On the Complexity of Wavelength Converters*

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Abstract. In this paper we present a greedy wavelength routing algorithm that allocates a total bandwidth of $w(l)$ wavelengths to any set of requests of load l (where load is defined as the maximum number of requests that go through any directed fiber link) and we give sufficient conditions for correct operation of the algorithm when applied to binary tree networks. We exploit properties of Ramanujan graphs to show that (for the case of binary tree networks) our algorithm increases the bandwidth utilized compared to the algorithm presented in [3]. Furthermore, we use another class of graphs called dispersers, to implement wavelength converters of asymptotically optimal complexity with respect to their size (the number of all possible conversions). We prove that their use leads to optimal and nearly–optimal bandwidth allocation even in a greedy manner.

1 Introduction

Optical fiber is rapidly becoming the standard transmission medium for networks. Networks using optical transmission and maintaining optical data paths through the nodes are called all–optical networks. Wavelength division multiplexing (WDM) technology establishes connectivity by finding transmitter–receiver paths and assigning a wavelength to each path such that no two paths going through the same link use the same wavelength. Optical bandwidth is the number of available wavelengths.

Current techniques for optical bandwidth allocation cannot guarantee high bandwidth utilization under the worst conditions. A promising solution for efficient use of bandwidth is wavelength conversion. Devices called wavelength converters are located at the nodes of the network and they can change the wavelength assigned to a transmitter–receiver path up to a node and allocate a different wavelength at the rest of the path.

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Related Work. Several authors have already addressed the case of no wavelength conversion in tree networks. Raghavan and Upfal [10] showed that routing requests of maximum load l per link of undirected trees can be satisfied using $3l/2$ optical wavelengths and their arguments extend to give a $2l$ bound for the directed case. Mihail et al. [8] address the directed case. Their main result is a $15l/8$ upper bound which was improved to $7l/4$ in [4] and independently in [6]. Kaklamanis et al. [5] present a greedy algorithm that routes a set of requests of maximum load l using at most $5l/3$ and prove that no greedy algorithm can go below $5l/3$ in general.

Models for wavelength routing with converters in trees have been studied in [1,2,3]. The authors present in [1] how to obtain optimal routing in binary tree networks that support l wavelengths using converters of degree $2\sqrt{l}-1$. The model used is actually the one addressed throughout the current paper. A model with many wavelength converters in each node of the network is studied in [2]. In that work it is shown how to obtain nearly optimal and optimal bandwidth allocation using converters of constant degree. This result refers to binary trees as well. A wavelength routing algorithm of any pattern of requests of load l in arbitrary tree networks with $3l/2 + o(l)$ wavelengths using converters of poly-logarithmic degree is also presented in [2]. Gargano in [3] presents an algorithm that guarantees efficient wavelength routing in arbitrary trees under a different network model of limited wavelength conversion. She also extends the optimal result of [2] in quasi-binary trees.

Network Model. We model the underlying fiber network as a directed graph. Connectivity requests are ordered pairs of nodes, to be thought of as transmitter–receiver paths. For networks with unique transmitter–receiver paths (such as trees), the load l of a directed fiber link is the number of paths going through the link. Each directed fiber link can support $w(l)$ wavelengths $\omega_1, \omega_2, ..., \omega_{w(l)}$, with distinct optical frequencies.

Current approaches to the wavelength assignment problem in trees use greedy algorithms [4,5,6,8]. Intuitively we can think of wavelengths as colors and the procedure of wavelength assignment as coloring. A greedy algorithm visits the network in a top to bottom manner and at each vertex v colors all requests that touch vertex v and are still uncolored. Moreover, once a request has been colored it is never recolored again. Although greedy algorithms are important as they are very simple and amenable of being implemented in a distributed setting, they cannot guarantee a bandwidth utilization higher than 60% [5]. Furthermore, no algorithm can guarantee bandwidth utilization better than 80% [6] if wavelength conversion is not supported.

A wavelength converter is represented by a bipartite graph $G(U, V, E)$. For each wavelength ω_i, there exist two vertices $u_i \in U$ and $v_i \in V$ in the bipartite graph ($|U| = |V| = w(l)$). The set of edges E is defined as follows: $(u_i, v_j) \in E \Leftrightarrow$ *the wavelength* ω_i *can be converted to the wavelength* ω_j.

Previous work on wavelength conversion consider that the cost of the converters depends on their *wavelength degree*, i.e. the maximum degree of any node $u \in U$ of the corresponding bipartite graph. Another factor that is expected to influence the cost of a converter is its *size*, i.e. the number of edges in its bipartite graph (the number of all possible conversions).

We can define the class of greedy algorithms for networks that support wavelength conversion. Such algorithms visit the network in a DFS manner but the functionality in a node u that supports wavelength conversion is different. In this case, the greedy algorithm colors the segments of a request that touch u. Thus, a greedy algorithm may assign a different color for a request that has been colored in a previous step, so that the wavelength corresponding to the old color can be converted to the new one by the wavelength converter located at node u which is responsible for the conversion of that request.

Converters are placed at network nodes as follows. The tree network is rooted at a predefined node. At each non–leaf node u of degree $d+1, d > 1$ with a parent f and children $v_1, ..., v_d$, there are $2d$ converters $C_1, C_2, ..., C_{2d-1}, C_{2d}$. Converter $C_{2i-1}, 1 \leq i \leq d$ is responsible for the conversion of the wavelengths assigned to the set of requests R_{2i-1}, which comes from the parent node f and goes to the child v_i. Converter $C_{2i}, 1 \leq i \leq d$ is responsible for the conversion of the wavelengths assigned to the set of requests R_{2i}, which comes from the child v_i and goes to the parent node f.

Summary of results. We start with some preliminary definitions and lemmas in section 2. In section 3 we present a greedy wavelength routing algorithm for binary tree networks with converters. As a first application, we use Ramanujan graphs as converters and prove that our algorithm utilizes the two thirds of the bandwidth wasted by the algorithm presented in [3] (when a Ramanujan graph of a given degree is used as converter in both cases). Next we exploit properties of dispersers to present upper bounds on the size of the converters in order to greedily achieve optimal and almost optimal ($w(l) = l+o(l)$) bandwidth allocation. These results are presented in section 4. In section 5 we prove that our upper bounds are asymptotically tight for greedy algorithms. Also, we prove that the $2\sqrt{l}-1$ upper bound on the degree of the converters presented in [1] is asymptotically tight as well.

2 Preliminaries

As in [2,3] we will exploit expansion properties of k–regular bipartite graphs to build wavelength converters. The following lemma states Tanner's inequality which relates the expansion of a graph with the value of its second eigenvalue.

Lemma 1. *Let $G(U, V, E)$ with $|U| = |V| = n$ be a k–regular bipartite graph. For any $X \subseteq U$,*

$$\frac{|N(X)|}{|X|} \geq \frac{k^2}{\lambda^2 + (k^2 - \lambda^2)|X|/n}$$

where λ is the second largest eigenvalue of the adjacency matrix of G in absolute value.

Ramanujan graphs have the property that their second eigenvalue is upper bounded by $2\sqrt{k-1}$. Furthermore, these graphs have been explicitly constructed in [7]. In this work we also exploit properties of another class of bipartite graphs called dispersers.

Definition 1. *[11] A bipartite graph $G = (U, V, E)$ is an (K, ε) disperser if for each subset $A \subseteq U$ of size K there are at least $(1 - \varepsilon)|V|$ vertices of V that are adjacent to A.*

Sipser showed in [11] that such graphs exist. An almost optimal explicit construction of dispersers is reported in [12].

Lemma 2. *[11] There exist a (K, ϵ)-disperser $G(U, V, E)$ with $|U| = N$ and $|V| = M$, such that each node $v \in U$ has degree $\max\{\frac{2M}{K}(1 + \ln 1/\epsilon), \frac{2}{\epsilon}(1 + \ln N/K\}$.*

The properties of dispersers have been used for the construction of asymptotically optimal depth–two superconcentrators.

Definition 2. *A N-superconcentrator is a directed graph with N distinguished vertices called inputs, and N other distinguished vertices called outputs, such that for any $1 \leq k \leq N$, any set X of k inputs and any set Y of k outputs, there exist k vertex-disjoint paths from X to Y. The size of a superconcentrator G is the number of edges in it, and the depth of G is the number of edges in the longest path from an input to an output.*

Lemma 3. *[9] Depth-two N-superconcentrators have size $\Theta\left(N\frac{\log^2 N}{\log\log N}\right)$.*

The construction of [9] produces a depth–two superconcentrator $H(U, V, W, E)$ with $|U| = |W| = N$ and $|V| = 2N$. We slightly extend their arguments to obtain the following lemma.

Lemma 4. *There exist a depth two superconcentrator $G(U, V, W, E)$ of size $O\left(N\frac{\log^2 N}{\log\log N}\right)$ with $U = \{u_i | 1 \leq i \leq N\}$, $V = \{v_i | 1 \leq i \leq N\}$, $W = \{w_i | 1 \leq i \leq N\}$, such that for any $1 \leq i, j \leq N$, $(u_i, v_j) \in E \Leftrightarrow (v_i, w_j) \in E$.*

3 The Wavelength Routing Algorithm

In this section we describe a greedy wavelength routing algorithm that allocates optical bandwidth of $w(l)$ available wavelengths to any set of communication requests of load l on a binary tree network. Four wavelength converters C_1,C_2,C_3,C_4 are placed at each node as described. We denote by S the set of available wavelengths (colors).

Starting from a node, the algorithm computes a DFS numbering of the nodes of the tree. The algorithm proceeds in phases, one per each node u of the tree.

The nodes are considered following their depth–first numbering. The phase associated with node u assumes that we already have a partial proper coloring where the segments of requests (paths) that touch nodes with numbers strictly smaller than u's have been colored and no other segments have been colored.

Consider a phase of the algorithm associated with a node u. Let f be the parent node of u and v,w its children. Let A_1 be the set of colors assigned to the set R_1 of the requests from f to v, A_2 the set of colors assigned to the set R_2 of the requests from v to f, A_3 the set of colors assigned to the set R_3 of requests from f to w, and A_4 the set of colors assigned to the set R_4 of requests from w to f. These colors are used only in segment (f,u). Also let R_5 be the set of requests from v to w and R_6 the set of requests from w to v. We ignore requests than start or end at u since they can be colored easily. The algorithm performs two independent steps:

Step 1: Converters C_1 and C_4 are set in such way that:

- C_1 converts the color assigned to the segment (f,u) of each request $r \in R_1$ to a color that is assigned to the segment (u,v) of r.
- C_4 converts a color that is assigned to the segment (w,u) of each request $r \in R_4$ to the color assigned to the segment (u,f) of r.

Let A'_1 and A'_4 the set of colors assigned to the segments (u,v) and (w,u) of the requests R_1 and R_4, respectively. The algorithm maintains the following invariants:

1. Segments (u,v) of the requests R_1 are assigned different colors ($|A'_1| = |A_1|$) and segments (w,u) of the requests R_4 are assigned different colors ($|A'_4| = |A_4|$).
2. $|A'_1 \cap A'_4| \geq \min\{|R_1|,|R_4|\} - w(l) + l$.

Requests R_6 are assigned colors from $S \backslash (A'_1 \cup A'_4)$.

Step 2: This step is symmetric to step 1. Converters C_2 and C_3 are set and the uncolored segments of requests R_2,R_3, and R_5 are colored in a similar way.

The following lemma gives sufficient conditions for the correctness of our algorithm.

Lemma 5. *Let $H(A,B,E(H))$ be the bipartite graph that corresponds to the wavelength converter ($A = \{a_i|1 \leq i \leq w(l)\}$ and $B = \{b_i|1 \leq i \leq w(l)\}$). Consider the three-level graph $G(U,V,W,E(G))$ such that $U = \{u_i|1 \leq i \leq w(l)\}$,$V = \{v_i|1 \leq i \leq w(l)\}$, $W = \{w_i|1 \leq i \leq w(l)\}$, and*

$$E(G) = \{(u_i,v_j)|(a_i,b_j) \in E(H)\} \cup \{(v_j,w_i)|(a_i,b_j) \in E(H)\}.$$

The wavelength routing algorithm correctly assigns $w(l)$ colors to any set of communication requests of load l on a binary tree network with wavelength converters H if for any sets $\Gamma_1 \subseteq U$ and $\Gamma_2 \subseteq W$ of cardinalities $|\Gamma_1|,|\Gamma_2| \leq l$, the following conditions hold:

1. *There exist sets* $X \subseteq \Gamma_1$ *and* $Z \subseteq \Gamma_2$ *of cardinality* $k = |X| = |Z| \geq \min\{|\Gamma_1|, |\Gamma_2|\} - w(l) + l$ *such that there exist* k *vertex disjoint paths from* X *to* Z.
2. *Let* $Y \subseteq V$ *the set of vertices of* V *that belongs to the* k *disjoint paths. The set* $\Gamma_1 \backslash X$ *has a matching of cardinality* $|\Gamma_1 \backslash X|$ *with vertices of* $V \backslash Y$, *and the set* $\Gamma_2 \backslash Z$ *has a matching of cardinality* $|\Gamma_2 \backslash Z|$ *with vertices of* $V \backslash Y$.

Proof. We concentrate on step 1 of the algorithm at a node u. The proof is identical for step 2. Assume that at the current step the algorithm has colored the segments (f, u) of the requests R_1 and R_4. Consider the color assigned to the segment (f, u) of a request $r_1 \in R_1$ as a vertex of U and a color assigned to the segment (u, f) of a request $r_4 \in R_4$ as a vertex of W. The mate vertex of V that is connected with an edge (implied by the two conditions) to a vertex of U is the color assigned to the segment (u, v) of r_1, while the mate vertex of V that is connected with an edge to a vertex of W is the color that will be assigned to the segment (w, u) of r_4.

Both conditions maintain that requests of R_1 are assigned different colors in segment (u, v) (similarly for the requests of R_4 in segment (w, u)). Furthermore, condition 1 guarantees that the invariant 2 of the algorithm is satisfied. Thus the algorithm can assign the remaining colors (corresponding to vertices of V that have no mates to U or V) to requests of R_6. □

The algorithm is correct even if sets Γ_1 and Γ_2 have the same cardinality. The second condition can be eliminated if the bipartite graph H of the wavelength converter has a perfect matching (which always holds). Formally

Lemma 6. *Let* H *and* G *be the graphs defined in lemma 5.* H *has a perfect matching and for any sets* $\Gamma_1 \subseteq U$, $\Gamma_2 \subseteq W$ *of the same cardinality* g, *there exist sets* $X \subseteq \Gamma_1$ *and* $Z \subseteq \Gamma_2$ *of cardinality* $k = |X| = |Z| \geq g - w(l) + l$ *such that there exist* k *vertex disjoint paths from* X *to* Z. *Let* $Y \subseteq V$ *the set of vertices of* V *that belongs to the* k *disjoint paths. Then the set* $\Gamma_1 \backslash X$ *has a matching of maximum cardinality* $g - k$ *with vertices of* $V \backslash Y$, *and the set* $\Gamma_2 \backslash Z$ *has a matching of maximum cardinality* $g - k$ *with vertices of* $V \backslash Y$.

As a corollary, when the cardinality of R_1 and R_4 is small, no vertex disjoint paths need to be found. In particular,

Corollary 1. *Consider a node* u *of a binary tree network of* $w(l) > l$ *available wavelengths and wavelength converters* H, *and a pattern of requests of maximum load* l. *If* $|R_1| = |R_4| \leq w(l) - l$, *then the uncolored segments of* R_1, R_4, *and* R_6 *have a proper wavelength assignment with* $w(l)$ *wavelengths.*

The following lemma gives a condition for the existence of vertex disjoint paths when the cardinality of R_1 and R_4 is large.

Lemma 7. *Let* $G(U, V, W, E)$ *be a three level graph with* $|U| = |W| = w(l)$. *If for any sets* $\Gamma_1 \subseteq U$ *and* $\Gamma_2 \subseteq W$ *of cardinality* k *with* $w(l) - l < k \leq l$ *there exist* $k - w(l) + l$ *common neighbors in* V, *then for any sets* $A \subseteq U, B \subseteq W$ *of cardinality* $k \leq l$, *there exist subsets* $X \subseteq A$ *and* $Z \subseteq B$ *such that there exist* $k - w(l) + l$ *vertex disjoint paths from* X *to* Z.

Proof. By Menger's theorem proving that the minimum cut has size at least $k - w(l) + l$. □

4 Upper Bounds

Theorem 1. *Let T be a binary tree network and $w(l)$ be the available number of wavelengths on each link. Using (explicitly constructible) converters of degree k, it is possible to greedily assign wavelengths to any set of requests of load $l \leq \left(1 - \frac{4(k-1)}{3(k-2)^2}\right) w(l)$.*

Proof. We use a k–regular Ramanujan graph H as converter with $w(l)$ wavelengths. We construct the three level graph G. Let $X \subseteq U$, such that $|X| > w(l) - l$. It can be verified that

$$|N(X)| \geq \frac{k^2|X|}{4(k-1) + (k-2)^2|X|/w(l)} \geq \frac{l + |X|}{2}$$

where $N(X)$ is the neighborhood of X in V. Thus, for any sets $\Gamma_1 \subseteq U$ and $\Gamma_2 \subseteq W$ of cardinality k with $w(l) - l < k \leq l$, there exist $k - w(l) + l$ common neighbors in V. H has a perfect matching, thus, by lemmas 7 and 6 the conditions of lemma 5 hold. The theorem follows. □

The result of [3] and theorem 1 imply the following.

Theorem 2. *Let $1 < f(l) = o(l)$. There exist converters of size $O(lf(l))$ that allow routing of requests of load l using at most $l + \frac{l}{f(l)}$ wavelengths.*

Next we show better tradeoffs between the unutilized bandwidth and the size of the converters under our network model.

Lemma 8. *Let $f(l) = o(l)$. There exists a three level graph $G(U, V, W, E)$ with $|U| = |V| = |W| = l + \frac{l}{f(l)}$ with size $O\left(l \frac{\log^2 f(l)}{\log\log f(l)}\right)$ such that for any sets $X \subseteq U$ and $Y \subseteq W$ with cardinality k with $\frac{l}{f(l)} < k \leq l$ there exist $l - \frac{l}{f(l)}$ common neighbors.*

Proof. The proof is based on [9]. Let $w(l) = l + \frac{l}{f(l)}$. We build a three level graph $(A = [w(l)], C = [w(l)], B = [w(l)], E)$. Let $C = C_{i_s}$ where C_{i_s} is defined as follows. Let $i_s = \frac{\log \frac{w(l)}{3}}{\log logf(l)} - 1$, $i_0 = \frac{\log l - \log f(l)}{\log logf(l)}$, and $C_i = \left[3 \log^{i+1} f(l)\right]$, $i = i_0, \ldots i_s$, such that $C_i \subseteq C_{i+1}$, for $i_0 \leq i \leq i_s - 1$. For every i put a $(K = \log^i f(l), \epsilon = \frac{1}{3})$–disperser $D_i = (A, C_i, E_i)$ and another (K, ϵ)–disperser between B and C_i. Also, put a copy of the edges between A, C between C, A and C, B and (symmetric) 15–regular ramanujan graphs $H_1 = (A, C, E(H_1))$ and $H_2 = (B, C, E(H_2))$ (in order graphs (A, C) and (C, B) can correspond to identical wavelength converters).

For $\log^i f(l) \leq K \leq \log^{i+1} f(l)$, for every set $X \subseteq A$ of cardinality K, its neighborhood $N(X)$ has size at least $\frac{2|C_i|}{3}$. Similarly for each set $Y \subseteq B$ of cardinality K, $|N(Y)| \geq \frac{2|C_i|}{3}$. Thus the number of the common neighbors of X and Y is at least $\frac{|C_i|}{3} \geq K \geq K - \frac{l}{f(l)}$. This holds for any K with $\frac{l}{f(l)} = \log^{i_0} f(l) \leq K \leq \log^{i_s+1} f\left(\frac{w(l)}{3}\right) = \frac{w(l)}{3}$.

Applying the Tanner's inequality to the ramanujan graph H_1, we have that for any set $X \subseteq A$ of cardinality $K > \frac{w(l)}{3}$, there are $\frac{w(l)+K}{2}$ neighbors in C. The same argument holds for H_2 and thus any two sets $X \subseteq A$ and $Y \subseteq B$ of cardinality $K > \frac{w(l)}{3}$ have at least $K > K - \frac{l}{f(l)}$ common neighbors.

It can be verified that the total size of the converters used in the above construction is $O\left(l \frac{\log^2 f(l)}{\log\log f(l)}\right)$. □

Theorem 3. *Let $f(l) = o(l)$. There exist converters of size $O\left(l \frac{\log^2 f(l)}{\log\log f(l)}\right)$ that allow greedy routing of requests of load l using at most $l + \frac{l}{f(l)}$ wavelengths.*

Proof. We use the converter implied by lemma 8. The theorem follows by lemmas 7, 6, and 5. □

Using as a converter the half part of the depth–two superconcentrator of lemma 4, we obtain the following.

Theorem 4. *There exist converters of size $O\left(l \frac{\log^2 l}{\log\log l}\right)$ that allow greedy routing of requests of load l using at most l wavelengths.*

5 Lower Bounds

In [1] it is shown that if the binary tree network has wavelength converters with degree $2\sqrt{l}-1$, under this model it is possible to route all possible sets of requests of load l with l wavelengths. Next we show that this result is asymptotically tight.

Theorem 5. *Let T be a binary tree network supporting l wavelengths. Then, wavelength converters of degree $\Omega(\sqrt{l})$ are necessary to guarantee that all sets of requests of load l can be routed on T by a greedy deterministic algorithm.*

Proof. Consider a node v of the tree. Assume that there is one request from the parent p of v to the left child u colored with a color c_1, one request from the right child w to the parent colored with c_2 and $l-1$ requests from w to u. Since no conversion is supported for requests from w to u, the color c_1 must be converted to a color that can be converted to c_2. We can create a pattern on a sufficiently large tree T such that a color c_1 must be converted to colors that can be converted to all colors $c_1, ...c_l$. Assume that the converter translates c_1 to k colors. Then there must be a color c_i that can be converted to at least l/k colors. Thus the degree of the converter must be at least $\min\{k, l/k\} = \sqrt{l}$. □

The following theorems state that the upper bounds of theorems 3 and 4 are asymptotically tight as well. The proofs are omitted.

Theorem 6. *Let T be a tree network supporting l wavelengths. Then, wavelength converters of size $\Omega\left(l\frac{\log^2 l}{\log\log l}\right)$ are necessary to guarantee that all sets of requests of load l can be routed on T by a greedy deterministic algorithm.*

Theorem 7. *Let $f(l) = o(l)$ and T be a tree network supporting $w(l) = l + \frac{l}{f(l)}$ wavelengths. Then, wavelength converters of size $\Omega\left(l\frac{\log^2 f(l)}{\log\log f(l)}\right)$ are necessary to guarantee that all sets of requests of load l can be routed on T by a greedy deterministic algorithm.*

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