A Little Charity Guarantees Fair Connected Graph Partitioning

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Abstract

Motivated by fair division applications, we study a fair connected graph partitioning problem, in which an undirected graph with m nodes must be divided between n agents such that each agent receives a connected subgraph and the partition is *fair*. We study approximate versions of two fairness criteria: α-*proportionality* requires that each agent receives a subgraph with at least $1/\alpha \cdot m/n$ nodes, and α-*balancedness* requires that the ratio between the sizes of the largest and smallest subgraphs be at most α. Unfortunately, there exist simple examples in which no partition is reasonably proportional or balanced. To circumvent this, we introduce the idea of *charity*. We show that by "donating" just $n-1$ nodes, we can guarantee the existence of 2-proportional and almost 2-balanced partitions (and find them in polynomial time), and that this result is almost tight. More generally, we chart the tradeoff between the size of charity and the approximation of proportionality or balancedness we can guarantee.

1 Introduction

The problem of fair division concerns the allocation of a set of goods (or chores) fairly between a set of agents. Perhaps the most canonical model is cake-cutting, in which a heterogeneous divisible good, called cake, is divided between n agents. Under minimal assumptions, this model allows providing compelling fairness guarantees. For example, one can ensure proportionality (Steinhaus 1948), which demands that each agent's value for her allocation be at least $1/n$-th of her value for the entire cake, or the stronger notion of envy-freeness (Gamow and Stern 1958; Dubins and Spanier 1961), which demands that no agent strictly prefers another agent's allocation to her own.

However, many real-world applications pose additional constraints, which often make such strong fairness notions impossible to guarantee. A common constraint, which has received increasing attention recently, is *indivisibility*. Here, one assumes that the goods cannot be split, i.e., each good must be allocated entirely to a single agent. For example, when dividing an inheritance between heirs, goods such as a house or a piece of jewelry are indivisible. In this case, one can no longer guarantee proportionality or envy-freeness; think of allocating a single indivisible good between two agents. Nonetheless, "up to one good"-style relaxations can be guaranteed (Budish 2011; Conitzer, Freeman, and Shah 2017; Caragiannis et al. 2019), which converge to providing exact proportionality or envy-freeness when each individual good is negligible compared to the set of all goods.

The situation becomes more dire when we impose another common constraint: *connectedness*. Bouveret et al. (2017) introduced a model where the indivisible goods are nodes of a graph and the goal is to allocate to each agent a subset of goods that forms a connected subgraph. Examples of real-world applications where connectedness is desirable include allocation of offices to research groups in an academic building, land division (Devulapalli 2014), congressional redistricting[1], power grid islanding (Soltan, Yannakakis, and Zussman 2020), and metadata partitioning in large-scale distributed storage systems (Wu, Liu, and Chung 2010).

While many of these applications have identical goods (meaning that all agents have the same value for each good), it is easy to see that even in this special case, no reasonable relaxation of proportionality or envy-freeness can be guaranteed, even if each individual good is negligible compared to the set of all goods. For example, consider $m \gg n$ identical goods connected via a star graph with a hub node connected to $m-1$ leaf nodes. Any way of partitioning the nodes into n connected bundles will produce a highly imbalanced partition in which one very large bundle has at least $m-n+1$ nodes while every other bundle has at most a single node.

This, in essence, is the fair graph partitioning problem that we study in this work. Formally, we are given an undirected graph $G=(V,E)$, where V is a set of m nodes and we want to partition it into $(V_1,\ldots,V_n)$ such that each V_i forms a connected subgraph. Borrowing from the fair division literature, we call this partition α-proportional if $\min_i \alpha \cdot |V_i| \geqslant m/n$, and α-balanced if $\max_i |V_i| \leqslant \alpha \cdot \min_i |V_i|$. It is easy to see that α-balancedness implies α-proportionality.[2] Balancedness and similar cardinality constraints have been investigated previously in various fair division contexts (Leroux and Leroux 2004; Biswas and Barman 2018; Bei et al. 2021b; Halpern and Shah 2021); in our case, note that 1-

[1]This is the process of re-drawing electoral district boundaries. Formally, a graph of electoral precincts is divided into a fixed number of connected subgraphs (districts) with approximately equal populations (Williams Jr 1995).

[2]Actually, it implies $(\alpha-(\alpha-1)/n)$-proportionality.

balancedness is equivalent to envy-freeness.

While the aforementioned star graph example rules out any reasonably fair partition, note that if we could keep just the hub node unallocated, we could partition the leaf nodes in a highly proportional and balanced manner. In the fair division literature, the idea of keeping a few goods unallocated, termed *charity*, has been used to achieve fairness guarantees that are even stronger than envy-freeness up to one good without the connectedness constraint (Chaudhury et al. 2021b; Caragiannis, Gravin, and Huang 2019; Chaudhury et al. 2021a; Berger et al. 2021). We borrow this idea and show that charity also helps improve fairness when connectedness is desired. In our context, the unallocated nodes can also be viewed as shared between agents; e.g., in land division, these can be public land accessible by all agents.

Formally, we seek a partition $(V_1, \ldots, V_n, R)$ of V, where the set of unallocated (or excluded) nodes R is small and each V_i is connected "via" R (i.e., there exists $R_i \subseteq R$ such that $V_i \cup R_i$ is connected). While α-balancedness definition remains unchanged, α-proportionality is now defined as $\alpha \cdot \min_i |V_i| \geqslant (m - |R|)/n$, so that α-balancedness still implies α-proportionality. Revisiting the star graph example, we can see that if we divide a star graph with a hub node connected to three leaf nodes between two agents, the best we can hope for with a single node exclusion is 2-balancedness and 1.5-proportionality. Generalizing this example, we later show (Theorem 1) that when dividing a graph between n agents, the best we can hope for with $n-1$ node exclusions is 2-balancedness and $(2 - 1/n)$-proportionality. This leads to our main research questions:

Is a 2-balanced or $(2 - 1/n)$-proportional partition of a graph between n agents guaranteed to exist with only $n-1$ node exclusions? If so, can we find such a partition in polynomial time? More generally, what is the tradeoff between the approximation of proportionality or balancedness we can achieve and the number of nodes we need to exclude?

Since the number of nodes m can be much greater than n, following the fair division literature (Chaudhury et al. 2021b), we view excluding $O(n)$ nodes as "a little charity".

1.1 Our Results

We begin by the case where at most $n-1$ node exclusions are allowed. We prove a lower bound which shows that α-balancedness and α-proportionality cannot be guaranteed for any $\alpha < 2$ and $\alpha < 2 - 1/n$, respectively (Theorem 1).

Next, for $n \in \{2, 3\}$, we show that this bound is tight and such partitions can be found in polynomial time (Theorems 2 and 3). For higher values of n, we provide three efficient algorithms which obtain generally incomparable approximation guarantees: one ensures $(3 + O(n/m))$-balancedness and 3-proportionality, another ensures 4-balancedness and 2-proportionality, and the final one ensures $(2 + O(n^2/m))$-balancedness and $(2 - 1/n + O(n^2/m))$-proportionality. In particular, for fixed n, when $m \to \infty$, the final result matches the lower bound from Theorem 1. We conjecture that it should be possible to achieve 2-balancedness and $(2 - 1/n)$-proportionality for any n and m.

We also consider the tradeoff between the charity (number of node exclusions allowed) and approximations to balancedness or proportionality which can be guaranteed. While we provide almost tight bounds on this tradeoff when more than $n-1$ exclusions are allowed, we leave behind interesting open questions when fewer than $n-1$ exclusions are allowed. We also show hardness of checking the existence of balanced partitions with at most $n-1$ exclusions or approximately balanced partitions with no exclusions. All missing proofs can be found in the full version, along with miscellaneous extensions to our framework.

1.2 Related Work

Our work is related to various models studied in mathematics, theoretical computer science, and multiagent systems.

In theoretical computer science, the problem of partitioning the nodes of a graph into connected subgraphs is well-studied. It is known that checking whether a partition into equal-sized connected subgraphs — hence, perfectly proportional and balanced — exists is NP-hard (Dyer and Frieze 1985); hence, this literature focuses on designing approximation algorithms for computing partitions that are close to optimal according to various criteria, such as maximizing the minimum size (related to proportionality) (Chlebíková 1996; Chataigner, Salgado, and Wakabayashi 2007) and minimizing the maximum size (Chen et al. 2020). However, when even the optimal partitions are highly imbalanced, as in the star graph example from the introduction, such approximations are also unsatisfactory. The focus of our work is to provide worst-case bounds on balancedness and proportionality by allowing the exclusion of a few nodes (charity).

In mathematics, the related problem of partitioning the *edges* rather than nodes of a graph has received attention. For the special case of trees, this problem was introduced by Wu et al. (2007), who proved the existence of 3-balanced and $(2 - 1/n)$-proportional edge partitions; note that this is without any edge exclusions. Later, Dye (2009) improved the balancedness approximation to 2 for $n \in \{2, 3, 4\}$, Chu et al. (2010) extended this result to all values of n, and Chu, Wu, and Chao (2013) showed how to achieve this in linear time even when the edges are weighted. In Section 5, we make an connection between edge partitions of trees with no edge exclusions and node partitions of general graphs with at most $n-1$ node exclusions, allowing us to leverage the above results to obtain upper bounds for our problem.

Our primary motivation stems from the fair division literature in multiagent systems, where the goal is to partition the available goods between agents in a way that each agent receives a connected subset. While envy-freeness and proportionality can be achieved exactly when the goods are divisible, as in cake-cutting (Stromquist 1980; Su 1999), as illustrated in the introduction, not even a reasonable approximation of these guarantees can be provided when the goods are indivisible, modeled as nodes of a graph. Hence, this literature focuses on special families of graphs, such as path graphs, for which such guarantees can be provided (Bouveret et al. 2017; Bilò et al. 2019; Bei et al. 2021a), and on the computational complexity of the existence of fair connected allocations (Deligkas et al. 2021; Greco and Scarcello

2020; Igarashi and Peters 2019). Our goal is to provide approximate fairness guarantees for *general graphs*, by using the idea of charity, which has been explored recently for fair division without the connectedness constraint (Chaudhury et al. 2021b; Caragiannis, Gravin, and Huang 2019; Chaudhury et al. 2021a; Berger et al. 2021).

We remark that connected fair division has also been studied for *chores* rather than goods, with both divisible chores (Heydrich and van Stee 2015; Dehghani et al. 2018) and indivisible ones (Bouveret, Cechlárová, and Lesca 2019).

2 Preliminaries

For $q \in \mathbb{N}$, define $[q] = \{1, \ldots, q\}$. Let $G = (V, E)$ be a graph with $|V| = m$. We denote with $G[X]$ the subgraph induced by $X \subseteq V$. We say that $(V_1, \ldots, V_n, R)$ is a *pseudo n-partition* of G if

1. $V = (\cup_{i \in [n]} V_i) \cup R$;
2. $V_i \cap V_j = \emptyset$ for distinct $i, j \in [n]$, and $V_i \cap R = \emptyset$ for all $i \in [n]$; and
3. $|R| \leqslant n - 1$.

When $|R| = 0$, we simply refer to it as an n-partition of G. A pseudo n-partition $(V_1, \ldots, V_n, R)$ is called *connected* if, for every $i \in [n]$, there exists $R_i \subseteq R$ such that the subgraph $G[V_i \cup R_i]$ is connected. Throughout the paper, we assume that G is connected and $m \geqslant n$, otherwise there may not exist any connected pseudo n-partition of G.

In our motivating fair division applications, the nodes of G are the goods, V_i is the set of goods allocated to agent i, and R is the set of goods left unallocated (charity). We are typically interested in the case where $n \ll m$, so a charity of $n - 1$ out of m nodes is very little.

With such little charity, our goal is to find a connected pseudo n-partition $(V_1 \ldots, V_n, R)$ of G that is reasonably *fair*. We consider the following fairness desiderata.

Definition 1 (Balancedness)**.** For $\alpha \geqslant 1$, we say that a connected pseudo n-partition $(V_1, \ldots, V_n, R)$ is α-balanced if $\max_{i \in [n]} |V_i| \leqslant \alpha \cdot \min_{i \in [n]} |V_i|$. We refer to 1-balancedness simply as balancedness.

Definition 2 (Proportionality)**.** For $\alpha \geqslant 1$, we say that a connected pseudo n-partition $(V_1, \ldots, V_n, R)$ is α-proportional if $\alpha \cdot \min_{i \in [n]} |V_i| \geqslant (m - |R|)/n$. We refer to 1-proportionality simply as proportionality.

Note that if a connected pseudo n-partition $(V_1, \ldots, V_n, R)$ is α-balanced, then we have $m = |R| + \sum_{i \in [n]} |V_i| \leqslant |R| + |V_i| + (n-1) \cdot \alpha \cdot |V_i|$ for any $i \in [n]$, which, after some simplification, implies that the partition is also $(\alpha - (\alpha - 1)/n)$-proportional. In particular, 2-balancedness implies $(2 - 1/n)$-proportionality.

We remark that the most difficult case of our problem is when G is a tree. Trivially, any lower bounds derived for trees apply to the general case as well. But note that any upper bounds derived for trees can also be translated to the general case. This is because, given any algorithm for trees and an input graph G, we can apply the algorithm to any spanning tree of G (which can be computed efficiently). Any pseudo n-partition produced by the algorithm that is connected under the spanning tree must also be connected under G. Hence, throughout the paper, we assume G to be a tree without loss of generality.

We will often work with rooted trees. Given a tree $G = (V, E)$ and a node $v \in V$, let $T = (G, v)$ denote the tree G rooted at v. Given a node $u \in V$, let $ST(u, T)$, $c(u, T)$, and $p(u, T)$ denote the subtree, the set of children nodes, and the parent node of u, respectively ($p(v, T)$ is undefined); let $\text{level}(u, T)$ denote the length of the (unique) path from u to the root v in T, with $\text{level}(v, T) = 1$. Define the height of tree T as $\text{height}(T) = \max_{u \in V} \text{level}(u, T)$. We drop T from the notation when it is clear from the context.

3 A Lower Bound

We begin by showing that we cannot hope to provide any guarantee better than 2-balancedness or $(2 - 1/n)$-proportionality. This uses a generalization of the example used in the introduction to establish these lower bounds for $n = 2$. In later sections, we design algorithms that (almost) achieve these bounds.

Theorem 1. *There exists an instance in which no connected pseudo n-partition is α-balanced for any $\alpha < 2$ or α-proportional for any $\alpha < 2 - 1/n$.*

Proof. Let $\ell \geqslant n$ be an integer. Consider the graph $G = (V, E)$ that consists of $2n-1$ paths of length ℓ each, denoted $P_1, \ldots, P_{2n-1}$, and $n - 1$ additional "hub" nodes, denoted $h_1, \ldots, h_{n-1}$. Hence, $|V| = \ell \cdot (2n-1) + n - 1$. For $j \in [n-2]$, h_j is connected to h_{j+1} as well as to one of the endpoints of paths P_{2j-1} and P_{2j}. Finally, h_{n-1} is connected to one of the endpoints of paths P_{2n-3}, P_{2n-2}, and P_{2n-1}.

First, we show that there is no connected pseudo n-partition $(V_1, ..., V_n, R)$ such that $|V_i| \geqslant \ell + 1$ for all $i \in [n]$. For the sake of contradiction, assume that such a partition exists. We show that each path intersects at most one of the parts. Indeed, if there exist $j \in [2n - 1]$ and distinct $i, i' \in [n]$ such that $P_j \cap V_i \neq \emptyset$ and $P_j \cap V_{i'} \neq \emptyset$, then the part that contains the node in $P_j \cap (V_i \cup V_{i'})$ farthest from the hub that P_j is attached to would have size at most $\ell - 1$, which is a contradiction. Since there are $2n - 1$ paths and each intersects at most one part, by the pigeonhole principle, there must exist $i^* \in [n]$ such that V_{i^*} intersects with at most one path P_{j^*}. Since $|V_{i^*}| \geqslant \ell + 1$, it must contain at least one hub node v. Since each hub node is attached to at least two paths, v must be attached to a path $P_{j'}$ different from P_{j^*}. Since $|R| \leqslant n - 1 < \ell = |P_{j'}|$, we have $P_{j'} \not\subseteq R$; hence, there must exist $i' \in [n] \setminus \{i\}$ such that $V_{i'} \cap P_{j'} \neq \emptyset$. However, since the hub node v that $P_{j'}$ is attached to is allocated to V_i, by the connectedness constraint we have $V_{i'} \subseteq P_{j'}$, implying $|V_{i'}| \leqslant \ell$, which is a contradiction.

We have established that in any connected pseudo n-partition, there exists $i \in [n]$ such that $|V_i| \leqslant \ell$. If it is α-proportional, then we need

$$\alpha \cdot \ell \geqslant \frac{m - |R|}{n} \geqslant \frac{(2n-1) \cdot \ell}{n},$$

which implies $\alpha \geqslant 2 - 1/n$. Since α-balancedness implies $(\alpha - (\alpha - 1)/n)$-proportionality for any $\alpha \geqslant 1$, the impossi-

Algorithm 1: 2-balancedness and 1.5-proportionality for $n = 2$

Input: Tree $G = (V, E)$ with $|V| = m$ nodes.
Output: A connected pseudo 2-partition.

1: $r \leftarrow$ arbitrary node in V
2: $T \leftarrow$ tree (G, r) rooted at r
3: Find a node u^* such that $|ST(u^*, T)| \geqslant \lceil m/3 \rceil > ST(v, T)$ for every child v of u^*
4: **if** $|ST(u^*, T)| = \lceil m/3 \rceil$ **then**
5: $\quad (V_1, V_2, R) \leftarrow (ST(u^*, T), V \setminus ST(u^*, T), \emptyset)$
6: **else**
7: $\quad R \leftarrow \{u^*\}, V_1 \leftarrow \emptyset$
8: $\quad$ **for** $v \in c(u^*, T)$ **do**
9: $\quad\quad V_1 \leftarrow V_1 \cup ST(v, T)$
10: $\quad\quad$ **if** $|V_1| \geqslant \lceil m/3 \rceil$ **then**
11: $\quad\quad\quad$ **break**
12: $\quad\quad$ **end if**
13: $\quad$ **end for**
14: $\quad V_2 \leftarrow V \setminus (V_1 \cup \{u^*\})$
15: **end if**
16: **return** (V_1, V_2, R)

bility of achieving α-proportionality for $\alpha < (2 - 1/n)$ implies the impossibility of getting α-balancedness for $\alpha < 2$. □

4 Optimal 2-Partitions and 3-Partitions

In this section, we show that the lower bound from Theorem 1 is tight when $n \in \{2, 3\}$. For these cases, we design efficient algorithms for finding connected pseudo n-partitions that are 2-balanced (and thus, $(2 - 1/n)$-proportional). The algorithm for $n = 2$, Algorithm 1, is of particular interest, as we will use it as a subroutine in the next section to derive bounds for higher values of n.

Algorithm 1 returns a connected 2-balanced pseudo 2-partition with $|R| \leqslant 1$ as follows. It roots the given tree arbitrarily, and then finds a node u^* at maximal depth whose subtree has at least $\lceil m/3 \rceil$ nodes. If the subtree has exactly $\lceil m/3 \rceil$ nodes, it assigns the subtree as one part and the rest of the tree as the other part (not excluding any node). Otherwise, it excludes u^*, and adds subtrees of its children iteratively to a part until the part has at least $\lceil m/3 \rceil$ nodes. The remaining nodes form the other part. A similar trick has been used previously in the literature; see, e.g., (Micha and Shah 2020) and (Li et al. 2021).

Theorem 2. *When $n = 2$, Algorithm 1 runs in polynomial time and returns a connected pseudo 2-partition that is 2-balanced and, hence, 1.5-proportional.*

Proof. We have already argued that Algorithm 1 can be implemented efficiently. It is also easy to check that it returns a connected pseudo 2-partition. Now, we show that it achieves 2-balancedness, which implies 1.5-proportionality, as argued in Section 2.

First, consider the case where $|ST(u^*, T)| = \lceil m/3 \rceil$. In this case, since $|R| = 0$, we need to show that $\min(|V_1|, |V_2|) \geqslant \lceil m/3 \rceil$. This is already satisfied for $V_1 = ST(u^*, T)$, and we have $|V_2| = m - \lceil m/3 \rceil \geqslant \lceil m/3 \rceil$.

Next, consider the case where $|ST(u^*)| > \lceil m/3 \rceil$. In this case, since $|R| = 1$, we need to show that $\min(|V_1|, |V_2|) \geqslant \lceil (m-1)/3 \rceil$. For V_1, this follows by its construction. Also, consider the last subtree $ST(v, T)$ added to V_1 in Line 9. Before adding this subtree, V_1 must have had at most $\lceil m/3 \rceil - 1$ nodes. Further, since u^* is a node of maximal height with $|ST(u^*, T)| \geqslant \lceil m/3 \rceil$, we must have $|ST(v, T)| \leqslant \lceil m/3 \rceil - 1$ for the child v of u^*. Hence, we have $|V_1| \leqslant 2(\lceil m/3 \rceil - 1)$, implying that $|V_2| \geqslant m - 1 - 2(\lceil m/3 \rceil - 1) \geqslant \lceil (m-1)/3 \rceil$. The theorem follows. □

We make a note of the following fact established in the proof of Theorem 2, which we will use in the next section when using Algorithm 1 as a subroutine and deriving bounds for higher values of n.

Corollary 1. *Algorithm 1 returns a connected 2-partition (V_1, V_2, R) such that $\min(|V_1|, |V_2|) \geqslant \lceil (m - |R|)/3 \rceil$.*

Next, we establish a similar result for $n = 3$.

Theorem 3. *When $n = 3$, there exists a connected pseudo 3-partition that is 2-balanced and, thus, 5/3-proportional, and it can be computed in polynomial time.*

The tightness of the lower bound from Theorem 1 for $n \in \{2, 3\}$ leads us to make the following conjecture:

Conjecture 1. *For any $n \geqslant 2$, every graph admits a connected pseudo n-partition that is 2-balanced (and hence, $(2 - 1/n)$-proportional), and it can be computed efficiently.*

In the next section, we present a series of results which almost resolve this conjecture.

5 Upper Bounds for Higher n

We present three key upper bounds that hold for all $n \geqslant 2$. The first is via a fairly straightforward algorithm that uses Algorithm 1 for $n = 2$ recursively to obtain $(3 + O(n/m))$-balancedness and 3-proportionality. The second algorithm uses the key idea from Algorithm 1 of finding a subtree of some desired size, and iteratively applies it to achieve 4-balancedness and 2-proportionality; while the balancedness approximation gets worse when $n \ll m$, the proportionality approximation improves and matches the lower bound of $2 - 1/n$ from Theorem 1 in the limit when $n \to \infty$. Finally, by making an interesting connection to the literature on *edge partitions* of a tree, we show that $(2 + O(n^2/m))$-balancedness and $(2 - 1/n + O(n^2/m))$-proportionality can be achieved, which matches the respective lower bounds from Theorem 1 for each n in the limit when $m \to \infty$.

Let us begin with our first result of this section. At a high level, Algorithm 2 works simply as follows: it starts with the entire input tree as a single part, and repeatedly divides the largest existing part into two using Algorithm 1 until there are n parts. One issue is that when Algorithm 1 excludes a node, the two parts it returns may become disconnected, preventing us from applying Algorithm 1 to them in future iterations; this is because Algorithm 1 assumes its input to be a tree. This is easily fixed by adding artificial edges between the neighbors of the excluded node to ensure that the

Algorithm 2: $(3 + O(n/m))$-balancedness and 3-proportionality for $n \geqslant 2$

Input: Tree $G = (V, E)$ and integer $n \geqslant 2$.
Output: A connected pseudo n-partition.

$C^0 \leftarrow \{G\}$; $R^0 \leftarrow \emptyset$
for $i = 1$ to $n - 1$ **do**
  $T^i \leftarrow$ largest tree in C^{i-1} (break ties arbitrarily)
  $(V_1^i, V_2^i, \hat{R}^i) \leftarrow$ Call Algorithm 1 on T^i
  $H_1^i \leftarrow T^i[V_1^i]$, $H_2^i \leftarrow T^i[V_2^i]$
  if $\hat{R}^i \neq \emptyset$ **then**
    Let $u^i \in \hat{R}^i$ {This is unique}
    For $j \in \{1, 2\}$, if H_j^i has at least two neighbors of u^i, connect an arbitrarily chosen neighbor to every other neighbor {This ensures that H_j^i is now a tree}
  end if
  $C^i \leftarrow C^{i-1} \cup \{H_1^i, H_2^i\} \setminus T^i$
  $R^i \leftarrow R^{i-1} \cup \hat{R}^i$
end for
return $(V_1, \ldots, V_n, R)$, where $V_1, \ldots, V_n$ are the sets of nodes of the trees in C^{n-1} and $R = R^{n-1}$.

parts returned by Algorithm 1 become trees. This is not a problem because if a part returned at the end of Algorithm 2 is connected due to the artificially added edges, it would also be connected via the excluded nodes.

Theorem 4. *When $n \geqslant 2$ and $m \geqslant n \cdot (n-1)$, Algorithm 2 runs in polynomial time and returns a connected pseudo n-partition that is $(3 + 6n/m)$-balanced and 3-proportional.*

Next, we show how to achieve 4-balancedness and 2-proportionality. We use the key idea from Algorithm 1 of finding a subtree of some desired size and iteratively apply it to separate out one part at a time from the tree. An interesting detail, and the driving force behind the balancedness guarantee, is that because we cannot exactly control the size of the parts being separated out, we keep adjusting the desired size of the next part based on the actual size of the previous part created. This ensures that when ℓ parts are created, their total size stays close to $\ell \cdot m/n$, leaving the size of the remaining tree close to $(n - \ell) \cdot m/n$. In particular, after $n - 1$ parts are created, the remaining tree, much of which forms the last part, is not too large.

To make our analysis work, we need to ensure that $|R| = n-1$. Hence, we need $m \geqslant 2n-1$, so that even after removing $n - 1$ nodes, we can always create an n-partition with non-empty parts. We remark that Line 6 can be implemented efficiently similarly to Line 3 of Algorithm 1.

When Line 20 of Algorithm 3 excludes node u_i in iteration i, T_i may become disconnected as the subtrees rooted at children of u_i become disconnected from each other and from the rest of the tree. This is fixed by adding artificial edges connecting every child of u_i that remains in T_i to the parent of u_i. As mentioned above, if a part is connected using these artificial edges, it is also connected using excluded nodes instead. If u_i is the root of the tree, we can imagine creating an artificial new root node, connecting it to all

Algorithm 3: 4-balancedness and 2-proportionality for $n \geqslant 2$

Input: Tree $G = (V, E)$ and integer $n \geqslant 2$.
Output: A connected pseudo n-partition.

1: $r \leftarrow$ arbitrary node in V
2: $T \leftarrow$ tree (G, r) rooted at r
3: $R \leftarrow \emptyset$;$\forall i \in [n], V_i \leftarrow \emptyset$
4: $s \leftarrow \frac{m-(n-1)}{n}$, $x_0 \leftarrow 0, T_1 \leftarrow T$
5: **for** $i = 1$ to $n - 1$ **do**
6:   Find a node u_i such that $|ST(u_i, T_i)| \geqslant \lceil s(1 + x_{i-1})/2 \rceil > |ST(v, T_i)|$ for all $v \in c(u_i, T_i)$
7:   **if** $|ST(u_i, T_i)| = \lceil s(1 + x_{i-1})/2 \rceil$ **then**
8:     $V_i \leftarrow ST(u_i, T_i)$
9:     $T_{i+1} \leftarrow T_i \setminus ST(u_i, T_i)$
10:   **else**
11:     $R = R \cup \{u_i\}$
12:     **for** $u' \in c(u_i, T_i)$ **do**
13:       $V_i \leftarrow V_i \cup ST(u', T_i)$
14:       **if** $|V_i| \geqslant \lceil s(1 + x_{i-1})/2 \rceil$ **then**
15:         **break**
16:       **end if**
17:     **end for**
18:     $T_i \leftarrow T_i \setminus V_i$
19:     Connect each $v \in c(u_i, T_i)$ to $p(u_i, T_i)$
20:     $T_{i+1} \leftarrow T_i \setminus \{u_i\}$
21:   **end if**
22:   $x_i \leftarrow 1 + x_{i-1} - |V_i|/s$
23: **end for**
24: $S \leftarrow$ set of $n - 1 - |R|$ arbitrary nodes from T_n
25: $V_n \leftarrow T_n \setminus S$, $R \leftarrow R \cup S$
26: **return** $(V_1, \ldots, V_n, R)$

children of u_i, but not counting this artificial root node in future computations of subtree sizes. Note that, unlike in Algorithm 2, we do not just connect an arbitrary neighbor of u_i in T_i to its remaining neighbors because this can alter the rooted tree structure, which we use in this algorithm.

Theorem 5. *When $n \geqslant 2$ and $m \geqslant 2n - 1$, Algorithm 3 runs in polynomial time and returns a connected pseudo n-partition that is 4-balanced and 2-proportional.*

Proof. As explained above, the addition of artificial edges in Line 19 ensure that the remaining graphs (T_i-s) are trees and the parts being created (V_i-s) are connected via the excluded nodes. Later, in Lemma 2, we will establish that for $i \in [n-1]$, $x_{i-1} \leqslant 1$ and $|T_i| \geqslant \lceil (n-i) \cdot s \rceil \geqslant \lceil s \rceil \geqslant \lceil s(1 + x_{i-1})/2 \rceil$. Hence, the algorithm will be able to successfully find node u_i in every iteration i and proceed without any issues. Since at most a single node is added to R in each of $n - 1$ iterations, we clearly have $|R| \leqslant n - 1$. This establishes that the algorithm is valid (i.e., it produces a connected pseudo n-partition at the end. It is also easy to see that the algorithm runs in polynomial time). Hence, it remains to establish its balancedness and proportionality guarantees.

As $m \geqslant 2n - 1$, we have that $s = \frac{m-(n-1)}{n} \geqslant 1$. Before proceeding further, we need the following observation.

Lemma 1. *For any $y \geqslant 0$, $s(1+y) \geqslant \lceil s(1+y)/2 \rceil$.*

Proof. As $s \geqslant 1$ and $y \geqslant 0$, we have $s(1+y) \geqslant 1$. Now, if $s(1+y) \geqslant 2$, then we have

$$\lceil s(1+y)/2 \rceil \leqslant s(1+y)/2 + 1 \leqslant s(1+y).$$

Otherwise, we have $2 > s(1+y) \geqslant 1$, so $s(1+y) \geqslant 1 = \lceil s(1+y)/2 \rceil$. □

Next, we prove the following lemma inductively, and establish several structural properties that hold during the execution of the algorithm.

Lemma 2. *For each $i \in \{0\} \cup [n-1]$, the following hold:*

- $0 \leqslant x_i \leqslant 1$,
- $\lceil s(1+x_{i-1})/2 \rceil \leqslant |V_i| \leqslant s(1+x_{i-1})$ *if $i \geqslant 1$,*
- $|\cup_{j \in [i]} V_j| = (i - x_i) \cdot s$, *and*
- $|T_{i+1}| \geqslant \lceil (n-i) \cdot s \rceil$.

Proof. We prove the lemma using induction on i. The base case of $i = 0$ trivially holds because $x_0 = 0$ and $T_1 = T$. Fix $i \geqslant 1$. Suppose the induction hypothesis holds for $0, 1, \ldots, i-1$.

Note that $|V_i| \geqslant \lceil s(1+x_{i-1})/2 \rceil$ holds by construction (Lines 7 and 14). If the condition in Line 7 works, then we have $|V_i| = \lceil s(1+x_{i-1})/2 \rceil \leqslant s(1+x_{i-1})$ by Lemma 1. Otherwise, since we keep adding subtrees of size at most $\lceil s(1+x_{i-1})/2 \rceil - 1$ until $|V_i| \geqslant \lceil s(1+x_{i-1})/2 \rceil$ (Line 14), we have $|V_i| \leqslant 2 \cdot (\lceil s(1+x_{i-1})/2 \rceil - 1) \leqslant s(1+x_{i-1})$. Hence, the second claim holds.

For the third claim, we use the fact that $|\cup_{j \in [i]} V_j| = |\cup_{j \in [i-1]} V_j| + |V_i| = (i-1-x_{i-1}) \cdot s + |V_i|$. To establish that this is equal to $(i - x_i) \cdot s$, we need $|V_i| = (1 + x_{i-1} - x_i) \cdot s$, which holds by the definition of x_i in Line 22.

For the fourth claim, since at most $n-1$ nodes are excluded at any point during the execution of the algorithm, we have

$$\begin{aligned} |T_{i+1}| &\geqslant m - (n-1) - |\cup_{j \in [i]} V_j| \\ &= n \cdot s - (i - x_i) \cdot s \geqslant (n-i) \cdot s. \end{aligned}$$

Since $|T_{i+1}|$ is an integer, we also have $|T_{i+1}| \geqslant \lceil (n-i) \cdot s \rceil$.

For the first claim, recall that $x_i = 1 + x_{i-1} - |V_i|/s$. But $(1+x_{i-1})/2 \leqslant |V_i|/s \leqslant 1 + x_{i-1}$ from the second claim. Hence, $0 \leqslant x_i \leqslant (1+x_{i-1})/2$. Using $x_{i-1} \leqslant 1$ from the induction hypothesis, we get $0 \leqslant x_i \leqslant 1$ as desired. □

Combining the first two claims from Lemma 2, we have that $\lceil s/2 \rceil \leqslant |V_i| \leqslant 2s$ for $i \in [n-1]$. Let us now estimate $|V_n|$. From the third claim of Lemma 2 applied at $i = n-1$, we have

$$\begin{aligned} |T_n| &= |T \setminus (\cup_{i \in [n-1]} V_i \cup R)| \\ &= m - (n-1-x_{n-1}) \cdot s - |R| \end{aligned}$$

Note that $V_n = T_n \setminus S$, where S is a set of $n-1-|R|$ arbitrary nodes from T_n. Hence,

$$|V_n| = m - (n-1-x_{n-1}) \cdot s - (n-1) = (1+x_{n-1}) \cdot s,$$

where the second transition follows since $m-(n-1) = n \cdot s$. Using $0 \leqslant x_{n-1} \leqslant 1$ from Lemma 2, we have $s \leqslant |V_n| \leqslant 2s$. Hence, in conclusion, we have $\lceil s/2 \rceil \leqslant |V_i| \leqslant 2s$ for all $i \in [n]$, which clearly implies 4-balancedness. Since we force $|R| = n-1$, we have $s = (m-(n-1))/n = (m - |R|)/n$, so this also implies 2-proportionality. □

Next, we show that $(2 + O(n^2/m))$-balancedness and $(2 - 1/n + O(n^2/m))$-proportionality can be obtained by making a connection to the literature on *edge partitions* of trees. We say that $(E_1, \ldots, E_n)$ is an *n-edge-partition* of a tree $G = (V, E)$ if $E_i \cap E_j = \emptyset$ for all distinct $i, j \in [n]$ and $\cup_{i \in [n]} E_i = E$. We say that it is *connected* if, for each $i \in [n]$, the subgraph formed by the edges in E_i is connected (hence, also a tree). For $\alpha \geqslant 1$, we say that it is α-balanced if $\max_{i \in [n]} |E_i| \leqslant \alpha \cdot \min_{i \in [n]} |E_i|$ and α-proportional if $\alpha \cdot \min_{i \in [n]} |E_i| \geqslant |E|/n$, where $|E_i|$ and $|E|$ refer to the number of *edges* in those sets. Observe that α-balancedness also implies $(\alpha - (\alpha-1)/n)$-proportionality in this context. In particular, 2-balancedness implies $(2-1/n)$-proportionality.

Note that edge partitions are similar to node partitions, except we seek to partition the edges without excluding any edges. For connected node partitions, we argued in Section 1, using the star graph as an example, that no reasonable approximation of balancedness or proportionality can be obtained without excluding any nodes. However, it turns out that there exist reasonably balanced and proportional edge partitions of a tree that do not require any edge exclusions.

Theorem 6 (Chu et al. 2010)**.** *For any $n \geqslant 2$, every tree admits a connected n-edge-partition that is 2-balanced and, hence, $(2 - 1/n)$-proportional, and such a partition can be computed in polynomial time.*

In the following lemma, we show that a connected n-edge-partition of a tree (with no edge exclusions) can be used to obtain a connected pseudo n-partition of the nodes (with at most $n-1$ node exclusions) while almost preserving the balancedness and proportionality guarantees.

Before we proceed further, recall that for node partitions, our assumption of the input graph being a tree was without loss of generality because a connected pseudo n-partition of a spanning tree of the graph is also a connected pseudo n-partition of the graph itself; both the graph and its spanning tree have the same set of nodes. This does not hold for edge partitions. In particular, an n-edge-partition of a spanning tree of a graph is not even an n-edge-partition of the graph, since the additional edges in the graph not included in the spanning tree also need to be partitioned. In that sense, we are using the aforementioned result on edge partitions for the *special case* of trees to derive a result on pseudo node partitions for general graphs.

Lemma 3. *For any $n \geqslant 2$, given a connected n-edge-partition $(E_1, \ldots, E_n)$ of a tree $G = (V, E)$, we can compute, in polynomial time, a connected pseudo n-partition $(V_1, \ldots, V_k, R)$ of V (i.e., with $|R| \leqslant n-1$) such that $|E_i| + 1 - |R| \leqslant |V_i| \leqslant |E_i| + 1$ for each $i \in [n]$.*

We now use Lemma 3 to translate the guarantee in Theorem 6 to our setting.

Theorem 7. *When $n \geqslant 2$ and $m \geqslant 4n^2$, every graph admits a connected pseudo n-partition of its nodes that is*

$(2+8n^2/m)$*-balanced and* $(2-1/n+8n^2/m)$*-proportional, and one such solution can be computed in polynomial time.*

For fixed n, in the limit when $m \to \infty$, Theorem 7 provides 2-balancedness and $(2-1/n)$-proportionality, matching the lower bound from Theorem 1 and settling Conjecture 1. However, when m is not too large, the guarantee provided by Theorem 4 or Theorem 5 can be better.

6 The Fairness-Charity Tradeoff

In this section, we consider the tradeoff between the limit on charity (the maximum number of nodes we are allowed to exclude) and the approximations to balancedness and proportionality we can guarantee. Given a graph $G = (V, E)$ and $d \in \{0\} \cup \mathbb{N}$, $(V_1, \ldots, V_n, R)$ is called a d-pseudo n-partition of G if it is a partition of V and $|R| \leqslant d$. As before, we say that it is connected if, for each $i \in [n]$, there exists $R_i \subseteq R$ such that $G[V_i \cup R_i]$ is a connected subgraph of G.

The next two results focus on $d > n-1$ and provide an almost tight tradeoff. Let us introduce the lower bound first.

Theorem 8. *Fix any* $m, n \geqslant 2$ *and* $c \geqslant 0$ *such that* $\ell = \frac{m-n+1}{2n-1} \in \mathbb{N}$ *and* $\ell > (c+1)\cdot(n-1)$*. Then, there exists an instance with* m *nodes in which no connected* d*-pseudo* n*-partition is* $(2-c/\ell)$*-balanced when* $d < (c+1)\cdot(n-1)$*, and no connected* d*-pseudo* n*-partition is* α*-balanced for any* $\alpha < 2 - c/\ell$ *when* $d = (c+1)\cdot(n-1)$*.*

One implication of this lower bound is that if we hope to achieve α-balancedness for any *constant* $\alpha < 2$, then we must have $c = \Omega(\ell)$, i.e., $d = \Omega(m)$. Hence, a little charity ($o(m)$ exclusions) would not suffice for this purpose. This shows that 2 is the best constant approximation to balancedness we can hope for with just a little charity. Next, we provide an upper bound via a simple algorithm which starts with any α-balanced connected pseudo n-partition (i.e., with at most $n-1$ exclusions) and repeatedly excludes a node from the largest part until either perfect balancedness is achieved or a total of d nodes are excluded.

Theorem 9. *Fix any* $m, n \geqslant 2$*,* $c \geqslant 0$*,* $d = (c+1)\cdot(n-1)$*,* $\alpha \geqslant 1$*, and* $\hat{\ell} = \frac{m-n+1}{\alpha n - (\alpha - 1)}$*. Given any graph of* m *nodes and any connected* $(n-1)$*-pseudo* n*-partition of it that is* α*-balanced, we can efficiently compute a* d*-pseudo* n*-partition that is* $(\alpha - c/\hat{\ell})$*-balanced.*

In Section 5, we established that α-balanced connected pseudo n-partitions exist for $\alpha \approx 2$ (in particular, with $\alpha \to 2$ when $m \to \infty$). Note that with $\alpha = 2$, the upper bound from Theorem 9 would precisely match the lower bound from Theorem 8. Thus, assuming that 2-balanced connected pseudo n-partitions exist, taking such a partition and repeatedly excluding a node from the largest part provides optimal balancedness for any $d > n-1$.

With $d < n-1$, the situation becomes more complex as it does not seem easy to start from a connected pseudo n-partition with (at most) $n-1$ exclusions and re-include some nodes while maintaining n connected parts. First, we show that decreasing the charity limit by just one increases the balancedness lower bound from 2 to 3.

Theorem 10. *For any* $n \geqslant 2$*,* $d < n-1$*, and* $\epsilon > 0$*, there exists an instance in which no connected* d*-pseudo* n*-partition is* α*-balanced for any* $\alpha < 3 - \epsilon$*.*

Next, we establish a different lower bound that is better when $d < n/3$.

Theorem 11. *For any* $n \geqslant 2$ *and* $d < n$*, there exists an instance in which no connected* d*-pseudo* n*-partition is* α*-balanced for any* $\alpha < n/d$*.*

We believe that this bound is tight up to a constant factor; that is, it should be possible to achieve $O(n/d)$-balancedness with d exclusions for any $d < n$. In particular, with a single exclusion, we believe it should be possible to achieve $O(n)$-balancedness. Below, we prove a weaker result: $O(n)$-proportionality can be achieved with a single exclusion. Note that this implies that the smallest part has size $\Omega(m/n^2)$. Since the largest part can have size at most m, this also implies $O(n^2)$-balancedness.

Theorem 12. *For any* $n \geqslant 2$*, every graph admits a connected* 1*-pseudo* n*-partition that is* $O(n)$*-proportional, and it can be computed in polynomial time.*

7 Complexity

In this section, we contemplate the complexity of checking whether an (approximately) balanced connected pseudo partition exists. To that end, we present two hardness results. The first one considers exact balancedness when $n-1$ exclusions are allowed.

Theorem 13. *Checking whether a balanced connected pseudo* n*-partition (with at most* $n-1$ *exclusions) exists is NP-complete.*

The second result considers exact as well as approximate balancedness when no exclusions are allowed.

Theorem 14. *For any* $\alpha < 2$*, checking whether an* α*-balanced connected* n*-partition (with no exclusions) exists is NP-complete.*

8 Discussion

Our work leaves open a number of directions for the future. For example, does there always exist a 2-balanced and $(2-1/n)$-proportional connected pseudo n-partition with at most $n-1$ node exclusions? While we chart out a tight fairness-charity tradeoff when more than $n-1$ exclusions are allowed, what happens when fewer exclusions are allowed? In particular, does there always exist an $O(n)$-balanced connected pseudo n-partition with at most a single exclusion? Do restricted families of graphs (especially those with higher connectivity) admit better fairness guarantees?

It would also be interesting to consider natural extensions and modifications of our model. What if, instead of excluding nodes, we are allowed to assign a few nodes to multiple parts? What if we allow nodes to have weights, and redefine proportionality and balancedness in terms of the total node weights of the different parts? In the full version, we provide some guarantees in both these cases when $n = 2$. More broadly, it would be exciting to investigate the effectiveness of charity in the general fair division framework, where agents can have *heterogeneous* valuations for the nodes.

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