Bounds on Optical Bandwidth Allocation on Directed Fiber Tree Topologies

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Abstract

In this paper, we address the issue of efficiently allocating wavelengths to communication requests in wavelength division multiplexing (WDM) optical networks. We present a simple greedy algorithm and study its performance on directed binary fiber trees. We prove that our algorithm guarantees a $5/3$-approximation for wavelengths assignment in binary tree shaped WDM optical networks. We also prove that no greedy algorithm can go below the ratio of $5/3$, even for the case of leaf–to–leaf communication.

1 Introduction

Optics is emerging as a key technology in state-of-the-art communication networks. A single optical wavelength supports rates of gigabits-per-second (which in turn support multiple channels of voice, data, and video [5, 10]. Multiple laser beams that are propagated over the same fiber on distinct optical wavelengths can increase this capacity much further; this is achieved through WDM (Wavelength Division Multiplexing).

The model we use for the underlying fiber network is a directed graph. Connectivity requests are ordered pairs of nodes, to be thought of as transmitter-receiver pairs. For networks with unique transmitter-receiver paths (such as trees), the load of a fiber link is the number of paths going through the link. WDM technology establishes connectivity by finding transmitter-receiver paths, and assigning a wavelength to each path, so that no two paths going through the same link use the same wavelength. Optical bandwidth is the number of available wavelengths. As state-of-the-art optics technology allows for a limited number of wavelengths (even in the laboratory) the engineering question to be solved can be expressed as follows: "Given fixed W-wavelength technology, what type of requests can we route?". Alternatively phrased, for unique transmitter-receiver path networks (like trees) the question becomes: "What is the minimum number of necessary wavelengths to route requests of maximum load L per fiber link?"

We consider tree topologies, with each edge of the tree consisting of two opposite directed fiber links. Raghavan and Upfal [11] considered trees with undirected fiber links and undirected paths. However, it has since become apparent that optical amplifiers placed on fiber will be directed devices [4]. Thus, directed graphs are essential to model the current optics technology.

Raghavan and Upfal [11] showed that routing requests of maximum load L per link of undirected trees can be satisfied using $3L/2$ optical wavelengths and their arguments extend to give a $2L$ bound for the directed case. Mihail et al. [9] address the directed case. Their main result is a $15L/8$ upper bound. This is done by reduction to a bipartite graph edge-coloring, which is achieved in phases by obtaining matchings of the bipartite graph, and coloring them in pairs using detailed potential and averaging arguments. The algorithm in [9] is a greedy algorithm. That is, it visits the tree in a top to bottom manner and at each vertex v colors all requests that touch vertex v and are still uncolored. Moreover, once a request has been colored it is never recolored again. Greedy algorithm are important as they are very simple and, more importantly, they are amenable of

*Partially supported by EU Esprit Project GEPPCOM and EU Esprit Project ALCOM-IT.
†Partially supported by Progetto MURST 40%, Algoritmi, Modelli di Calcolo e Strutture Informative.

being implemented in a distributed enviroment.

Kaklamanis and Persiano [6] improve the upper bound for directed trees to $7L/4$. The main idea of their algorithm is similar to the one of [9] but new techniques are used for partitioning the bipartite graph matchings into groups that can be colored and accounted for independently. Recently, Kaklamanis and Persiano [7] have obtained a greedy algorithm that routes a set of requests of maximum load L using at most $5L/3$ and have proved that no greedy algorithm can go below $5L/3$. Their lower bound holds for binary trees also.

1.1 Results of this paper

The contribution of this paper is two-fold. We present a greedy algorithm for binary trees that routes a set of requests of maximum load L using at most $5L/3$. The algorithm presented does not make use of tools of matching theory (that were instead used in [6, 9]) and thus is very efficient and easy to implement and of potential practical interest. We also stress that, by the lower bound of [7], the algorithm for binary trees presented is actually optimal with respect to the approximation obtained.

We also prove that for the important special case of leaf-to-leaf communication the best approximation that can be guaranteeed by a greedy algorithm is $5L/3$ for set of leaf-to-leaf requests of maximum load L. The lower bound argument holds also for binary trees.

2 The Algorithm

We consider binary tree topologies (although a similar algorithm can be applied to any tree topology), with each edge of the tree consisting of two opposite directed fiber links. Connectivity requests between pairs of nodes are represented as a set of paths on the tree, such that the maximum number of paths going through any directed fiber link is at most L. Wavelength routing for such a network is equivalent to assigning to each path a proper color in such way that no two paths going through the same directed link have the same color (i.e., no two requests going through the same directed link use the same wavelength). We prove the following theorem.

Theorem 1 *There exists a polynomial time algorithm that routes connection requests with maximum load L per fiber link of a directed binary fiber tree using at most* $5L/3$ *wavelengths.*

The algorithm proceeds in phases, one per each node v of the fiber tree. The nodes are considered following their depth-first numbering. The phase associated with node v assumes that we already have a partial proper coloring of all paths that touch nodes with number strictly smaller than the number of v. During this phase, the partial coloring is extended to one that assigns proper colors to all paths that touch v but have not been colored yet. Throughout all phases the algorithm preserves that:

- The total number of colors used is at most $5L/3$.
- The total number of colors on two opposite directed fiber links is at most $4L/3$.

The proposed algorithm is a greedy algorithm. Once a connection request has been colored, the greedy algorithm never recolors it. Greedy algorithms do not require global control and are thus amenable of being implemented in a distributing setting: each node, starting from the root of the tree, locally decides the colors of the requests that go through it and are still uncolored and then passes the control to each of its children that can then proceed independently. In other words, a greedy algorithm does not need a "central authority" that has knowledge of the overall request pattern of the network to decide upon the colors. All known algorithms for the problem of wavelength routing on directed trees are indeed greedy algorithms [6, 8, 9].

3 Binary Fiber Trees

We consider the application of the algorithm on a binary tree. So far, the algorithm has produced a partial coloring for paths touching vertices with numbers strictly smaller than the number of v. In the current step, the algorithm will color the paths touched by v that have not been colored yet. We call K_1 the set of colors used for coloring the paths from the vertex w to the vertex u_1, K_2 the set of colors for paths from u_1 to w, K_3 the set of colors for paths from w to u_2 and K_4 the set of colors assigned to paths from u_2 to w. See Figure 1.

The algorithm has to produce a proper coloring for the requests going between u_1 and u_2 and the requests stopping and starting from v.

The colors of K_1, K_2, K_3, K_4 can be divided in *single* and *double* colors depending on whether they appear on one fiber link or on two opposite fiber links. For $1 \leq i \leq 4$, we denote by A_i the set of single colors that belong to K_i and, for $1 \leq i \neq j \leq 4$, by A_{ij} the set of double colors that belong to K_i and K_j. Since the coloring is correct, we have that A_{12}, A_{23}, A_{34} and A_{14} are disjoint and are the only non empty sets among the A_{ij}'s. In other words, we have

$$|K_1| = |A_1| + |A_{12}| + |A_{14}|,$$

$$|K_2| = |A_{12}| + |A_2| + |A_{23}|,$$

$$|K_3| = |A_{23}| + |A_3| + |A_{34}|,$$

$$|K_4| = |A_{14}| + |A_{34}| + |A_4|.$$

We start by considering a basic instance of the problem which can be solved easily. Then, we show how to reduce every other case to the basic problem.

4 The Basic Problem

In this section we show how to assign legal colors to communication requests in the case in which $|A_{12}| = |A_{34}|$.

We assume, without loss of generality, that the set of requests going from u_1 to u_2, which we denote by K_5, and the set of requests going from u_2 to u_1, which we denote by K_6, are of maximum possible size. In other words, we assume that:

$$|K_5| = L - \max\{|K_2|, |K_3|\},$$

$$|K_6| = L - \max\{|K_1|, |K_4|\}.$$

We consider four cases depending on the relative sizes of sets K_2 and K_3 and K_1 and K_4.

Case I.1: $|K_2| \geq |K_3|$ and $|K_1| \geq |K_4|$.
In this case we have:

$$|K_5| = L - |K_2| \text{ and } |K_6| = L - |K_1|.$$

Moreover observe that the link going from u_1 to v and the link going from v to u_1 have both load L and thus no other request can be assigned. The two links between v and u_2 instead can still accomodate some requests. We denote by K_7 the set of requests starting at v and going to u_2 and by K_8 the set of requests coming from u_2 and stopping at v (see Figure 1). Assuming that both K_7 and K_8 have maximum size, we have that:

$$|K_7| = L - |K_5| - |K_3| \text{ and } |K_8| = L - |K_6| - |K_4|.$$

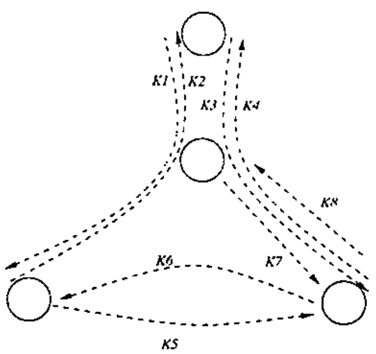

Figure 1: Case I.1.

Now observe that the old colors which do not belong to K_2 nor K_3 can be used to color requests from K_5. Therefore, for coloring the requests from u_1 to u_2 we need $\max\{0, T_5\}$ new colors, where $T_5 = L - |K_2| - |A_1| - |A_{14}| - |A_4|$. A similar argument shows that $\max\{0, T_6\}$ new colors are needed to color requests from K_6, where $T_6 = L - |K_1| - |A_3| - |A_{23}| - |A_2|$. Without loss of generality, we assume that $T_5 \geq T_6$. We also assume that $T_5 > 0$, for otherwise no new colors would be needed and the inductive hypotheses are maintained trivially.

We thus use T_5 new colors and color K_6, K_7 and K_8 without resorting to any other new colors. Therefore the total number of colors used is equal to the total number of old colors (which we denote by M) plus the total number of new colors (which is T_5). We thus have that the total number of colors is equal to:

$$\begin{aligned} M + T_5 &= L + M - |K_2| - |A_1| - |A_{14}| - |A_4| \\ &= L + |A_3| + |A_{34}| \\ &= L + |K_3| - |K_3 \cap K_2|. \end{aligned}$$

Now observe that, since $|K_3| \leq |K_2|$, $|K_3| - |K_3 \cap K_2| \leq |K_3|/2 \leq M/2$. By inductive hypothesis $M \leq 4L/3$, and thus $M + T_5 \leq L + M/2 \leq 5L/3$.

Let us now show that each pair of opposite links does not see more than $4L/3$ colors and to do this we need to specify how requests of K_6, K_7 and K_8 are colored. After we have colored the requests in K_5, the opposite links (v, u_1) and (u_1, v) see L colors, while the links (v, u_2) and (u_2, v) see $L - |K_2| + |K_3|$ colors. Some of the requests of K_6 can be colored recycling the T_5 new colors used to color the requests of K_5 and the colors from A_{23}. Observe that these colors are already seen in one of the two opposite links and thus we are not increasing the number of colors seen by the opposite links. After this coloring, the number of requests of K_6 that are still uncolored is:

$$\begin{aligned} |K_6| - T_5 - |A_{23}| &= L - |K_1| - L + |K_2| + |A_1| + \\ & \quad |A_{14}| + |A_4| - |A_{23}| \\ &= |A_2| + |A_4| \end{aligned}$$

We color these leftover requests using colors from A_2 and A_3. Simple algebraic manipulations show that $T_5 \geq T_6$ implies $|A_3| \geq |A_4|$, and thus we have enough colors to choose from. We color the still uncolored requests in the following way. $|K_2| - |K_3|$ of these requests are colored using colors from A_2. Notice that, $|A_{12}| = |A_{34}|$ implies $|K_2| - |K_3| = |A_2| - |A_3| \leq |A_2|$. After this coloring, the number of colors seen by links (u_2, v) and (v, u_2) increases to L and we are left with $|A_2| + |A_4| - |K_2| + |K_3|$ requests to color. Since $|A_3| + |A_2| - |K_2| + |K_3| \geq 1/2(|A_2| + |A_4| - |K_2| + |K_3|)$, the remaining paths can be colored half with colors of A_2 and half with colors of A_3.

Therefore each pair of opposite directed links now sees the following number of colors:

$$\begin{aligned} & L + 1/2(|A_2| + |A_4| - |K_2| + |K_3| \\ = \; & L + 1/2(|A_3| + |A_4|) \\ = \; & L + 1/2(|K_1 \cup K_2 \cup K_3 \cup K_4| - |K_1| - |K_2|) \\ \leq \; & L + 1/2 \cdot 2L/3 = 4L/3 \end{aligned}$$

Finally the requests in the sets K_7 and K_8 are colored as follows. Since $|K_7| = |K_2| - |K_3|$, the requests of K_7 can be colored using the colors of A_2 that have been employed for the coloring of K_6 without increasing the number of colors seen by the opposite arcs (v, u_2) and (u_2, v). Moreover, since $|K_8| = |K_1| - |K_4| \leq |A_1|$, K_8 can be colored using colors of A_1 (the number of colors seen by the opposite arcs (v, u_2) and (u_2, v) is not increased since the colors of A_1 have already been used for K_5).

Case I.2: $|K_2| \geq |K_3|$ and $|K_4| \geq |K_1|$. In this case we have:

$$|K_5| = L - |K_2| \text{ and } |K_6| = L - |K_4|.$$

Moreover observe that the link going from u_1 to v and the link going from u_2 to v have both load L and thus no other request can be assigned. Instead the links from v to u_2 and from v to u_1 can still accomodate some requests. We denote the set of requests from v to u_2 by K_7 and the set of requests from v to u_1 by K_8 and we have:

$$|K_7| = L - |K_5| - |K_3| \text{ and } |K_8| = L - |K_6| - |K_1|.$$

As done before, we define $T_5 = L - |K_2| - |A_1| - |A_{14}| - |A_4|$ and $T_6 = L - |K_4| - |A_3| - |A_{23}| - |A_2|$ and assume that $T_5 \geq T_6$ and that $T_5 > 0$. We use T_5 new colors to color the requests in K_5 and show how the requests in K_6, K_7 and K_8 can be colored without resorting to new colors. The total number of colors employed is thus:

$$\begin{aligned} M + T_5 &= L + M - |K_2| - |A_1| - |A_{14}| - |A_4| \\ &= L + |A_3| + |A_{34}| \\ &= L + |K_3| - |K_3 \cap K_2| \end{aligned}$$

which is $\leq 5L/3$ since $|K_3| - |K_3 \cap K_2| \leq 2L/3$.

Next we show how to color the set of requests K_6, K_7 and K_8 and bound the number of colors seen on opposite directed links. Before K_6 is colored, the opposite links (v, u_1) and (u_1, v) see L colors, while the links (v, u_2) and (u_2, v) see $L - |K_2| + |K_3|$ colors. We recycle colors from T_5 and A_{23} to color $|T_5| + |A_{23}|$ requests form K_6. Using these colors does not increase the number of colors seen by a pair of opposite directed links and leaves us with $|K_6| - T_5 - |A_{23}| = |A_1| + |A_2|$ uncolored requests in K_6. Now $T_5 \geq T_6$ and $|A_{12}| = |A_{34}|$ imply that $|A_3| \geq |A_1|$, so the uncolored paths can be colored using colors from A_2 and A_3.

We perform the coloring as before. We color $|K_2| - |K_3|$ of the uncolored requests of K_6 using colors from A_2 (this is possible since $|A_{12}| = |A_{34}|$ implies $|K_2| - |K_3| = |A_2| - |A_3| \leq |A_2|$). After this coloring, the number of colors seen by links (u_2, v) and (v, u_2) is L. The remaining requests are then colored half with colors of A_2 and half with colors of A_3.

Therefore each pair of opposite arcs now sees the following number of colors:

$$\begin{aligned} & L + 1/2(|A_1| + |A_2| - |K_2| + |K_3| \\ = \; & L + 1/2(|A_1| + |A_3|) \\ = \; & L + 1/2(|K_1 \cup K_2 \cup K_3 \cup K_4| - |K_2| - |K_4|) \\ \leq \; & L + 1/2 \cdot 2L/3 = 4L/3 \end{aligned}$$

Similarly to the previous case, K_7 and K_8 can be colored using colors of A_2 and A_4, respectively, without increasing the number of colors seen by the opposite arcs.

Case I.3: $|K_3| \geq |K_2|$ and $|K_1| \geq |K_4|$. This case is symmetric to case I.2.

Case I.4: $|K_3| \geq |K_2|$ and $|K_4| \geq |K_1|$. This case is symmetric to case I.1.

5 Solving the General Problem

In this section, we show how to assign colors to requests in the case in which $|A_{12}| \neq |A_{34}|$. We assume that $|A_{12}| \geq |A_{34}|$ and define $s = |A_{12}| - |A_{34}|$ (the symmetric case is handled similarly).

We select s colors from A_{12} and remove the paths that have been colored using these colors. In this way we have a new instance of the coloring problem, that can be solved as shown in the previous section. Then, we show how the solution to the new instance can be modified to obtain a solution for the original instance.

We let Λ_1 be the set of colors used for coloring the paths from the vertex w to the vertex u_1, Λ_2 the set of colors used for coloring the paths from u_1 to w, Λ_3 the set of colors used for coloring the paths from w to u_2, and Λ_4 the set of colors used for coloring the paths from u_2 to w in the new instance of the problem. We define the sets of colors B_i and B_{ij} for $1 \leq i < j \leq 4$ similary to the sets A_i's and A_{ij}'s.

Since the coloring in the new problem is still correct, we have:

$$\Lambda_1 = |B_1| + |B_{12}| + |B_{14}|$$

$$\Lambda_2 = |B_2| + |B_{12}| + |B_{23}|$$

$$\Lambda_3 = |B_3| + |B_{34}| + |B_{23}|$$

$$\Lambda_4 = |B_4| + |B_{34}| + |B_{14}|$$

and, obviously, $|B_{34}| = |B_{12}|$. Also, as before, we assume that requests along the links (u_1, u_2) and (u_2, u_1) fill the links to capacity. Denoting these two sets of requests by Λ_5 and Λ_6, we have $|\Lambda_5| = L - \max\{|\Lambda_2|, |\Lambda_3|\}$ and $|\Lambda_6| = L - \max\{|\Lambda_1|, |\Lambda_4|\}$.

Case II.1: $|K_1| \geq |K_4|$ and $|K_2| \geq |K_3|$.

Case II.1.1: $|\Lambda_1| \geq |\Lambda_4|$ and $|\Lambda_2| \geq |\Lambda_3|$.

In this case we have $|\Lambda_5| = L - |\Lambda_2|$ and $|\Lambda_6| = L - |\Lambda_1|$. Observe that, since we removed s requests along the two links between v and u_1, we have that $|K_5| = |\Lambda_5| - s$ and $|K_6| = |\Lambda_6| - s$ where K_5 and K_6 are the sets of requests from u_1 to u_2 and from u_2 to u_1 in the original instance that we can assume, without loss of generality, to be the maximum possible size.

We then fill to capacity the remaining links by $|\Lambda_7|$ and $|\Lambda_8|$ requests to the links (v, u_2) and (u_2, v), respectively, where

$$|\Lambda_7| = L - |\Lambda_5| - |\Lambda_3|$$

and

$$|\Lambda_8| = L - |\Lambda_6| - |\Lambda_4|.$$

In this way, we obtain the basic instance of a bandwidth allocation problem in a directed binary fiber tree with maximum load L per fiber link. The problem corresponds to the first case of the basic instance (remember we have $|B_{34}| = |B_{12}|$). We define X_5 and X_6 to be the number of new colors needed by paths of Λ_5 and Λ_6, and assume that $X_5 \geq X_6$. Simple algebraic manipulation shows that $X_5 = |\Lambda_5| - |B_1| - |B_{14}| - |B_4| = T_5 + s$ and $X_6 = |\Lambda_6| - |B_2| - |B_{23}| - |B_3| = T_6 + s$. The total number of colors used is thus

$$|\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4| + X_5 = L + |\Lambda_3| - |\Lambda_2 \cap \Lambda_3|.$$

Now, since $|\Lambda_2 \cup \Lambda_3| \leq 4L/3 - s$ and $|\Lambda_2| \geq |\Lambda_3|$, the total number of colors is at most $5L/3 - s/2$.

The coloring of the requests is performed in manner similar to the cases in the previous sections. The requests in Λ_5 are colored using all the colors of B_1, B_{14}, and B_4, along with the X_5 new colors. The requests in Λ_6 are instead colored using all the new colors and the colors of B_{23} and $|\Lambda_2| - |\Lambda_3| + 1/2(|B_2| + |B_4| - |\Lambda_2| + |\Lambda_3|)$ colors of B_2 and $1/2(|B_2| + |B_4| - |\Lambda_2| + |\Lambda_3|)$ colors of B_3. With this coloring, each pair of opposite directed fiber links sees

$$\begin{aligned} & L + 1/2(|B_2| + |B_4| - |\Lambda_2| + |\Lambda_3|) \\ = \; & L + 1/2(|B_3| + |B_4|) \\ = \; & L + 1/2(|\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4| - |\Lambda_1| - |\Lambda_2|) \\ \leq \; & L + 1/2(2L/3 - s/2) = 4L/3 - s/4. \end{aligned}$$

Let us show now how we compute the solution to the original instance. The number of new colors $X_5 = T_5 + s$ is certainly greater than s for otherwise the original problem would need no new colors at all. We select s new colors and remove the paths of Λ_5 and Λ_6 that have been colored with these colors. In addition, we put back in the paths that were removed at the beginning and which were colored with s colors of A_{12}. The number of colors, clearly does not increase. The opposite directed fiber links (v, u_1) and (u_1, v) now see no more than $4L/3 - s/4$ colors, while the fiber links (v, u_2) and (u_2, v) see at most $4L/3 - 5s/4$ colors.

Since $|A_1| + s \geq |K_8| = |K_1| - |K_4|$, K_8 can be colored using the colors of A_1 and s colors of A_{12}. Λ_7 will be colored using the s colors of A_{12} used also in Λ_8 and $|A_2| - |A_3|$ colors of A_2 that have already been used in Λ_6. Thus, the number of colors seen by pairs of opposite arcs is less than $4L/3 - s/4 \leq 4L/3$.

Case II.1.2: $|\Lambda_2| \geq |\Lambda_3|$ and $|\Lambda_4| \geq |\Lambda_1|$.

In this case we have $|\Lambda_5| = L - |\Lambda_2|$, $|\Lambda_6| = L - |\Lambda_4|$, and sets Λ_7 and Λ_8 of requests along the directed links (v, u_2) and (v, u_1) of size $|\Lambda_7| = L - |\Lambda_5| - |\Lambda_3|$ and $|\Lambda_8| = L - |\Lambda_6| - |\Lambda_1|$.

In this way, we obtain an instance of the bandwidth allocation problem in a directed binary fiber tree with maximum load L per fiber link which corresponds to the second case of the basic instance. We define $X_5 = |\Lambda_5| - |B_1| - |B_{14}| - |B_4| = T_5 + s$ and $X_6 = |\Lambda_6| - |B_2| - |B_{23}| - |B_3| = T_6 + |K_1| - |K_4|$ to be the number of new colors needed by paths of Λ_5 and Λ_6, and assume that $X_6 \geq X_5$, without loss of generality. The total number of colors needed used is thus

$$|\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4| + X_6 = L + |B_1| + |B_{34}|.$$

The requests in Λ_6 are colored using all the colors of B_2, B_{23}, and B_3, and X_6 new colors. The requests in Λ_5 are colored using all the new colors along with the colors of B_{14}. The number of paths that remain uncolored is:

$$|\Lambda_5| - X_6 - |B_{14}| = |B_3| + |B_4|$$

But $X_6 \geq X_5$ implies that $|B_1| \geq |B_3|$ so the remaining paths can be colored using colors of B_1 and B_4. Thus, $|\Lambda_4| - |\Lambda_1| + 1/2(|B_3| + |B_4| - |\Lambda_4| + |\Lambda_1|)$ and $1/2(|B_3| + |B_4| - |\Lambda_4| + |\Lambda_1|)$ colors are used from B_4 and B_1, respectively, and each pair of opposite directed arcs sees $L + 1/2(|B_3| + |B_4| - |\Lambda_4| + |\Lambda_1|) = L + 1/2(|B_1| + |B_3|)$ colors.

We will now return to the real problem. The number of new colors is greater than $|K_1| - |K_4|$, for otherwise the real problem would need no new colors at all. We thus pick $|K_1| - |K_4|$ of the X_6 new colors introduced and remove the paths of Λ_5 and Λ_6 that have been colored with these colors. In addition, we remove $s - |K_1| + |K_4| = |\Lambda_4| - |\Lambda_1|$ paths of Λ_5 that have been colored using colors of A_4 and add the paths we removed at the beginning which were colored with the s colors of A_{12}. The two opposite links (v, u_1) and (u_1, v) now see no more than $L + 1/2(|A_1| + |A_3|)$ colors, while the opposite arcs (v, u_2) and (u_2, v) see at most $L + 1/2(|A_1| + |A_3) - |K_1| + |K_4|$ colors.

Since $s \geq |K_8| = |K_1| - |K_4|$, the requests in K_8 can be colored using $|K_1| - |K_4|$ colors of A_{12}. Note that it is $|K_2| - |K_3| = |K_1| - |K_4| + |A_2| - |A_3| + |A_4| - |A_1|$, so K_7 can be colored using the $|K_1| - |K_4|$ colors of A_{12} used also in K_8, $|A_2| - |A_3|$ colors of A_2, and $|A_4| - |A_1|$ of A_4.

The total number of colors is $L + |A_1| + |A_{34}| - |K_1| + |K_4| + s = L + |A_4| + |A_{34}| = L + |K_4| - |K_1 \cap K_4|$. But $|K_1 \cup K_4| \leq 4L/3$ and $|K_1| \geq |K_4|$ implies that $|K_4| - |K_1 \cap K_4| \leq 2L/3$ and the total number of colors is less than $5L/3$. Each pair of opposite directed arcs sees $L + 1/2(|A_1| + |A_3|)$ colors. Since $|A_1| + |A_2| + |A_3| + |A_4| \leq 4L/3, |A_1| \leq |A_4|$ and $|A_3| \leq |A_2|$, the number of colors seen by each pair of opposite directed arcs is at most $4L/3$.

Case II.1.3: $|\Lambda_1| \geq |\Lambda_4|$ and $|\Lambda_3| \geq |\Lambda_2|$.

This case is symmetric to case II.1.2.

Case II.1.4: $|\Lambda_4| \geq |\Lambda_1|$ and $|\Lambda_3| \geq |\Lambda_2|$.

In this case we have $|\Lambda_5| = L - |\Lambda_3|$, $|\Lambda_6| = L - |\Lambda_4|$, and sets Λ_7 and Λ_8 of requests along the directed links (u_1, v) and (v, u_1) of size $|\Lambda_7| = L - |\Lambda_5| - |\Lambda_2|$ and $|\Lambda_8| = L - |\Lambda_6| - |\Lambda_1|$.

In this way, we obtain a simple bandwidth allocation problem in a directed binary fiber tree with maximum load L per fiber link which corresponds to the fourth case of the basic instance. We define $X_5 = |\Lambda_5| - |B_1| - |B_{14}| -$

$|B_4| = T_5 + |K_2| - |K_3|$ and $X_6 = |\Lambda_6| - |B_2| - |B_{23}| - |B_3| = T_6 + |K_1| - |K_4|$ to be the number of new colors needed by paths of Λ_5 and Λ_6, and assume that $X_5 \geq X_6$, without loss of generality. The total number of colors needed is thus:

$$|\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4| + X_5 = L + |B_2| + |B_{34}|$$

The requests in Λ_5 will be colored using all the colors of B_1, B_{14}, and B_4, and X_5 new colors. The requests in Λ_6, are colored using all the new colors along with the colors of B_{14}. The number of paths that remain uncolored is:

$$|\Lambda_6| - X_5 - |B_{23}| = |B_1| + |B_3|$$

But $X_5 \geq X_6$ implies that $|B_2| \geq |B_1|$ so the remaining paths can be colored using colors of B_2 and B_3. Thus, $|\Lambda_3|-|\Lambda_2|+1/2(|B_1|+|B_3|-|\Lambda_3|+|\Lambda_2|)$ and $1/2(|B_1|+|B_3| - |\Lambda_3| + |\Lambda_2|)$ colors are used from B_3 and B_2, respectively, and each pair of opposite directed arcs sees $L + 1/2(|B_1| + |B_3| - |\Lambda_3| + |\Lambda_2|) = L + 1/2(|B_1| + |B_2|)$ colors.

We will now return to the real problem. The number of new colors is greater than $|K_2|-|K_3|$, for otherwise the real problem would need no new colors at all. We pick $|K_2| - |K_3|$ of the X_5 new colors introduced and remove the paths of Λ_5 and Λ_6 that have been colored with these colors. In addition, we remove $s - |K_2| + |K_3| = |\Lambda_3| - |\Lambda_2|$ paths of Λ_6 that have been colored using colors of A_3 and add the paths we removed at the beginning which was colored with s colors of A_{12}. The opposite arcs (v, u_1) and (u_1, v) now see no more than $L + 1/2(|A_1| + |A_2|)$ colors, while the opposite arcs (v, u_2) and (u_2, v) see at most $L+1/2(|A_1|+|A_2) - |K_2| + |K_3|$ colors.

Since $s \geq |K_7| = |K_2| - |K_3|$, the requests in K_7 can be colored using $|K_2| - |K_3|$ colors of A_{12}. Note that it is $|K_1|-|K_4| \leq |K_2|-|K_3|+|A_1|+|A_3|-|A_2| = |A_1|+s$, so K_8 can be colored using the $|K_2| - |K_3|$ colors of A_{12} used also in K_7, colors of A_1 and colors of A_3 that have not been used in K_7 (there are at least $|\Lambda_3| - |\Lambda_2|$ such colors).

Thus, the total number of colors is $L + |A_2| + |A_{34}| - |K_2| + |K_3| + s = L + |A_3| + |A_{34}| = L + |K_3| - |K_2 \cap K_3| \leq 5L/3$, while each pair of opposite directed arcs sees $L + 1/2(|A_1| + |A_2|) \leq 4L/3$ colors.

Case II.2: $|K_1| \geq |K_4|$ and $|K_3| \geq |K_2|$.

Case II.2.1: $|\Lambda_1| \geq |\Lambda_4|$ and $|\Lambda_3| \geq |\Lambda_2|$.

In this case we have $|\Lambda_5| = L - |\Lambda_3|$, $|\Lambda_6| = L - |\Lambda_1|$, and sets Λ_7 and Λ_8 of requests along the directed links (u_1, v) and (u_2, v) of size $|\Lambda_7| = L-|\Lambda_5|-|\Lambda_2|$ and $|\Lambda_8| = L - |\Lambda_6| - |\Lambda_4|$.

In this way, we obtain a simple bandwidth allocation problem in a directed binary fiber tree with maximum load L per fiber link which corresponds to the third case of the basic instance. We define $X_5 = |\Lambda_5| - |B_1| - |B_{14}| - |B_4| = T_5$ and $X_6 = |\Lambda_6| - |B_2| - |B_{23}| - |B_3| = T_6 + s$ to be the number of new colors needed by paths of Λ_5 and Λ_6, and assume that $X_5 \geq X_6$, without loss of generality. The total number of colors needed will be:

$$|\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4| + X_5 = L + |B_2| + |B_{34}|$$

The requests in Λ_5 are colored using all the colors of B_1, B_{14}, and B_4, and X_5 new colors. The requests in Λ_6 are colored using all the new colors along with the colors of B_{23}. The number of paths that remain uncolored is:

$$|\Lambda_6| - X_5 - |B_{23}| = |B_3| + |B_4|$$

But $X_5 \geq X_6$ implies that $|B_2| \geq |B_4|$ so the remaining paths can be colored using colors of B_2 and B_3. Thus, $|\Lambda_3|-|\Lambda_2|+1/2(|B_3|+|B_4|-|\Lambda_3|+|\Lambda_2|)$ and $1/2(|B_3|+|B_4| - |\Lambda_3| + |\Lambda_2|)$ colors are used from B_3 and B_2, respectively, and each pair of opposite directed arcs sees $L + 1/2(|B_3| + |B_4| - |\Lambda_3| + |\Lambda_2|) = L + 1/2(|B_2| + |B_4|)$ colors.

We will now return to the real problem. We remove s paths of Λ_6 that have been colored using colors of A_3 (note that it is $|A_3| - |A_2| \geq s$) and add the paths we removed at the beginning which was colored with s colors of A_{12}. The requests in K_7 can be colored using the $|A_3| - |A_2| - s$ colors that have been used in K_6 while K_8 can be colored with the colors of A_1 and colors of A_3 that have not been used in K_6 since $|A_1|+|A_3|-|A_3|+|A_2|+s-1/2(|A_2|+|A_4|) \geq |K_1| - |K_4|$.

Thus, the total number of colors is $L+|A_2|+|A_{34}|+s = L + |K_2| - |K_2 \cap K_3| \leq 5L/3$, while each pair of opposite directed arcs sees $L + 1/2(|A_2| + |A_4|) \leq 4L/3$ colors.

Case II.2.2: $|\Lambda_4| \geq |\Lambda_1|$ and $|\Lambda_3| \geq |\Lambda_2|$.

In this case we have $|\Lambda_5| = L - |\Lambda_3|$, $|\Lambda_6| = L - |\Lambda_4|$, and sets Λ_7 and Λ_8 of requests along the directed links (u_1, v) and (v, u_1) of size $|\Lambda_7| = L-|\Lambda_5|-|\Lambda_2|$ and $|\Lambda_8| = L - |\Lambda_6| - |\Lambda_1|$.

In this way, we obtain a simple bandwidth allocation problem in a directed binary fiber tree with maximum load L per fiber link which corresponds to the fourth case of the basic instance. We define $X_5 = |\Lambda_5| - |B_1| - |B_{14}| - |B_4| = T_5$ and $X_6 = |\Lambda_6| - |B_2| - |B_{23}| - |B_3| = T_6 + |K_1| - |K_4|$ to be the number of new colors needed by paths of Λ_5 and Λ_6, and assume that $X_5 \geq X_6$, without loss of generality. The total number of colors needed will be:

$$|\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4| + X_5 = L + |B_2| + |B_{34}|$$

The requests in Λ_5 are colored using all the colors of B_1, B_{14}, and B_4, and X_5 new colors. The requests in Λ_6 are colored using all the new colors along with the colors of B_{23}. The number of paths that remain uncolored is:

$$|\Lambda_6| - X_5 - |B_{23}| = |B_1| + |B_3|$$

But $X_5 \geq X_6$ implies that $|B_2| \geq |B_1|$ so the remaining paths can be colored using colors of B_2 and B_3. Thus,

$|\Lambda_3|-|\Lambda_2|+1/2(|B_1|+|B_3|-|\Lambda_3|+|\Lambda_2|)$ and $1/2(|B_1|+|B_3|-|\Lambda_3|+|\Lambda_2|)$ colors are used from B_3 and B_2, respectively, and each pair of opposite directed arcs sees $L+1/2(|B_1|+|B_3|-|\Lambda_3|+|\Lambda_2|) = L+1/2(|B_1|+|B_2|)$ colors.

We will now return to the real problem. We remove $|K_1|-|K_4|$ paths of Λ_6 that have been colored using colors of A_3 (note that it is $|A_3|-|A_2| \geq s \geq |K_1|-|K_4|$) and add the paths we removed at the beginning which was colored with s colors of A_{12}. K_8 can be colored using the $|K_1|-|K_4|$ colors of A_3 that are not used in K_6 while K_7 can be colored with the colors of A_3 that have been used in K_6 since $|A_3|-|A_2|-|K_1|+|K_4|+1/2(|A_1|+|A_2|) \geq |A_3|-|A_2|-s = |K_3|-|K_2|$

Thus, the total number of colors is $L+|A_2|+|A_{34}|+s = L+|K_2|-|K_2 \cap K_3| \leq 5L/3$, while each pair of opposite directed arcs sees $L+1/2(|A_1|+|A_2|) \leq 4L/3$ colors.

Case II.3: $|K_4| \geq |K_1|$ and $|K_2| \geq |K_3|$.
This case is symmetric to case II.2.

Case II.4: $|K_4| \geq |K_1|$ and $|K_3| \geq |K_2|$.
In this case we have $|\Lambda_5| = L - |\Lambda_3|$, $|\Lambda_6| = L - |\Lambda_4|$, and sets Λ_7 and Λ_8 of requests along the directed links (u_1, v) and (v, u_1) of size $|\Lambda_7| = L-|\Lambda_5|-|\Lambda_2|$ and $|\Lambda_8| = L-|\Lambda_6|-|\Lambda_1|$.

In this way, we obtain a simple bandwidth allocation problem in a directed binary fiber tree with maximum load L per fiber link which corresponds to the fourth case of the basic instance. We define $X_5 = |\Lambda_5|-|B_1|-|B_{14}|-|B_4| = T_5$ and $X_6 = |\Lambda_6|-|B_2|-|B_{23}|-|B_3| = T_6$ to be the number of new colors needed by paths of Λ_5 and Λ_6, and assume that $X_5 \geq X_6$, without loss of generality. The total number of colors needed will be:

$$|\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4| + X_5 = L + |B_2| + |B_{34}|$$

The requests in Λ_5 will be colored using all the colors of B_1, B_{14}, and B_4, and X_5 new colors. The requests in Λ_6 are colored using all the new colors along with the colors of B_{23}. The number of paths that remain uncolored is:

$$|\Lambda_6| - X_5 - |B_{23}| = |B_1| + |B_3|$$

But $X_5 \geq X_6$ implies that $|B_2| \geq |B_1|$ so the remaining paths can be colored using colors of B_2 and B_3. Thus, $|\Lambda_3|-|\Lambda_2|+1/2(|B_1|+|B_3|-|\Lambda_3|+|\Lambda_2|)$ and $1/2(|B_1|+|B_3|-|\Lambda_3|+|\Lambda_2|)$ colors are used from B_3 and B_2, respectively, and each pair of opposite directed arcs sees $L+1/2(|B_1|+|B_3|-|\Lambda_3|+|\Lambda_2|) = L+1/2(|B_1|+|B_2|)$ colors.

We will now return to the real problem. We add the paths we removed at the beginning which was colored with s colors of A_{12}. The opposite arcs (v, u_1) and (u_1, v) see $L+1/2(|A_1|+|A_2|)+s$ colors, while the opposite arcs vu_2 and u_2v see $L+1/2(|A_1|+|A_2|)$ colors. We insert $s/2$ new colors and replace colors of A_3 in paths of K_6 and colors of A_4 in paths of K_5 with the new colors (we can do so because $|A_3|-|A_2|+1/2(|A_1|+|A_2|) \geq s/2$ and $|A_4| \geq s/2$). Each pair of opposite directed arcs see $L+1/2(|A_1|+|A_2|+s)$ colors. But $|A_1|+|A_2| \leq |A_3|+|A_4|-2s \leq 4L/3-3s-|A_1|-|A_2| \Rightarrow |A_1|+|A_2| \leq 2L/3-3s/2$ so each pair of opposite directed arcs see at most $4L/3-s/4$ colors. The total number of colors is $L+|A_2|+|A_{34}|+s+s/2 = L+|A_2|+|A_{12}|+s/2$. But $|A_{12}| \leq |A_3|-|A_2|+|A_{34}|$ and $|A_{12}| \leq |A_4|-|A_1|+|A_{34}|$ meaning that $|A_{12}| \leq |A_{34}| + \frac{|A_3|+|A_4|-|A_1|-|A_2|}{2}$ so the total number of colors is at most $L+\frac{|A_2|+|A_3|+|A_4|-|A_1|}{2}+|A_{34}|+s/2 \leq L+\frac{|A_2|+|A_3|+|A_4|+|A_{12}|+|A_{34}|-|A_1|}{2} \leq 5L/3$.

K_8 can be colored using the colors of A_4 that have been used in K_5 since $|A_4|-s/2 \geq |A_4|-|A_1|-s = |K_4|-|K_1|$ and K_7 can be colored using the colors of A_3 that have been used in K_6 since $|A_3|-|A_2|+1/2(|A_1|+|A_2|)-s/2 \geq |A_3|-|A_2|-s = |K_3|-|K_2|$.

6 Binary Trees of Rings

The tree of rings is an important architecture for optical networks. Local area optical networks based on the ring architecture (i.e. SONET rings [10]) are connected together via a tree shaped backbone network. In particular, if the topology of the backbone network is a binary tree, wavelength routing can be done using our 5/3-aproximation. According to the methodology presented in [9], for assigning wavelengths to requests on a binary tree of rings, we remove one edge from every ring so to obtain a tree and we solve the problem using our algorithm. The coloring obtained uses $5/3 \cdot 2L = 10L/3$ colors.

7 Lower Bound for Leaf-to-Leaf Communication

In this section we show that for each greedy algorithm and for each $\epsilon > 0$ there exists a tree and a set of leaf-to-leaf requests of maximum load L such that the algorithm uses more than $(5/3-\epsilon)L$ wavelengths. We first prove our lower bound for ternary trees and then briefly discuss how to modify the argument for the case of binary trees.

We prove the lower bound inductively. We consider a tree in which each vertex has three children and one of the children is a leaf. We call the two non leaf children of a vertex the left and the right children. We assume inductively that, for a vertex V there are $\alpha_n L/2$ requests along each link to its parent C and that all of these requests are colored using different colors. Then we choose a set of requests between the two children A and B of V so to enforce the inductive hypothesis between V and one of its children for $\alpha_{n+1} = 1+\frac{\alpha}{4}$. It is easy to see that $\lim_{n\to\infty} \alpha_n = 4/3$, where

$$\alpha_n = \begin{cases} 1, & \text{for } n = 1; \\ 1+\frac{\alpha_{n-1}}{4}, & \text{for } n > 1 \end{cases}.$$

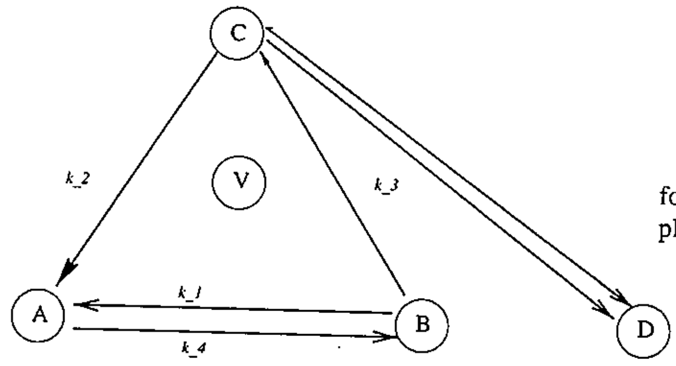

Figure 2: The lower bound.

Therefore, for any $\epsilon > 0$ and any greedy algorithm G, it is possible to construct a set of communication requests of maximum load L so that G uses at least $(5/3 - \epsilon)L$ colors. To prove the lower bound we have to enforce the above described strategy with all the requests starting and arriving to a leaf.

The base of our induction for $\alpha = 1$ is established in the following way. We start from the root R and consider its 2 children R_1 and R_2. We have $2L$ requests (i.e., L requests on each direction) between two children R_1 and R_2 of the the root R and we assume that R_2 is a leaf. The greedy algorithm colors these requests using at least L colors in each direction. We then choose two sets of $L/2$ requests along each direction with each request colored with a different color, propagate them to one of the children of R_1 and send the remaining requests to the child of R_1 that is a leaf.

In general let us consider a vertex V, and let C be its parent, A and B and D its children (see Figure 2) and assume D is a leaf. We denote by K_2 the set of colors used along the link (C, A), by K_3 the set of colors used along the link (B, C), by K_1 the set of colors used along the link (B, A) and by K_4 the set of colors used along the link (A, B).

We inductively assume that

Assumption 1. $K_2 \cap K_3 = \emptyset$;

Assumption 2. $|K_2| = |K_3| = \alpha L/2$;

and thus $|K_2 \cup K_3| = \alpha L$. We fill the link (B, A) to capacity by assigning $|K_1| = L(1 - \alpha/2)$ requests. These requests need to be colored with new colors and thus the total number of colors used increases to $L(1 + \frac{\alpha}{2})$. Next we assign L requests to the link (A, B). The best that the greedy algorithm can do is to color these L requests colored using all the new colors employed for the link (B, A), plus half of the colors of K_3 and half of the colors of K_2. The edge (A, V) thus sees

$$|K_1 \cup K_2 \cup K_4| =$$

$$= |K_1| + |K_2| + |K_4| - |K_2 \cap K_4| - |K_1 \cap K_4| =$$

$$= (1 - \frac{\alpha}{2})L + \frac{\alpha}{2}L + L - \frac{\alpha}{4} - (1 - \frac{\alpha}{4})L =$$

$$= (1 + \frac{\alpha}{4})L$$

In other to complete the inductive step we have to enforce for A the same situation as in V only with $(1 + \frac{\alpha}{4})$ in place of α. This is achieved in the following way:

1. among the $|K_1| + |K_2| = L$ requests coming from V, we let only the following requests continue to the left child of A.
   - S_1: $(\frac{1}{2} - \frac{\alpha}{8})\, L$ requests from K_1.
   - S_2: $\frac{\alpha}{4}L$ requests from K_2.

   for a total of $\frac{1}{2}\,(1 + \frac{\alpha}{4})$ colors.
2. and the $|K_4| = L$ requests coming up from A to V all originate from the leaf child of A except for the following ones which instead come from the right child of A.
   - R_1: $\frac{\alpha}{4}L$ requests that were colored with colors used of K_2 and which were not considered in S_2 above.
   - R_2: $\frac{\alpha}{4}L$ requests that were colored with colors used of K_3.
   - R_3: $(\frac{1}{2} - \frac{3}{8}\alpha)\, L$ requests that were colored with colors of K_1 and which were not considered in S_1 above.

   for a total of $\frac{1}{2}\,(1 + \frac{\alpha}{4})$ colors.

All other requests instead go to the child of A that is a leaf.

To obtain the lower bound for binary trees we need to split the vertex V with three vertices, one vertex and its two children, with one of the children being the leaf to which we send all requests that were going to be sent to the leaf child of A.

Finally, observe that the requests going down to the left child of A and those coming up from the right child of A are colored with different colors (i.e. the sets of colors are disjoint).

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