Improved Lower Bounds on the Price of Stability of Undirected Network Design Games*

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Abstract. Bounding the price of stability of undirected network design games with fair cost allocation is a challenging open problem in the Algorithmic Game Theory research agenda. Even though the generalization of such games in directed networks is well understood in terms of the price of stability (it is exactly H_n, the n-th harmonic number, for games with n players), far less is known for network design games in undirected networks. The upper bound carries over to this case as well while the best known lower bound is $42/23 \approx 1.826$. For more restricted but interesting variants of such games such as broadcast and multicast games, sublogarithmic upper bounds are known while the best known lower bound is $12/7 \approx 1.714$. In the current paper, we improve the lower bounds as follows. We break the psychological barrier of 2 by showing that the price of stability of undirected network design games is at least $348/155 \approx 2.245$. Our proof uses a recursive construction of a network design game with a simple gadget as the main building block. For broadcast and multicast games, we present new lower bounds of $20/11 \approx 1.818$ and 1.862, respectively.

1 Introduction

Network design is among the most well-studied problems in the combinatorial optimization literature. A natural definition is as follows. We are given a graph consisting of a set of nodes and edges among them representing potential links. Each edge has an associated cost which corresponds to the cost for establishing the corresponding link. We are also given connectivity requirements as pairs of source-destination nodes. The objective is to compute a subgraph of the original graph of minimum total cost that satisfies the connectivity requirements. In other words, we seek to establish a network that satisfies the connectivity requirements

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at the minimum cost. This optimization problem is known as Minimum Steiner Forest and generalizes well-studied problems such as the Minimum Spanning Tree and Minimum Steiner Tree.

In this paper, we consider a game-theoretic variant of network design that was first considered in [2]. Instead of considering the connectivity requirements as a global goal, we assume that each connectivity requirement is desired by a different player. The players participate in a non-cooperative game; each of them selects as her strategy a path from her source to the destination and is charged for part of the cost of the edges she uses. According to the fair cost sharing scheme we consider in the current paper, the cost of an edge is shared equally among the players using the edge. The *social cost* of an *assignment* (i.e., a snapshot of players' strategies) is the cost of the edges contained in at least one path. An optimal assignment would contain a set of edges of minimum cost so that the connectivity requirements of the players are satisfied. Unfortunately, this does not necessarily mean that all players are satisfied with this assignment since a player may have an incentive to deviate from its path to another one so that her individual cost is smaller. Eventually, the players will reach a set of strategies (and a corresponding network) that satisfies their connectivity requirements and in which no player has any incentive to deviate to another path; such outcomes are known as *Nash equilibria*. Interestingly, even though the optimal solution is always a forest, Nash equilibria may contain cycles.

The non-optimality of the outcomes of *network design games* (which is typical when selfish behavior comes into play) leads to the following question that has been a main line of research in Algorithmic Game Theory: How is the system performance affected by selfish behavior? The notion of the *price of anarchy* (introduced in [8]; see also [10]) quantifies the deterioration of performance. In general terms, it is defined as the ratio of the social cost of the worst possible Nash equilibrium over the optimal cost. Hence, it is pessimistic in nature and (as its name suggests) provides a worst-case guarantee for conditions of total anarchy. Instead, the notion of the *price of stability* (introduced in [2]) is optimistic in nature. It is defined as the ratio of the social cost of the best equilibrium over the optimal cost and essentially asks: What is the best one can hope for the system performance given that the players are selfish?

The aim of the current paper is to determine better lower bounds on the price of stability for network design games in an attempt to understand the effect of selfishness on the efficiency of outcomes in such games. We usually refer to network design games as multi-source network design games in order to capture the most general case in which players may have different sources. An interesting variant is when each player wishes to connect a particular common node, which we call the *root*, with her destination node; we refer to such network design games as multicast games. An interesting special case of multicast games is the class of broadcast games: in such games, there is a player for each non-root node of the network that has this node as her destination.

The existence of Nash equilibria in network design games is guaranteed by a potential function argument. Rosenthal [11] defined a potential function over all assignments of a network design game so that the difference in the potential of two assignments that differ in the strategy of a single player equals the difference of the cost of that player in these assignments; hence, an assignment that locally minimizes the potential function is a Nash equilibrium. So, the price of stability is well-defined in network design games. Anshelevich *et al.* [2] considered network design games in directed graphs and proved that the price of stability is at most H_n. Their proof considers a Nash equilibrium that can be reached from an optimal assignment when the players make arbitrary selfish moves. The main argument used is that the potential of the Nash equilibrium is strictly smaller than that of the optimal assignment and the proof follows due to the fact that the potential function of Rosenthal approximates the social cost of an assignment within a factor of at most H_n. This approach suggests a general technique for bounding the price of stability and has been extended to other games as well; see [3,5]. For directed graphs, the bound of H_n was also proved to be tight [2]. Although the upper bound proof carries over to undirected network design games, the lower bound does not. The bound of H_n is the only known upper bound for multi-source network design games in undirected graphs. Better upper bounds are known for single-source games. For broadcast games, Fiat *et al.* [7] proved an upper bound of $O(\log \log n)$ while Li [9] presented an upper bound of $O(\log n / \log \log n)$ for multicast games. These bounds are not known to be tight either and, actually, the gap with the corresponding lower bounds is large. For single-source games, in the full version of [7] Fiat *et al.* present a lower bound of $12/7 \approx 1.714$; their construction uses a broadcast game. This was the best lower bound known for the multi-source case as well until the recent work of Christodoulou *et al.* [6] who presented an improved lower bound of $42/23 \approx 1.826$. Higher (i.e., super-constant) lower bounds are only known for weighted variants of network design games (see [1,4]).

In this paper, we present better lower bounds for general undirected network design games, as well as for the restricted variants of broadcast and multicast games. For the general case, we present a game that has price of stability at least $348/155 \approx 2.245$, improving the previously best known lower bound of [6]. Our proof uses a simple gadget as the main building block which is augmented by a recursive construction to our lower bound instance. The particular recursive construction of the game has two advantages. Essentially, the recursive construction blows up the price of stability of the gadget used as the main building block. Furthermore, recursion allows to handle successfully the technical difficulties in the analysis. We believe that our construction could be extended to use more complicated gadgets as building blocks that would probably lead to better lower bounds on the price of stability For multicast games, we present a lower bound of 1.862. Our proof uses a game on a graph with a particular structure. For this game, we prove sufficient conditions on the edge costs of the graph so that a particular assignment is the unique Nash equilibrium. Then, the construction that yields the lower bound is the solution of a linear program which has the edge costs as variables, the sufficient conditions as constraints, an additional constraint that upper-bounds the optimal cost by 1, and its objective is to maximize the cost of the unique Nash equilibrium. The particular lower bound was obtained in a game with 100 players using the linear programming solver of Matlab. A slight variation of this construction yields our lower bound for broadcast games. In this case, we are able to obtain a more compact set of sufficient conditions so that there is a unique Nash equilibrium. As a result, we have a formal proof that the price of stability approaches $20/11 \approx 1.818$ when the number of players is large.

2 Preliminaries

In an undirected network design game, we are given an undirected graph $G = (V, E)$ in which each edge $e \in E$ has a non-negative cost c_e. There are n players; player i wishes to establish a connection between two nodes $s_i, t_i \in V$ called the source and destination node of player i, respectively. The set of strategies available to player i consists of all paths connecting nodes s_i and t_i in G. We call an *assignment* any set of strategies σ, with one strategy per player. Given an assignment σ, let $n_e(\sigma)$ be the number of players using edge e in σ. Then, the cost of player i in σ is defined as $\text{cost}_i(\sigma) = \sum_{e \in \sigma_i} \frac{c_e}{n_e(\sigma)}$. Let $G(\sigma)$ be the subgraph of G which contains the edges of G that are used by at least one player in assignment σ. The social cost of the assignment σ is simply the total cost of the edges in $G(\sigma)$ which coincides with the sum of the costs of the players.

An assignment σ is called a *Nash equilibrium* if for any player i and for any other assignment σ' that differs from σ only in the strategy of player i, it holds $\text{cost}_i(\sigma) \leq \text{cost}_i(\sigma')$. It can be easily seen that any Nash equilibrium is a *proper* assignment, in the sense that the edges used by any pair of players do not form any cycle. The *price of stability* of a network design game is defined as the ratio of the minimum social cost among all Nash equilibria over the optimal cost.

Network design games with $s_i = s$ for any player i are called *multicast* games. We refer to node s as the *root* node. Multicast games in which there is one player for any non-root node that has this node as destination are called *broadcast* games. We also use the term *multi-source* games to refer to the general class of undirected network design games and the term *single-source* games in order to refer to multicast and broadcast games.

3 The Lower Bound for Multi-source Games

In this section, we prove the following theorem.

Theorem 1. *For any $\delta > 0$, there exists an undirected network design game with price of stability at least $348/155 - \delta$.*

We will construct a network design game on a connected undirected graph so that there is a distinct player associated with each edge of the graph that wishes to connect the endpoints of the edge. The construction uses integer parameters $k \geq 3$ and $t \geq 2$. We start with the *gadget* construction depicted in Figure 1a.

We use the terms *left* and *right gadget player* for the players associated with the left and right gadget edge of a gadget, respectively. We also use the term *floor players* for the players associated with floor edges. Given an edge e, we build a *block under this edge* by putting k gadgets so that the leftmost node of the first gadget coincides with the left endpoint of e, the rightmost node of i-th gadget coincides with the leftmost node of the $(i+1)$-th gadget for $i = 1, ..., k-1$, and the rightmost node of the k-th gadget coincides with the right endpoint of e (see Figure 1b). We refer to e as the *ceiling edge* of the block.

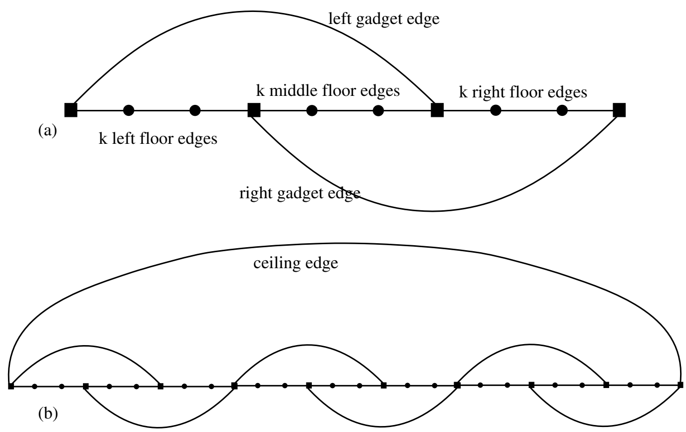

Fig. 1. (a) The gadget used in the proof of Theorem 1. (b) The construction of a block under a ceiling edge (with $k = 3$).

We set $x = 28/109$, $y = 33/109$, $z = 30/109 - \epsilon$, $w = 35/109 - \epsilon$, and $\alpha = 63/218 - \epsilon$, where ϵ is a negligibly small but strictly positive number. If g denotes the cost of the ceiling edge, then the cost of the edges in each gadget of the block under it are defined as follows: $\frac{xg}{\alpha k^2}$ for each of the left floor edges, $\frac{(1-x-y)g}{\alpha k^2}$ for each of the middle floor edges, $\frac{yg}{\alpha k^2}$ for each of the right floor edges, $\frac{zg}{\alpha k}$ for the left gadget edge, and $\frac{wg}{\alpha k}$ for the right gadget edge. So, the total cost of the floor edges of the block is g/α while the total cost of all edges of the block is $g(1+z+w)/\alpha$.

Now, our construction starts with a *roof edge* of cost 1 (and an associated *roof player*) and a block under it. The roof edge has level t and the block under it has level $t-1$. We build blocks of level $t-2$ by building a block under each of the floor edges of the block of level $t-1$. We continue recursively and define all blocks down to level 1. Clearly, for $j = 1, ..., t-1$, the total cost of the floor edges of level j is $g\alpha^{j-t}$ while the total cost of all edges of level j is $g(1+z+w)\alpha^{j-t}$.

Hence, the total cost of the edges in the graph is

$$1 + \sum_{i=1}^{t-1} (1+z+w)\alpha^{-i} = \frac{348 - 436\epsilon}{155 + 218\epsilon}\alpha^{1-t} - \frac{193 - 654\epsilon}{155 + 218\epsilon}$$

while the cost of the floor edges of level 1 is α^{1-t} and upper-bounds the optimal cost (since the floor edges of level one constitute a spanning tree of the whole graph). For any $\delta > 0$, we can set t and ϵ appropriately so that the ratio of the total cost of edges over the optimal cost is at least $348/155 - \delta$.

In order to complete the proof of the theorem, it suffices to prove that the assignment in which each player uses her direct edge is the unique Nash equilibrium; the rest of this section is devoted to proving this claim. We will refer to the players associated to floor edges (respectively, gadget edges) at blocks of level j as the floor players of level j (respectively, the gadget players of level j). A floor player of level j follows a *non-increasing* strategy if she uses neither a gadget edge of her gadget nor any edge of level $j' > j$. A gadget player of level j follows a non-increasing strategy if she does not use any edge of level $j' > j$. In the opposite case, we say that the player follows an *increasing* strategy.

In an assignment, a player may use a floor edge or connect its endpoints by being routed through the block under the edge. In the latter case, we say that the player *crosses* the floor edge. We also say that a player is *external* to a gadget (respectively, external to a block) if she does not correspond to any edge of the gadget (respectively, block) and uses or crosses its edges.

In a proper assignment, the sets of non-increasing strategies of the gadget players of a gadget can belong to one of the following types (Figure 2); any other set of non-increasing strategies violates the fact that the assignment is proper.

- Type A: Both gadget players use their direct edges.
- Type B: The left gadget player uses her direct edge and the right gadget player uses or crosses the middle and right floor edges.
- Type C: Both gadget players use the left gadget edge. The right gadget player uses or crosses the left and right floor edges as well.
- Type D: The right gadget player uses her direct edge and the left gadget player uses or crosses the left and middle floor edges.
- Type E: Both gadget players use the right gadget edge. The left gadget player uses or crosses the left and right floor edges as well.
- Type F: The left gadget player uses or crosses the left and middle floor edges and the right gadget player uses or crosses the middle and right floor edges.

We are ready to significantly restrict the structure of assignments we have to consider as candidates to be Nash equilibria.

Lemma 1. *At any Nash equilibrium, all players besides the roof player follow non-increasing strategies. Furthermore, at each block: either there are no external players and the gadget players have strategies of type A or there are $h > 0$ external players and each of them experiences cost more than g/h, where g is the cost of the ceiling edge of the block.*

Proof. Consider a Nash equilibrium. We will prove the claim inductively (on the block level). We will first prove it for the blocks of level 1. In this case, there is no block under any floor edge and players do not cross the floor edges.

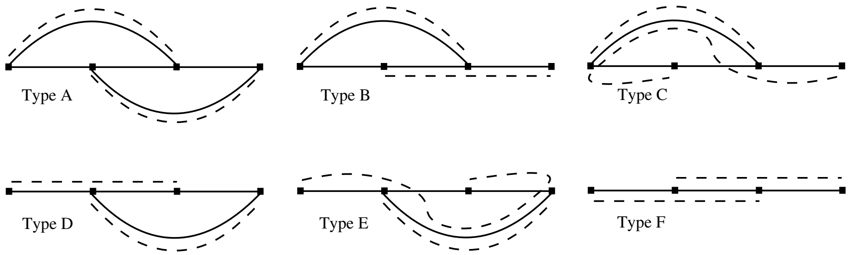

Fig. 2. The six possible types for the players of a gadget that follow non-increasing strategies. The dashed lines denote the paths used by the left and the right gadget player. Only the gadget edges that are used by some player are shown.

Consider a block of level 1 and assume that a floor player p follows an increasing strategy. Then, she should connect the endpoints of her floor edge to the two closest gagdet edge endpoints by using $k-1$ floor edges. Furthermore, observe that neither a gadget player of the same gadget nor an external player to this gadget uses these floor edges (since this would imply that they also use the direct edge of player p and the assignment would not be proper). Similarly, the players associated to the $k-1$ floor edges use their direct edges. Hence, player p uses each of the $k-1$ floor edges together with one floor player. Since $k \geq 3$, this means that the cost she experiences at the $k-1 \geq 2$ floor edges plus the non-zero cost she experiences at the other edges she uses is strictly larger than the cost of her direct edge and she would have an incentive to move to its direct edge. So, all floor players of the block follow non-increasing strategies.

Now, assume that a gadget player p follows an increasing strategy, i.e., her path contains the endpoints of her gadget. This means that there are no external players to the current block nor other gadget players within the current block that follow increasing strategies (any such player should connect the endpoints of the gadget of p and the assignment would not be proper). So, there are at least $k-1$ gadgets whose gadget (and floor) players follow non-increasing strategies.

We focus on such a gadget of the current block and assume that there are $h \geq 0$ external players; these can be players that are external to the block or a player from another gadget of the same block that follows an increasing strategy. In the inequalities below, we use the following claim.

Claim. Let ζ, η be positive integers. Then, $\frac{1}{\zeta+h} \geq \frac{\eta}{(\zeta+\eta)h}$ for any integer $h \geq \eta$.

We consider the six different cases for the strategies of the gadget players. If the strategies of the gadget players are of type A, then all the external players (if any) are routed either through the left gadget edge and the right floor edges of the gadget, or through the left floor edges and the right gadget edge, or through the left gadget edge, the middle floor edges, and the right gadget edge (any other case violates the fact that the assignment is proper). In the first subcase, the cost of each external player at the edges of the gadget is

$$\frac{g}{\alpha k}\left(\frac{z}{1+h}+\frac{y}{1+h}\right) \geq \frac{g}{\alpha k h}\left(\frac{z}{2}+\frac{y}{2}\right) = \frac{g}{\alpha k h}\left(\frac{63}{218}-\frac{\epsilon}{2}\right) > \frac{g}{kh}.$$

In the second subcase, the cost of each external player is

$$\frac{g}{\alpha k}\left(\frac{x}{1+h}+\frac{w}{1+h}\right) \geq \frac{g}{\alpha kh}\left(\frac{x}{2}+\frac{w}{2}\right) = \frac{g}{\alpha kh}\left(\frac{63}{218}-\frac{\epsilon}{2}\right) > \frac{g}{kh}.$$

In the third subcase, the cost of each external player is again

$$\frac{g}{\alpha k}\left(\frac{z}{1+h}+\frac{1-x-y}{1+h}+\frac{w}{1+h}\right) \geq \frac{g}{\alpha kh}\left(\frac{w}{2}+\frac{1-x-y}{2}+\frac{w}{2}\right) > \frac{g}{kh}.$$

If the strategies of the gadget players are of type B, all the external players are routed through the left gadget edge and the right floor edges. We will first show that $h \geq 2$. Indeed, if at most one external player is routed through the gadget, the cost of the right gadget player would be at least

$$\frac{g}{\alpha k}\left(\frac{1-x-y}{2}+\frac{y}{3}\right) = \frac{g}{\alpha k}\cdot\frac{35}{109} > \frac{gw}{\alpha k},$$

i.e., this player would have an incentive to move and use her direct edge. So, since $h \geq 2$, the cost of each external player at the edges of the gadget is

$$\frac{g}{\alpha k}\left(\frac{z}{1+h}+\frac{y}{2+h}\right) \geq \frac{g}{\alpha kh}\left(\frac{2z}{3}+\frac{y}{2}\right) = \frac{g}{\alpha kh}\left(\frac{73}{218}-\frac{2\epsilon}{3}\right) > \frac{g}{kh}.$$

If the strategies of the gadget players are of type C, all the external players are routed through the left gadget edge and the right floor edges. We will show again that $h \geq 2$. Indeed, if at most one external player is routed through the gadget, the cost of the right gadget player would be at least

$$\frac{g}{\alpha k}\left(\frac{x}{2}+\frac{z}{3}+\frac{y}{3}\right) = \frac{g}{\alpha k}\left(\frac{35}{109}-\frac{\epsilon}{3}\right) > \frac{gw}{\alpha k},$$

i.e., this player would have an incentive to move. So, since $h \geq 2$, the cost of each external player at the edges of the gadget is

$$\frac{g}{\alpha k}\left(\frac{z}{2+h}+\frac{y}{2+h}\right) \geq \frac{g}{\alpha kh}\left(\frac{z}{2}+\frac{y}{2}\right) = \frac{g}{\alpha kh}\left(\frac{63}{218}-\frac{\epsilon}{2}\right) > \frac{g}{kh}.$$

If the strategies of the gadget players are of type D, all the external players are routed through the left floor edges and the right gadget edge. We will show again that $h \geq 2$. Indeed, if at most one external player is routed through the gadget, the cost of the left gadget player would be at least

$$\frac{g}{\alpha k}\left(\frac{x}{3}+\frac{1-x-y}{2}\right) = \frac{g}{\alpha k}\cdot\frac{100}{327} > \frac{gz}{\alpha k},$$

i.e., this player would have an incentive to move. So, since $h \geq 2$, the cost of each external player at the edges of the gadget is

$$\frac{g}{\alpha k}\left(\frac{x}{2+h}+\frac{w}{1+h}\right) \geq \frac{g}{\alpha kh}\left(\frac{x}{2}+\frac{2w}{3}\right) = \frac{g}{\alpha kh}\left(\frac{112}{327}-\frac{2\epsilon}{3}\right) > \frac{g}{kh}.$$

If the strategies of the gadget players are of type E, then all the external players are routed through the left floor edges and the right gadget edge. We will show again that $h \geq 2$. Indeed, if at most one external player is routed through the gadget, the cost of the left gadget player would be at least

$$\frac{g}{\alpha k}\left(\frac{x}{3}+\frac{w}{3}+\frac{y}{2}\right)=\frac{g}{\alpha k}\left(\frac{75}{218}-\frac{\epsilon}{3}\right)>\frac{gz}{\alpha k},$$

i.e., this player would have an incentive to move. So, since $h \geq 2$, the cost of each external player at the edges of the gadget is

$$\frac{g}{\alpha k}\left(\frac{x}{2+h}+\frac{w}{2+h}\right)\geq\frac{g}{\alpha kh}\left(\frac{x}{2}+\frac{w}{2}\right)=\frac{g}{\alpha kh}\left(\frac{63}{218}-\frac{\epsilon}{2}\right)>\frac{g}{kh}.$$

If the strategies of the gadget players are of type F, then all the external players are routed through the floor edges. We will show that $h > 0$ in this case. Indeed, if there were no external players that are routed through the gadget, the cost of the left gadget player would be

$$\frac{g}{\alpha k}\left(\frac{x}{2}+\frac{1-x-y}{3}+\frac{y}{2}\right)=\frac{g}{\alpha k}\cdot\frac{93}{218}>\frac{gz}{\alpha k},$$

i.e., this player would have an incentive to move. So, the cost of each external player at the edges of the gadget is

$$\frac{g}{\alpha k}\left(\frac{x}{2+h}+\frac{1-x-y}{3+h}+\frac{y}{2+h}\right)\geq\frac{g}{\alpha kh}\left(\frac{x}{3}+\frac{1-x-y}{4}+\frac{y}{3}\right)>\frac{g}{kh}.$$

Now, consider again the gadget player p which follows an increasing strategy. In each of the other $k-1$ gadgets of the same block, the gadget players have strategies of types A or F and the cost player p experiences at the edges of the gadget is more than $\frac{g}{k}$. Her total cost through the edges of the $k-1 \geq 2$ gadgets different than her own one would be $\frac{g(k-1)}{k} \geq \frac{g}{\alpha k}\max\{z, w\}$, i.e., she would have an incentive to move and use her direct edge instead. So, all gadget players of the block follow non-increasing strategies as well.

Now, assume that no external player is routed through the block. Then, by the above discussion, the only case in which the gadget players of a gadget do not have an incentive to move is when they follow strategies of type A. If one external player is routed through the block, then the gadget players follow strategies of type A or F and the cost experienced by the external player at each gadget is more than g/k, i.e., more than g in total. If $h \geq 2$ external players are routed through the block, then each of them experiences cost more than $\frac{g}{kh}$ at each gadget, i.e., more than g/h in total.

We have completed the proof of the base of the induction. Now, assuming that the statement is true for blocks of levels up to j, we have to prove it for blocks of level $j+1$. The proof of the induction step is almost identical to the proof of the induction base. The only difference is that, now, a player may cross a floor edge in order to connect its endpoints. Then, when h players cross a floor edge, they are external to the block under the edge and (by the induction hypothesis) the cost they experience when crossing the edge is more than its cost over h (as opposed to exactly its cost over h which we had in the induction base). This inequality (instead of equality) does not affect any of the inequalities above and the proof of the induction step completes in the very same way. □

Lemma 2. *At any Nash equilibrium, there are no external players at any block.*

Proof. Assume that this is not the case and consider a Nash equilibrium with external players at some block. Consider the block of highest level that has some external player routed through it. Then, it is either the block of level $t-1$ or (by Lemma 1) some block under a floor edge of a gadget of the higher-level block whose gadget players follows strategies of type A. In both cases, exactly one player is routed through the block (i.e., the player corresponding to its ceiling edge) and, by Lemma 1, her cost at the edges of the block is more than the cost of the ceiling edge of the block. Hence, this player has an incentive to move and use the ceiling edge instead. The lemma follows. □

Now, Theorem 1 follows by Lemmas 1 and 2 since they imply that the assignment in which every player uses her direct edge is the unique Nash equilibrium.

4 Lower Bounds for Single-Source Games

In this section, we present our lower bounds for multicast and broadcast games. We note that since all players have a common source node in such games, in any proper assignment the set of edges that are used by at least one player is a tree that is rooted at the source node and spans the destinations of all players. Also, any such tree defines in a unique way the strategies of the players in a proper assignment. So, when considering Nash equilibria in multicast or broadcast games, it suffices to restrict our attention to assignments defined by trees spanning the root node and the destination nodes of all players. We refer to them as *multicast* or *broadcast trees* depending on whether the game is a multicast or a broadcast game.

Our lower bound for multicast games uses the graph M_n depicted in Figure 3. There are n players; player i wishes to connect node s to node t_i. The cost of the edges is defined by the tuple $C = (x_2, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n)$. We denote by τ the multicast tree formed by the edges (s, t_i) for $i = 1, ..., n$. The next lemma provides a sufficient condition so that the assignment defined by tree τ is the unique Nash equilibrium of the multicast game on M_n; its formal proof is omitted due to lack of space.

Lemma 3. *The assignment defined by tree τ is the unique Nash equilibrium of the multicast game on graph M_n if C is such that for $i = 2, ..., n$ and for $k = 1, ..., i-1$ it holds*

$$z_k < \frac{z_i}{\min\{2i-2k, n-k\}+1} + \frac{y_i}{\min\{2i-2k, n-k\}} + \sum_{p=0}^{i-k-1} \frac{x_{i-p}}{i-k-p} + y_k,$$

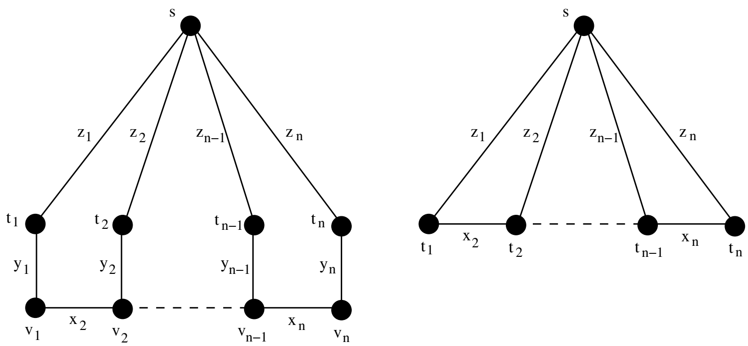

Fig. 3. The graphs M_n (left) and B_n (right)

and for $i = 1, ..., n-1$ *and for* $j = i+1, ..., n$ *it holds*

$$z_j < \frac{z_i}{\min\{2j-2i, j\}} + \frac{y_i}{\min\{2j-2i, j\}-1} + \sum_{p=1}^{j-i} \frac{x_{i+p}}{j-i-p+1} + y_j.$$

Now, we can use Lemma 3 to obtain lower bounds on the price of stability of multicast games by solving the following linear program. The variables of the linear program are the edge costs of the tuple C. The objective is to maximize the cost $\sum_{i=1}^{n} z_i$ of tree τ subject to the two sets of constraints in the statement of Lemma 3 and the additional constraint $z_1 + \sum_{i=2}^{n} x_i + \sum_{i=1}^{n} y_i \leq 1$ which upper-bounds the optimal cost by 1 (observe that the left-hand side of this constraint is the cost of the multicast tree containing all edges of M_n besides (s, t_i) for $i = 2, ..., n$). Then, the objective value of this linear program denotes the price of stability of the multicast game on M_n for the particular values of the edge costs that correspond to the solution of the linear program. We obtained our lower bound on the price of stability using the linear programming solver of Matlab. Note that we have used $n = 100$ and have simulated the strict inequalities in the conditions of Lemma 3 by using standard inequalities and adding a constant of 10^{-6} on their left-hand side. The following statement summarizes our best observed lower bound.

Theorem 2. *There exists a multicast game with price of stability at least* 1.862*.*

Our lower bound for broadcast games uses the graph B_n depicted at the right part of Figure 3. In this case, the cost of the edges is defined by the tuple $C = (x_2, \ldots, x_n, z_1, \ldots, z_n)$. Again, there are n players; player i wishes to connect node s to node t_i. Denote by τ the broadcast tree formed by the edges (s, t_i) for $i = 1, ..., n$. Observe that the graph B_n is obtained from M_n by contracting the edges (t_i, v_i). Hence, any Nash equilibrium of the multicast game on graph M_n with $y_i = 0$ for $i = 1, ..., n$ corresponds to a Nash equilibrium of the broadcast game on graph B_n of the same cost (and vice versa) while the cost of the optimal assignment is the same in both cases. So, we can apply the same technique we used above by further constraining the variable y_i to be zero for $i = 1, ..., n$. Fortunately, we are able to define a much more compact set of conditions for C in order to guarantee that the assignment defined by τ is the unique Nash equilibrium of the broadcast game on B_n. Our related result is the following; due to lack of space, the formal proof is omitted.

Theorem 3. *For any $\delta > 0$, there exists a broadcast game with price of stability at least $20/11 - \delta$.*

We remark that the graph B_n has the same structure with the lower bound construction of [7] albeit with a different definition of the edge costs that yields the improved lower bound on the price of stability.

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