
NO	NO
?	?

Envy-freeness vs. Pareto-optimality

goods

agents

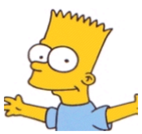
				
	**$3**	$0	$5	**$12**
	$0	**$8**	**$8**	$4

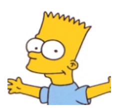
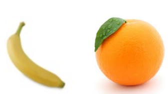
		**PO**	**EF**
		YES	NO
		NO	NO
		YES	YES

Envy-freeness vs. Pareto-optimality

goods

agents

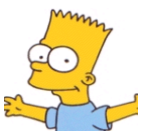
				
	**$3**	$0	$5	$12
	$0	**$8**	**$8**	**$4**

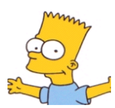
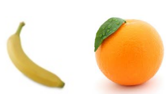
		**PO**	**EF**
		**YES**	**NO**
		**NO**	**NO**
		**YES**	**YES**
		**?**	**?**

Envy-freeness vs. Pareto-optimality

goods

agents

				
	**$3**	$0	$5	$12
	$0	**$8**	**$8**	**$4**

		**PO**	**EF**
		YES	NO
		NO	NO
		YES	YES
		YES	NO

Envy-freeness vs. Pareto-optimality

- **Theorem**: Consider an allocation instance with 2 agents that has at least one EF allocation. Then, **there is an EF allocation that is simultaneously PO**.

Envy-freeness vs. Pareto-optimality

- **Theorem**: Consider an allocation instance with 2 agents that has at least one EF allocation. Then, **there is an EF allocation that is simultaneously PO**.
- **Proof**. Sort the EF allocations in lexicographic order of agents' valuations. The first allocation in this order is clearly PO.

Envy-freeness vs. Pareto-optimality

- **Theorem**: Consider an allocation instance with 2 agents that has at least one EF allocation. Then, **there is an EF allocation that is simultaneously PO**.
- **Proof**. Sort the EF allocations in lexicographic order of agents' valuations. The first allocation in this order is clearly PO.
- **Question**: What about 3-agent instances?
- **Question**: What about Proportionality vs PO?
  - See Bouveret & Lemaitre (2016)

Social welfare

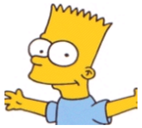
- **Social welfare** is a measure of global value of an allocation A = (A_1, …, A_n)
- **Utilitarian social welfare** of an allocation A:
  - the total value of the agents for the goods allocated to them in A, i.e., $$\mathrm{uSW}(A) = \sum_{i \in N} v_i(A_i)$$
- **Egalitarian social welfare**: $\mathrm{eSW}(A) = \min_{i \in N} v_i(A_i)$
- **Nash social welfare**: $\mathrm{nSW}(A) = \prod_{i \in N} v_i(A_i)$

An example

- SW-maximizing allocations?

goods

agents

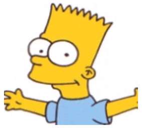
				
	15	0	40	45
	0	30	30	40

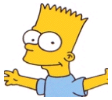
An example

- SW-maximizing allocations?

goods

agents

				
	15	0	40	45
	0	30	30	40

		
uSW	**?**	**?**
eSW	**?**	**?**
nSW	**?**	**?**

An example

- SW-maximizing allocations?

good

agents

	15	0	**40**	**45**
	0	**30**	30	40

Give each good to the agent who values it the most
uSW=130

uSW

eSW **?** **?**

nSW **?** **?**

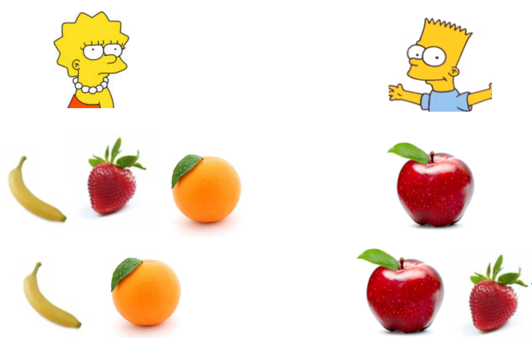
An example

- SW-maximizing allocations?

goods

agents

	15	0	40	**45**
	0	**30**	**30**	40



uSW

eSW

nSW **?** **?**

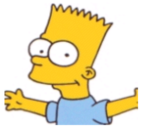
eSW=60

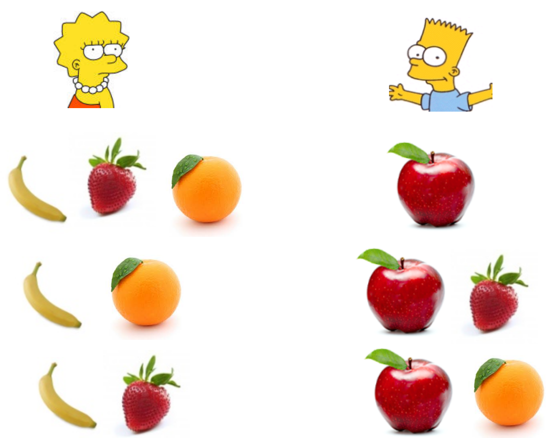
An example

- SW-maximizing allocations?

goods

agents

				
	**15**	0	**40**	45
	0	**30**	30	**40**



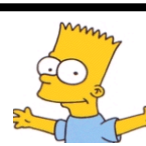
nSW=3850

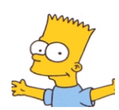
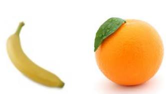
An example

- SW-maximizing allocations?

goods

agents

				
	15	0	40	45
	0	30	30	40

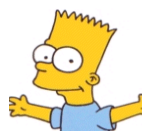
			**EF**
uSW			**?**
eSW			**?**
nSW			**?**

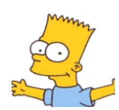
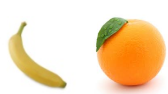
An example

- SW-maximizing allocations?

goods

agents

				
	15	0	40	45
	0	30	30	40

			**EF**
uSW			**NO**
eSW			**YES**
nSW			**YES**

Price of fairness

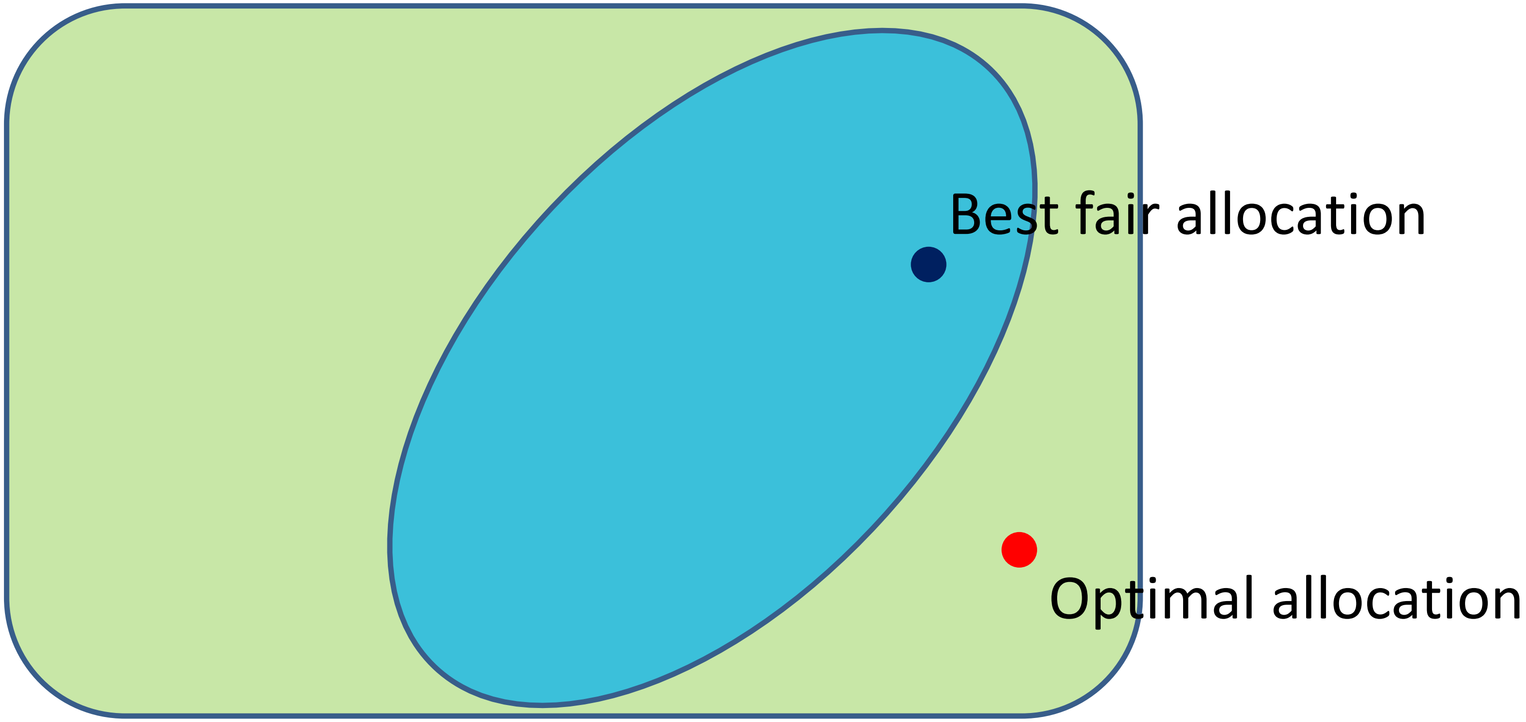
- **Price of fairness** (in general)
  - how far from its maximum value can the social welfare of the best fair allocation be?

- More specifically:
  - Which definition of social welfare to use?
  - Which fairness notion to use?

- Answer:
  - **Any combination of them**

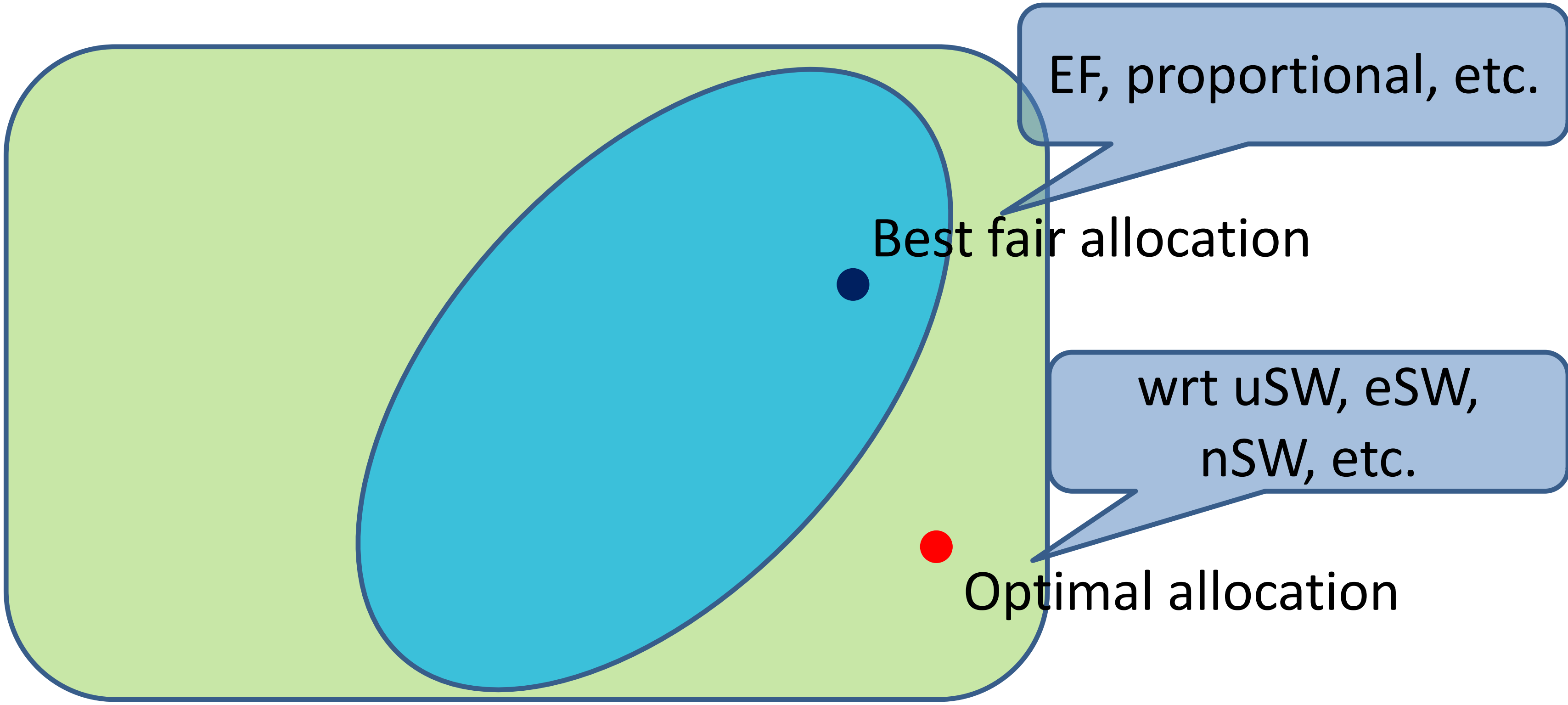
Price of fairness

- How **large** the social welfare of a **fair** allocation can be?
  - C., Kaklamanis, Kanellopoulos, and Kyropoulou (2012)



Price of fairness

- How **large** the social welfare of a **fair** allocation can be?
  - C., Kaklamanis, Kanellopoulos, and Kyropoulou (2012)

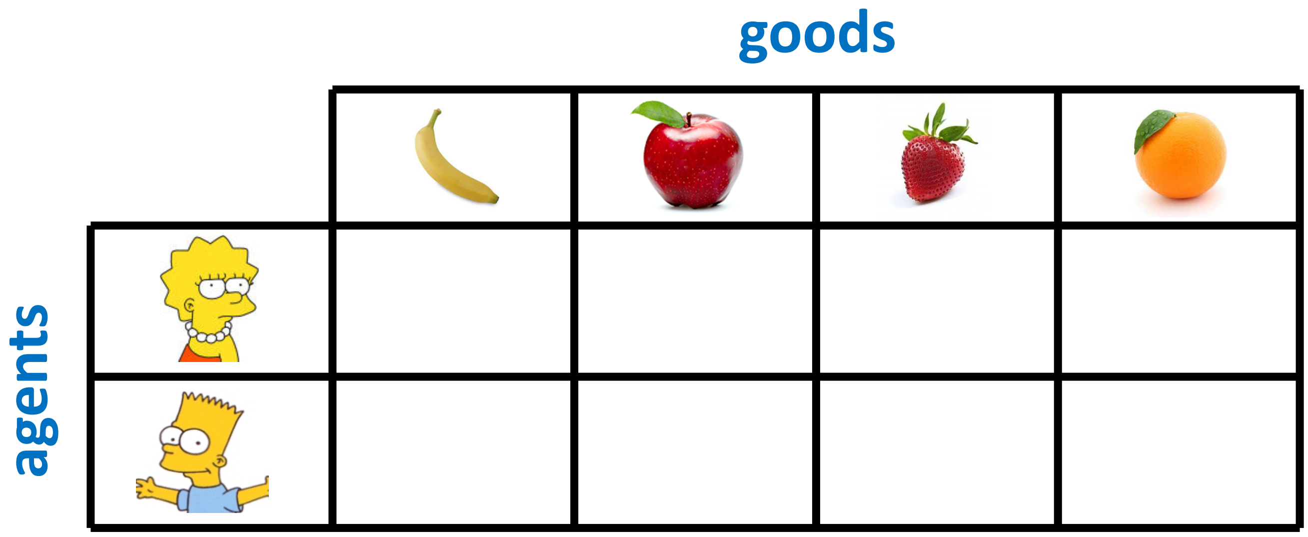


PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is 3/2 (**tight bound**)

PoP & uSW for 2 agents

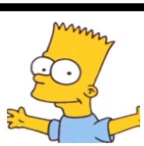
- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is **at least 3/2**.



PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is **at least 3/2**.

goods

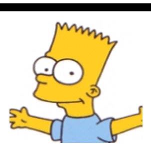
agents					
					
agents		0.5-ε	0.5-ε	ε	ε
agents		0.25+ε	0.25+ε	0.25-ε	0.25-ε

PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is **at least 3/2**.

goods

agents

				
	**0.5-ε**	**0.5-ε**	ε	ε
	0.25+ε	0.25+ε	**0.25-ε**	**0.25-ε**

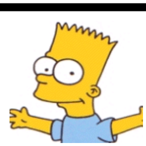
- Optimal allocation (uSW ≈ 1.5)

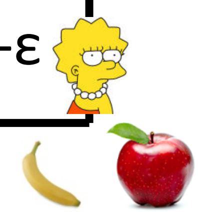
PoP & uSW for 2 agents

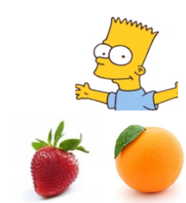
- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is **at least 3/2**.

goods

agents

				
	0.5-ε	0.5-ε	ε	ε
	0.25+ε	0.25+ε	0.25-ε	0.25-ε





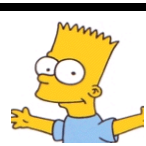
- Optimal allocation (uSW ≈ 1.5)
- Best proportional allocation

? ?

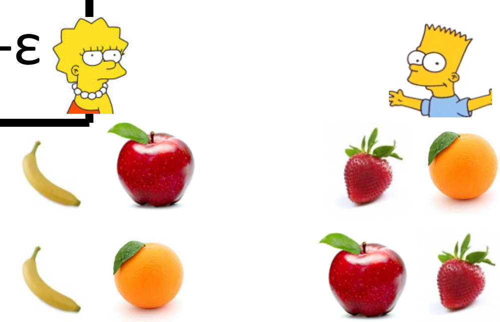
PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is **at least 3/2**.

goods

agents				
	**0.5-ε**	0.5-ε	ε	**ε**
	0.25+ε	**0.25+ε**	**0.25-ε**	0.25-ε

- Optimal allocation (uSW ≈ 1.5)
- Any prop. allocation has uSW ≈ 1



PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is **at most 3/2**.

PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is **at most 3/2**.
- **Proof**: If the uSW-maximizing allocation is proportional, then PoP=1.

PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is **at most 3/2**.
- **Proof**: If the uSW-maximizing allocation is proportional, then PoP=1. So, assume otherwise. Then, some agent has value less than 1/2 for a total of at most 3/2. In any proportional allocation, uSW=1.

PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is **at most 3/2**.
- **Proof**: If the uSW-maximizing allocation is proportional, then PoP=1. So, assume otherwise. Then, some agent has value less than 1/2 for a total of at most 3/2. In any proportional allocation, uSW=1.
- **Question:** PoP/PoEF wrt uSW for many agents?

Computational (in)efficiency

- Computing a proportional/EF allocation is **NP-hard**
- Reduction from **Partition**:
  - Partition instance: given items with weights w_1, w_2, …, w_m, decide whether they can be partitioned into two sets with equal total weight
  - Proportionality/EF instance: A good for each item; 2 agents with identical valuation of w_i for good i

EF1: a relaxed version of EF

spliddit

PROVABLY FAIR SOLUTIONS.

Spliddit offers quick, free solutions to everyday fair division problems, using methods that provide indisputable fairness guarantees and build on decades of research in economics, mathematics, and computer science.

Share Rent

Split Fare

Assign Credit

spliddit

- Fairness hierarchy
  1. Envy-freeness
  2. Proportionality
  3. Maxmin share guarantee
- Previous spliddit protocol
  - Find best fairness criterion
  - Maximize **social welfare** (subject to that criterion)

spliddit

Spliddit Feedback - [redacted]

admin@spliddit.org Jan 7

to admin

Hi! Great app :) We're 4 brothers that need to divide an inheritance of 30+ furniture items. This will save us a fist fight ;) I played around with the demo app and it seems there are non-optimal results for at least two cases where everyone distributes the same amount of value onto the same goods. Try it with either 3 people distributing 1000 points to good A and 0 to the 5 remaining goods, OR try 3 people, 5 goods, with everyone placing 200 on every good. The first case gives 0 to one person, 1 to another and 5 to the third. The second case gives 3 to one person and 1 to each of the others. Why is that? All the best, [redacted]

spliddit

Spliddit Feedback - [redacted]

admin@spliddit.org Jan 7

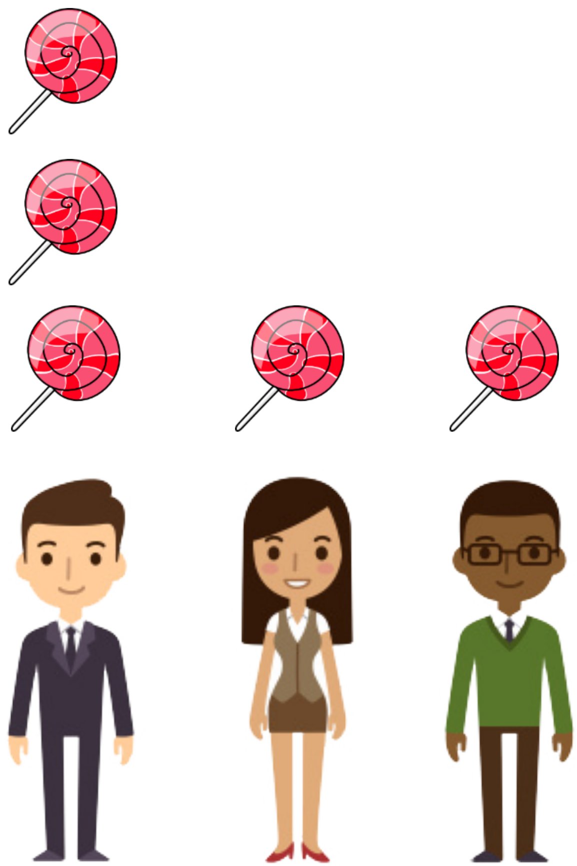
to admin

Hi! Great app :) We're 4 brothers that need to divide an inheritance of 30+ furniture items. This will save us a fist fight ;)

… *try 3 people, 5 goods, with everyone placing 200 on every good*.

… *gives 3 to one person and 1 to each of the others*. Why is that?

…



Relaxing EF

- **Envy-freeness up to one good (EF1)**:
  - There is a good that can be removed from the bundle of agent j so that any envy of agent i for agent j is eliminated

$$\forall i, j, \exists g \in A_j: v_i(A_i) \geq v_i(A_j - g)$$

Relaxing EF

- **Envy-freeness up to one good (EF1)**:
  - There is a good that can be removed from the bundle of agent j so that agent i is not envious for agent j
  - Budish (2011)
  - Easy to achieve: **draft mechanism**
  - Also: Lipton, Markakis, Mossel, and Saberi (2004)

The draft mechanism

- Drafting order: 

			ROLLS ROYCE		
	$1200	$200	$300	$200	$100
	$800	$500	$200	$300	$200
	$800	$400	$400	$300	$100

The draft mechanism

- Drafting order: 

	$1200	$200	$300	$200	$100
	**$800**	$500	$200	$300	$200
	$800	$400	$400	$300	$100

The draft mechanism

- Drafting order: 

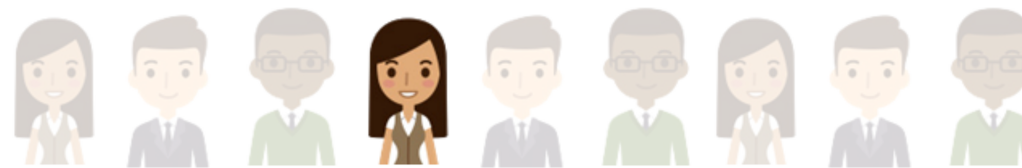
$1200	$200	**$300**	$200	$100
$800	$500	$200	$300	$200
$800	$400	$400	$300	$100

The draft mechanism

- Drafting order: 

	$1200	$200	**$300**	$200	$100
	**$800**	$500	$200	$300	$200
	$800	**$400**	$400	$300	$100

The draft mechanism

- Drafting order: 

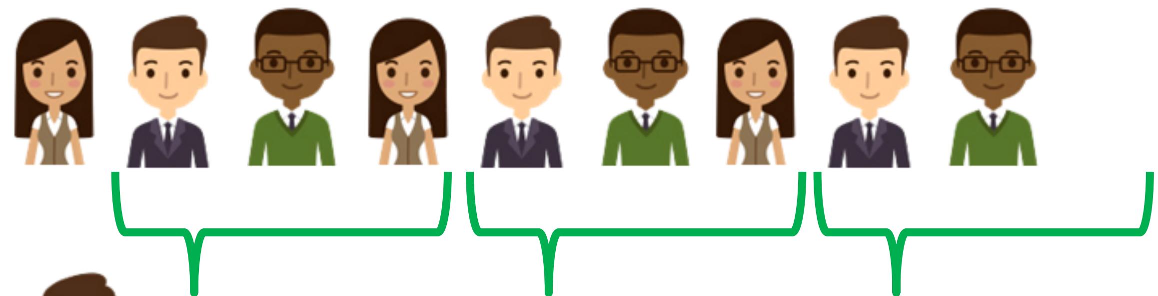
	$1200	$200	**$300**	$200	$100
	**$800**	$500	$200	**$300**	$200
	$800	**$400**	$400	$300	$100

The draft mechanism

- Drafting order: 

	$1200	$200	**$300**	$200	**$100**
	**$800**	$500	$200	**$300**	$200
	$800	**$400**	$400	$300	$100

The draft mechanism

- Drafting order: 
- **Phases** for agent 
- In each phase,  prefers the good he gets to the good every other agent gets
- So, ignoring the good picked by an agent at the very beginning of the sequence,  is EF

Local search

- **Allocate goods one by one**
- In each step j:
  - Allocate good j **to an agent that nobody envies**
  - If this creates a "cycle of envy", **redistribute the bundles along the cycle**
- Crucial property:
  - Envy can be eliminated by removing just a **single good**
  - Implies **EF1**
- Lipton, Markakis, Mossel, & Saberi (2004)

Adding an efficiency objective

- **Pareto optimality (PO)**:
  - No alternative allocation exists that makes some agent better off without making any agents worse off
  - An allocation $A = (A_1, A_2, \ldots, A_n)$ is called **Pareto-optimal** if there is no allocation $B = (B_1, B_2, \ldots, B_n)$ such that $v_i(B_i) \geq v_i(A_i)$ for every agent i and $v_{i'}(B_{i'}) > v_{i'}(A_{i'})$ for some agent i'
- Easy to achieve: give each good to the agent that values it the most

EF1+PO?

EF1+PO?

- **Maximum Nash welfare (MNW)** allocation:
  - the allocation that maximizes the Nash welfare (**product of agent valuations**)
- **Theorem**: the MNW solution is EF1 and PO
  - C., Kurokawa, Moulin, Procaccia, Shah, & Wang (2016)

Theorem: MNW solution is EF1+PO

Theorem: MNW solution is EF1+**PO**

- **PO** is trivial since MNW maximizes $\prod_{i \in N} v_i(A_i)$

Theorem: MNW solution is **EF1**+PO

- Assume MNW is not EF1

Theorem: MNW solution is **EF1**+PO

- Assume MNW is not EF1
- Agent i envies agent j even after any single good is removed from j's bundle

Theorem: MNW solution is **EF1**+PO

- Assume MNW is not EF1
- Agent i envies agent j even after any single good is removed from j's bundle
- For good $g^* = \underset{g \in A_j : v_i(g) > 0}{\operatorname{argmin}} \frac{v_j(g)}{v_i(g)}$

  we have $v_i(A_i) < v_i(A_j) - v_i(g^*)$

Theorem: MNW solution is **EF1**+PO

- Recall that $g^* = \underset{g \in A_j : v_i(g) > 0}{\operatorname{argmin}} \frac{v_j(g)}{v_i(g)}$

Theorem: MNW solution is EF1+PO

- Recall that $g^* = \underset{g \in A_j : v_i(g) > 0}{\operatorname{argmin}} \frac{v_j(g)}{v_i(g)}$

$$v_j(A_j) \geq \sum_{g \in A_j : v_i(g) > 0} v_j(g)$$

Theorem: MNW solution is **EF1**+PO

- Recall that $g^* = \underset{g \in A_j : v_i(g) > 0}{\operatorname{argmin}} \frac{v_j(g)}{v_i(g)}$

$$v_j(A_j) \geq \sum_{g \in A_j : v_i(g) > 0} v_j(g) \geq \sum_{g \in A_j : v_i(g) > 0} v_i(g) \frac{v_j(g^*)}{v_i(g^*)}$$

Theorem: MNW solution is **EF1**+PO

- Recall that $g^* = \underset{g \in A_j : v_i(g) > 0}{\operatorname{argmin}} \frac{v_j(g)}{v_i(g)}$

$$v_j(A_j) \geq \sum_{g \in A_j : v_i(g) > 0} v_j(g) \geq \sum_{g \in A_j : v_i(g) > 0} v_i(g) \frac{v_j(g^*)}{v_i(g^*)}$$

$$= \frac{v_j(g^*)}{v_i(g^*)} \sum_{g \in A_j : v_i(g) > 0} v_i(g)$$

Theorem: MNW solution is **EF1**+PO

- Recall that $g^* = \underset{g \in A_j : v_i(g) > 0}{\operatorname{argmin}} \frac{v_j(g)}{v_i(g)}$

$$v_j(A_j) \geq \sum_{g \in A_j : v_i(g) > 0} v_j(g) \geq \sum_{g \in A_j : v_i(g) > 0} v_i(g) \frac{v_j(g^*)}{v_i(g^*)}$$

$$= \frac{v_j(g^*)}{v_i(g^*)} \sum_{g \in A_j : v_i(g) > 0} v_i(g) = \frac{v_j(g^*)}{v_i(g^*)} v_i(A_j)$$

Theorem: MNW solution is **EF1**+PO

- Recall that $g^* = \underset{g \in A_j : v_i(g) > 0}{\operatorname{argmin}} \frac{v_j(g)}{v_i(g)}$

$$v_j(A_j) \geq \sum_{g \in A_j : v_i(g) > 0} v_j(g) \geq \sum_{g \in A_j : v_i(g) > 0} v_i(g) \frac{v_j(g^*)}{v_i(g^*)}$$

$$= \frac{v_j(g^*)}{v_i(g^*)} \sum_{g \in A_j : v_i(g) > 0} v_i(g) = \frac{v_j(g^*)}{v_i(g^*)} v_i(A_j)$$

- Hence, $\boxed{v_i(g^*) v_j(A_j) - v_j(g^*) v_i(A_j) \geq 0}$

Theorem: MNW solution is **EF1**+PO

$v_i(A_i) < v_i(A_j) - v_i(g^*)$

$v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0$

Theorem: MNW solution is **EF1**+PO

$v_i(A_i) < v_i(A_j) - v_i(g^*)$

$v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0$

$v_i(A_i)\ v_j(A_j)$

Theorem: MNW solution is **EF1**+PO

$v_i(A_i) < v_i(A_j) - v_i(g^*)$

$v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0$

$$v_i(A_i)\ v_j(A_j)$$

$$\leq v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j)$$

Theorem: MNW solution is **EF1**+PO

$v_i(A_i) < v_i(A_j) - v_i(g^*)$

$v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0$

$$v_i(A_i)\ v_j(A_j)$$

$$\leq v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j)$$

$$< v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_i) - v_j(g^*)v_i(g^*)$$

Theorem: MNW solution is **EF1**+PO

$$v_i(A_i) < v_i(A_j) - v_i(g^*)$$

$$v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0$$

$$v_i(A_i)\, v_j(A_j)$$

$$\leq v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j)$$

$$< v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_i) - v_j(g^*)v_i(g^*)$$

$$= (v_i(A_i) + v_i(g^*)) \cdot (v_j(A_j) - v_j(g^*))$$

Theorem: MNW solution is **EF1**+PO

$v_i(A_i) < v_i(A_j) - v_i(g^*)$ $v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0$

$$v_i(A_i)\, v_j(A_j)$$
$$\leq v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j)$$
$$< v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_i) - v_j(g^*)v_i(g^*)$$
$$= (v_i(A_i) + v_i(g^*)) \cdot (v_j(A_j) - v_j(g^*))$$

- So A is not a MNW solution, a contradiction.

- **QED**

Computational issues

- **EF1**+**PO** in **polynomial time**?
  - Yes for two agents (using a restricted MNW solution)
  - Open for more agents (e.g., three agents)
  - Several attempts (e.g., rounding a fractional MNW solution) miserably failed
  - Some progress in very recent work by Barman, Murthy, & Vaish (2018)

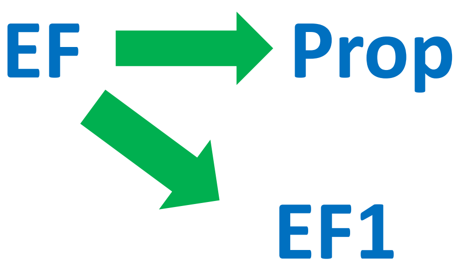
More fairness notions

What does "**fairly**" mean?

- **Fairness notions**
  - Envy freeness (EF)
  - Proportionality
  - Envy-freeness up to one good (EF1)

What does “fairly” mean?

- **Fairness notions**
  - Envy freeness (EF)
  - Proportionality
  - Envy-freeness up to one good (EF1)



What does "fairly" mean?

- **Fairness notions**
  - Envy freeness (EF)
  - Proportionality
  - Envy-freeness up to one good (EF1)
  - **Maxmin share (MmS) allocation**
  - **Minmax share (mMS) allocation**
  - **Envy-freeness up to any good (EFX)**
  - **Pairwise MmS allocation**

What does “fairly” mean?

- **Fairness notions**
  - Envy freeness (EF)
  - Proportionality
  - Envy-freeness up to one good (EF1)
  - **Maxmin share (MmS) allocation**: each agent’s value is at least the best guarantee when dividing the goods into n bundles and getting the least valuable bundle

$$\forall i, v_i(A_i) \geq \theta_i = \max_{A'} \min_{j \in N} v_i(A'_j)$$

MmS: an example

	$500	$600	$200	$400	$300
	$700	$700	$300	$200	$100
	$900	$600	$200	$200	$100

MmS: an example

Let’s compute the MmS thresholds first

						θ_i
	$500	$600	$200	$400	$300	
	$700	$700	$300	$200	$100	
	$900	$600	$200	$200	$100	

MmS: an example

Let’s compute the MmS thresholds first

						θ_i
	\$500	\$600	\$200	\$400	\$300	\$600
	\$700	\$700	\$300	\$200	\$100	\$600
	\$900	\$600	\$200	\$200	\$100	\$500

MmS: an example

Now, let's compute the allocation

						θ_i
	$500	$600	$200	$400	$300	$600
	$700	$700	$300	$200	$100	$600
	$900	$600	$200	$200	$100	$500

MmS: an example

Now, let's compute the allocation

						θ_i
	\$500	**\$600**	\$200	\$400	\$300	\$600
	**\$700**	\$700	\$300	\$200	\$100	\$600
	\$900	\$600	**\$200**	**\$200**	**\$100**	\$500

An implication

- **Theorem**: Proportionality implies MmS

An implication

- **Theorem**: Proportionality implies MmS
- **Proof**: Let A be a proportional allocation. Then,

$$\forall i, v_i(A_i) \geq \frac{1}{n} v_i(G)$$

But the MmS threshold for agent i is

$$\theta_i = \max_{A'} \min_{j \in N} v_i(A'_j) \leq \frac{1}{n} v_i(G)$$

Hence,

$$\forall i, v_i(A_i) \geq \theta_i$$

What does "fairly" mean?

- **Fairness notions**
  - Envy freeness (EF)
  - Proportionality
  - Envy-freeness up to one good (EF1)
  - Maxmin share (MmS) allocation
  - **Minmax share (mMS) allocation**: each agent's value is at least the worst guarantee when dividing the goods into n bundles and getting the most valuable bundle

$$\forall i, v_i(A_i) \geq \theta_i = \min_{A'} \max_{j \in N} v_i\left(A'_j\right)$$

mMS: an example

			ROLLS ROYCE		
	\$500	\$600	\$200	\$400	\$300
	\$700	\$700	\$300	\$200	\$100
	\$900	\$600	\$200	\$200	\$100

mMS: an example

						θ_i
	$500	$600	$200	$400	$300	
	$700	$700	$300	$200	$100	
	$900	$600	$200	$200	$100	

mMS: an example

Let’s compute the mMS thresholds first

					θ_i
\$500	\$600	\$200	\$400	\$300	\$700
\$700	\$700	\$300	\$200	\$100	\$700
\$900	\$600	\$200	\$200	\$100	\$900

mMS: an example

Now, let's compute the allocation

						θ_i
	\$500	\$600	\$200	\$400	\$300	\$700
	\$700	\$700	\$300	\$200	\$100	\$700
	\$900	\$600	\$200	\$200	\$100	\$900

mMS: an example

Now, let's compute the allocation

						θ_i
	$500	$600	$200	**$400**	**$300**	$700
	$700	**$700**	**$300**	$200	$100	$700
	**$900**	$600	$200	$200	$100	$900

An implication

- **Theorem**: EF implies mMS

An implication

- **Theorem**: EF implies mMS
- **Proof**: Let A be an EF allocation. Then,

$$\forall i, v_i(A_i) \geq \max_{j \in N} v_i(A_j) \geq \theta_i = \min_{A'} \max_{j \in N} v_i(A'_j)$$

Another implication

- **Theorem**: mMS implies Proportionality

Another implication

- **Theorem**: mMS implies Proportionality
- **Proof**: Let A be an mMS allocation. Then,

$$\forall i, v_i(A_i) \geq \theta_i = \min_{A'} \max_{j \in N} v_i\left(A'_j\right)$$

But the mMS threshold for agent i is

$$\theta_i = \min_{A'} \max_{j \in N} v_i(A'_j) \geq \frac{1}{n} v_i(G)$$

Hence,

$$\forall i, v_i(A_i) \geq \frac{1}{n} v_i(G)$$

What does “fairly” mean?

- **Fairness notions**
  - Envy freeness (EF)
  - Proportionality
  - Envy-freeness up to one good (EF1)
  - Maxmin share (MmS) allocation
  - Minmax share (mMS) allocation

What does "fairly" mean?

- **Fairness notions**
  - Envy freeness (EF)
  - Proportionality
  - Envy-freeness up to one good (EF1)
  - Maxmin share (MmS) allocation
  - Minmax share (mMS) allocation
  - **Envy-freeness up to any good (EFX)**: agent i is either not envious of agent j initially or s/he is not envious after removing any good from the bundle of agent j

$$\forall i, j, \forall g \in A_j \text{ with } v_i(g) > 0: v_i(A_i) \geq v_i(A_j - g)$$

EFX: an example

$500	$600	$200	**$400**	**$300**
$700	**$700**	**$300**	$200	$100
$900	$600	$200	$200	$100

EFX: another example

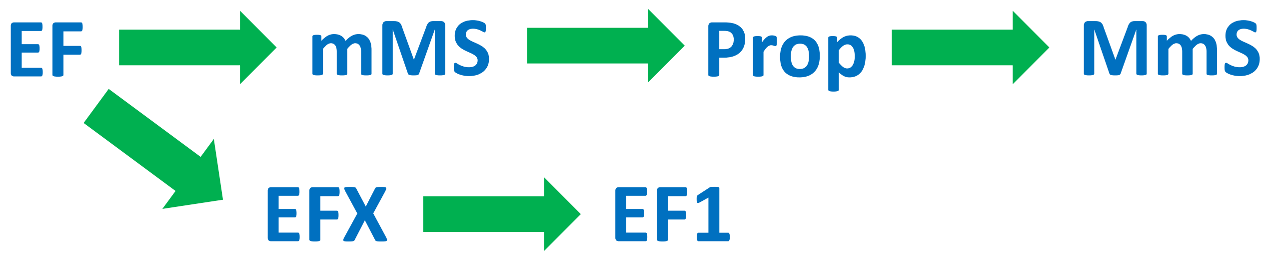
- Drafting order: 

Can the draft mechanism compute EFX allocations?

	$1200	$200	**$300**	$200	**$100**
	**$800**	$500	$200	**$300**	$200
	$800	**$400**	$400	$200	$200

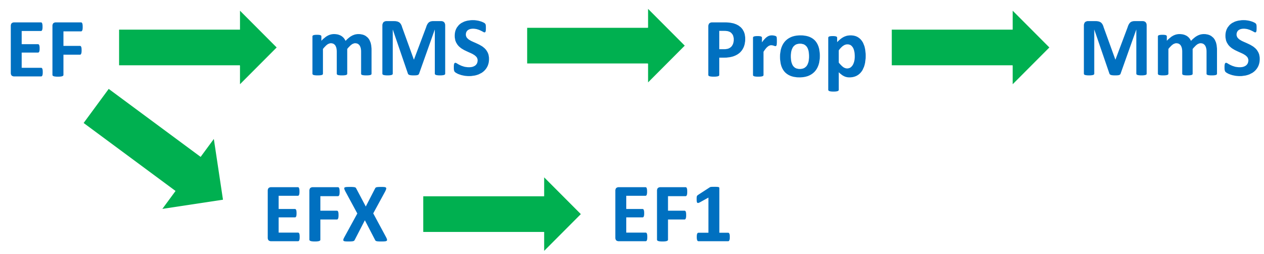
More implications

- **Theorem**: EF implies EFX, which implies EF1



More implications

- **Theorem**: EF implies EFX, which implies EF1



- **Open question**: Does an EFX allocation always exist?
- So, is the implication EFX => EF1 strict?

What does "**fairly**" mean?

- **Fairness notions**
  - Envy freeness (EF), Proportionality, Envy-freeness up to one good (EF1), Maxmin share (MmS) allocation, Minmax share (mMS) allocation, Envy-freeness up to any good (EFX)
  - **Pairwise MmS allocation**: an allocation A is pairwise MmS if for every pair of agents i and j, the allocation (A_i, A_j) between the two agents is MmS

Pairwise MmS: an example

						θ_i
	$500	$600	$200	**$400**	**$300**	$700
	$700	**$700**	**$300**	$200	$100	$600
	$900	$600	$200	$200	$100	

Pairwise MmS: an example

			ROLLS RR ROYCE			θ_i
	\$500	\$600	\$200	**\$400**	**\$300**	\$500
	\$700	**\$700**	**\$300**	\$200	\$100	
	**\$900**	\$600	\$200	\$200	\$100	\$300

Pairwise MmS: an example

						θ_i
	$500	$600	$200	**$400**	**$300**	
	$700	**$700**	**$300**	$200	$100	$700
	**$900**	$600	$200	$200	$100	$800

Pairwise MmS: another example

- Drafting order: 

Can the draft mechanism compute pMmS allocations?

					θ$_i$
$1200	$200	**$300**	$200	**$100**	$700
$800	$500	$200	**$300**	$200	$800
$800	**$400**	$400	$200	$200	

Yet another implication

- **Theorem**: EF implies pairwise MmS, which implies EFX

Yet another implication

- **Theorem**: EF implies pairwise MmS, which implies EFX
- **Proof**: The first implication is trivial.

Yet another implication

- **Theorem**: EF implies pairwise MmS, which implies EFX
- **Proof**: The first implication is trivial.

  Let A be a pMmS allocation that is not EFX.

Yet another implication

- **Theorem**: EF implies pairwise MmS, which implies EFX
- **Proof**: The first implication is trivial.

  Let A be a pMmS allocation that is not EFX.

  I.e., there are agents i, j so that for a good $g \in A_j$ with $v_i(g)>0$, it holds that $v_i(A_i) < v_i(A_j - g)$.

Yet another implication

- **Theorem**: EF implies pairwise MmS, which implies EFX
- **Proof**: The first implication is trivial.

  Let A be a pMmS allocation that is not EFX.

  I.e., there are agents i, j so that for a good $g \in A_j$ with $v_i(g)>0$, it holds that $v_i(A_i) < v_i(A_j\text{-}g)$.

  Then, the pairwise MmS threshold for agent i should be higher than either $v_i(A_i+g)$ or $v_i(A_j\text{-}g)$.

Yet another implication

- **Theorem**: EF implies pairwise MmS, which implies EFX
- **Proof**: The first implication is trivial.

  Let A be a pMmS allocation that is not EFX.

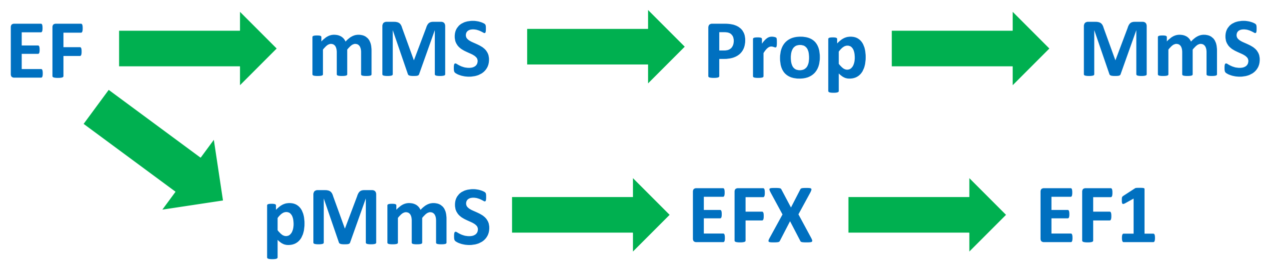
  I.e., there are agents i, j so that for a good $g \in A_j$ with $v_i(g)>0$, it holds that $v_i(A_i) < v_i(A_j-g)$.

  Then, the pairwise MmS threshold for agent i should be higher than either $v_i(A_i+g)$ or $v_i(A_j-g)$.

  This contradicts the assumptions that A is pMmS.

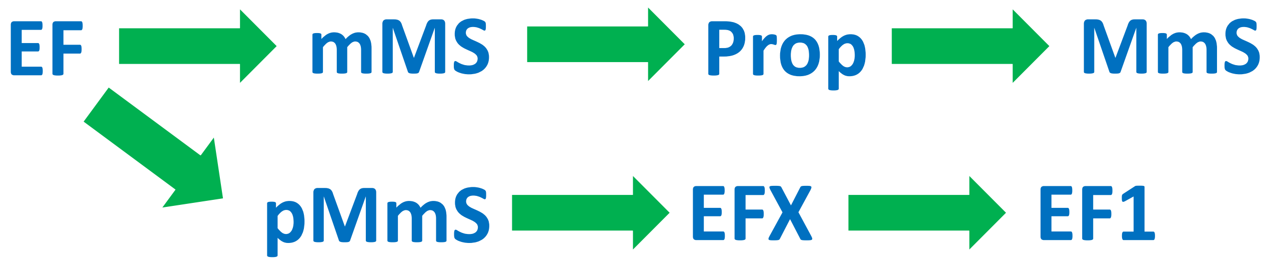
Yet another implication

- **Theorem**: EF implies pairwise MmS, which implies EFX



Yet another implication

- **Theorem**: EF implies pairwise MmS, which implies EFX



- **Open question**: Does a pairwise MmS allocation always exist?
- So, is the implication pMmS => EFX strict?

Further reading

- **Fairness notions**
  – MmS, EF1: Budish (2011)
  – MmS: Kurokawa, Procaccia, & Wang (2018), Amanatidis, Markakis, Nikzad, & Saberi (2017), Barman & Murthy (2017), Ghodsi, Hajiaghayi, Seddighin, Seddighin, & Yami (2018)
  – mMS: Bouveret & Lemaitre (2016)
  – EFX, pairwise MmS: C., Kurokawa, Moulin, Procaccia, Shah, & Wang (2016)
  – EFX: Plaut & Roughgarden (2018), C., Gravin, & Huang (2019)

Fairness, knowledge, and social constraints

Fairness and knowledge

- What kind of **knowledge** do the agents need to have?
- Knowledge about the **goods** and the **number of agents** only:
  - Proportionality, MmS, mMS
- Knowledge about the whole **allocation**:
  - EF, EFX, EF1, pairwise MmS

Envy-freeness?

			ROLLS ROYCE	
	$1000	$600	$600	$100
	$1000	$600	$600	$100
	$100	$600	$600	$1000

Epistemic envy-freeness (EEF)

			ROLLS ROYCE	
	**$1000**	$600	$600	$100
	$1000	**$600**	**$600**	$100
	$100	$600	$600	**$1000**

Epistemic envy-freeness (EEF)

- Informally: a **relaxation of EF** with a definition that uses only knowledge about goods and number of agents
- Formal definition:
  - the allocation (A_1, A_2, …, A_n) is **EEF** if, for every agent i, there is a **reallocation** (**B_1, …, B_{i-1}, A_i, B_{i+1}, …, B_n**) in which agent i is not envious, i.e., **$v_i(A_i) \geq v_i(B_j)$** for every other agent j
- Aziz, C., Bouveret, Giagkousi, & Lang (2018)

Epistemic envy-freeness (EEF)

- Formal definition:
  - the allocation (A_1, A_2, …, A_n) is **EEF** if, for every agent i, there is a **reallocation** (**B_1, …, B_{i-1}, A_i, B_{i+1}, …, B_n**) in which agent i is not envious, i.e., **$v_i(A_i) \geq v_i(B_j)$** for every other agent j
- **Theorem**: EF implies EEF, which implies mMS

Epistemic envy-freeness (EEF)

- Formal definition:
  - the allocation $(A_1, A_2, \ldots, A_n)$ is **EEF** if, for every agent i, there is a **reallocation** $(\mathbf{B_1}, \ldots, \mathbf{B_{i-1}}, \mathbf{A_i}, \mathbf{B_{i+1}}, \ldots, \mathbf{B_n})$ in which agent i is not envious, i.e., $\mathbf{v_i(A_i) \geq v_i(B_j)}$ for every other agent j
- **Theorem**: EF implies EEF, which implies mMS
- **Proof**: EF trivially implies EEF (with B = A).

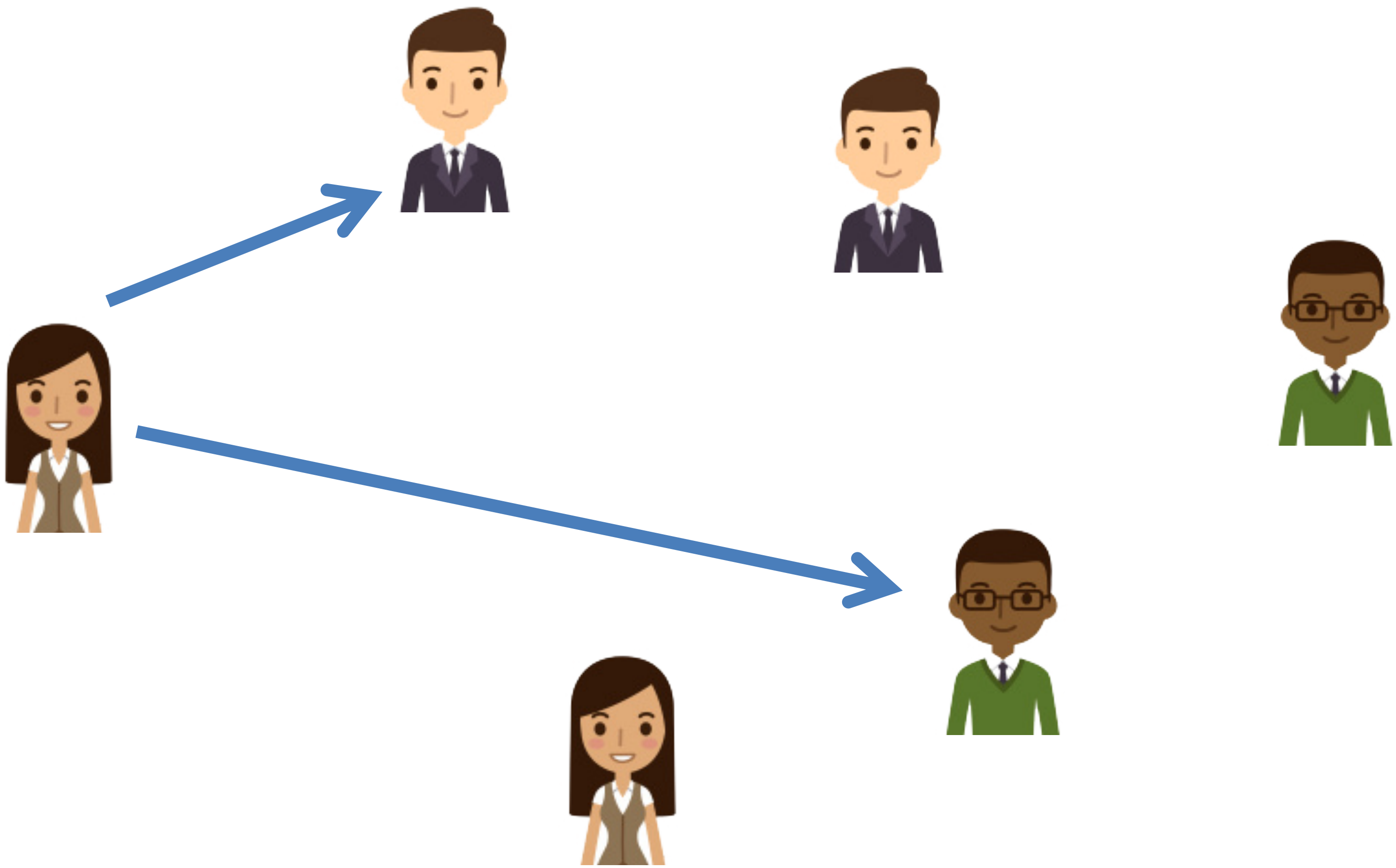
Also, $v_i(A_i) \geq \theta_i = \min_{A'} \max_{j \in N} v_i\left(A'_j\right)$

Epistemic envy-freeness (EEF)

- Formal definition:
  - the allocation $(A_1, A_2, \ldots, A_n)$ is **EEF** if, for every agent i, there is a **reallocation** $(B_1, \ldots, B_{i-1}, A_i, B_{i+1}, \ldots, B_n)$ in which agent i is not envious, i.e., $v_i(A_i) \geq v_i(B_j)$ for every other agent j
- **Theorem**: EF implies EEF, which implies mMS

Fairness with social constraints

- Existence of an **underlying social graph**



Fairness with social constraints

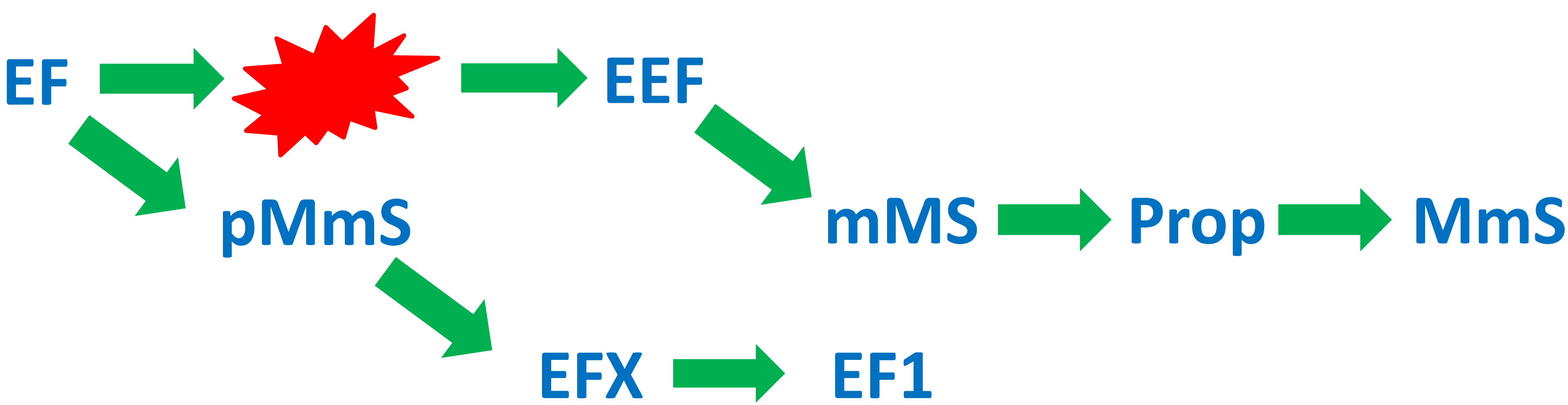
- Existence of an **underlying social graph**, which represents the knowledge each agent has for the bundles allocated to other agents
- Recent related papers (graph-EF/Proportionality):
  - Abebe, Kleinberg, & Parkes (2017)
  - Bei, Qiao, & Zhang (2017)
  - Chevaleyre, Endriss, & Maudet (2017)
  - Aziz, C., Bouveret, Giagkousi, & Lang (2018)

Graph-EEF

- **Social graph G**: directed graph having the agents as nodes
- **G-EEF**:
  - agent i is **EF wrt her neighbors** and
  - **EEF wrt to her non-neighbors**
- G-EEF is
  - **EF** if **G is the complete graph** (or every node has degree ≥ n-2)
  - **EEF** if **G is the empty graph**

More implications

- Social graphs G and H over the same set of nodes
  - Rich hierarchy of fairness notions between EF and EEF
  - If **G is a subgraph of H**, then **H-EEF implies G-EEF**
  - Otherwise, there is an n-agent allocation instance that has an **H-EEF but no G-EEF** allocation



More fairness notions

- **G-PEF**
  - Again, using a social graph G
  - P stands for **proportionality**
  - Combined with **EF**
- See also:
  - Aziz, C., Bouveret, Giagkousi, & Lang (2018)

Summary

- Basic notions
- Fairness vs. efficiency
- EF1: a relaxed version of envy-freeness
- More fairness notions
- Fairness, knowledge, and social constraints

What didn't we cover?

- Algorithms for EFX allocations with **item donations**
  - C., Gravin, & Huang (2019)
- **Connected bundles**
  - Bilo, C., Flammini, Igarashi, Monaco, Peters, Vince, & Zwicker (2019)
- Chores or mixed settings with **chores and goods**
  - Aziz, C., Igarashi, & Walsh (2019)

Last slide

- Please, send me any questions, remarks, or proofs at **caragian@ceid.upatras.gr**

Last slide

- Please, send me any questions, remarks, or proofs at **caragian@ceid.upatras.gr**

Thank you!