Solving Guarded Domain Equations in Presheaves Over Ordinals and Mechanizing It

Sergei Stepanenko
Aarhus University, Denmark

Amin Timany
Aarhus University, Denmark

Abstract

Constructing solutions to recursive domain equations is a well-known, important problem in the study of programs and programming languages. Mathematically speaking, the problem is finding a fixed point (up to isomorphism) of a suitable functor over a suitable category. A particularly useful instance, inspired by the step-indexing technique, is where the functor is over (a subcategory of) the category of presheaves over the ordinal ω and the functors are locally-contractive, also known as *guarded functors*. This corresponds to step-indexing over natural numbers. However, for certain problems, *e.g.*, when dealing with infinite non-determinism, one needs to employ trans-finite step-indexing, *i.e.*, consider presheaf categories over higher ordinals. Prior work on trans-finite step-indexing either only considers a very narrow class of functors over a particularly restricted subcategory of presheaves over higher ordinals, or treats the problem very generally working with sheaves over an arbitrary complete Heyting algebra with a well-founded basis.

In this paper we present a solution to the guarded domain equations problem over *all* guarded functors over the category of *presheaves* over ordinal numbers, as well as its mechanization in the Rocq Prover. As the categories of sheaves and presheaves over ordinals are equivalent, our main contribution is simplifying prior work from the setting of the category of sheaves to the setting of the category of presheaves and mechanizing it — presheaves are more amenable to mechanization in a proof assistant.

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1 Introduction

Recursive Domain Equations and Step-Indexing Recursive definitions are prevalent in computer programming. Thus, one of the important problems in the study of programs and programming languages is finding recursive mathematical objects to construct models of

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programs or the mathematical tools to study them, *e.g.*, program logics. This problem is often stated as a so-called *domain equation* [44] in terms of a fixed point (up to isomorphism) of an endo-functor $F : \mathcal{C} \to \mathcal{C}$ on a suitable category $\mathcal{C}$, *i.e.*, an object X of $\mathcal{C}$ such that $F(X) \simeq X$.[1] The problem was first studied by Dana Scott [42, 41] in the category of continuous lattices in order to give denotational semantics to untyped λ-calculi. Scott's construction [42] takes the inverse limit of an ω-tower of continuous lattices obtained through successive applications of the functor. As observed by Lawvere [42], the essence of the proof showing that this construction indeed constructs a fixed point is that the inverse limit coincides with the direct limit (of a related diagram). This has since been named the limit-co-limit coincidence theorem [44]. Wand [51] later observed that the essential point, rather than the category itself, is the structure of its hom-sets. Wand [51], Smyth, and Plotkin [44] give an abstract account of solving domain equations for endo-**O**-functors on **O**-categories, *i.e.*, categories that are enriched over the category of ω-cpos and ω-continuous functions and functors over them whose actions on morphisms is ω-continuous.[2] America and Rutten [3] solve domain equations in a certain category of metric spaces. Birkedal *et al.* [11] generalize the results of America and Rutten by constructing solutions to domain equations over so-called **M**-categories, categories enriched over the category of ultra-metric spaces and non-expansive maps, where the functors considered are locally contractive in the sense that the functors' action on morphisms is a contractive function. This generalization is inspired [11] by the relationship between bounded bisected ultra-metric spaces (where the distances belong to the set $\{0\} \cup \left\{\frac{1}{2^n} \middle| n \in \mathbb{N}\right\}$) and the technique of step-indexing [37, 4, 1, 10]. Birkedal *et al.* [9] later generalize these results further to a setting where the category is enriched over the category of sheaves over a complete Heyting algebra with a well-founded basis; ordinals in general, and ω in particular, being such complete Heyting algebras. These results [11, 9] have served as a foundation for a multitude of works based on step-indexing, *e.g.*, to give denotational semantics to programs [39, 14], to construct the model of the Iris program logic framework [29].

The Need for Step-Indexing Over Higher Ordinals It is well known [7, 13] that if one uses the step-indexing technique to reason about a programming language with countable non-determinism, it is no longer sufficient to consider step-indexing over ω. One must [13] instead use step-indexing over ω_1 (the first uncountable ordinal). Another way to look at this issue is through the lens of the step-indexed logic. The pertinent problem to consider here is the question of when existence of an object inside the step-indexed logic implies its existence outside. This is dubbed "the existential property" by Spies *et al.* [45]. Say we are given a predicate ϕ over a set A for which we have that $\exists x : A.\ \phi(x)$ is a valid sentence in the model of the step-indexed logic, *i.e.*, $\models \exists x : A.\ \phi(x)$. The existential property states that $\models \exists x : A.\ \phi(x)$ implies that there exists some $a \in A$ for which we have $\models \phi(a)$. Question: when does the existential property hold? Answer: if the cardinality $|A|$ is strictly smaller than that of the ordinal γ we are step-indexing over (provided that γ is a regular ordinal). Indeed, if $|A|$ is not smaller than the step-indexing ordinal γ, there are predicates for which the existential property does not hold. For a detailed, formal discussion of the need for step-indexing over higher-ordinals see Appendix A.

[1] In this paper we assume the reader is familiar with basic concepts in category theory found in standard textbooks [5, 34].

[2] See Section 3 for a brief explanation of enriched categories and functors. More details can be found in the book of Kelly [30] on the subject.

Spies *et al.* [45], motivated by this need, extend the step-indexed logic underlying the Iris program logic to trans-finite ordinals. Their work is also mechanized on top of the Rocq Prover. However, their domain equation solver can only be used to solve domain equations of functors of a special form. This special form is suitable for constructing Iris's higher-order resources [28], but is not sufficient for arbitrary locally contractive functors, *e.g.*, the functor for constructing so-called guarded interaction trees [17]. We will further discuss the relation between our work and Spies *et al.* [45] in Section 6.

Step-Indexing Over All Ordinals In the rest of this paper we talk about step-indexing over (all) ordinals, *e.g.*, we speak of sheaves or presheaves over **Ord**, the set of all ordinals (which we also consider to be a preorder category under the usual order). This is to be understood as the set of all ordinals definable in a certain Grothendieck universe. (In our Rocq formalization, the type **Ord** is a universe polymorphic definition corresponding to the type of all ordinals in the universe.) In terms of (pre)-sheaves over **Ord**, this should be understood as (pre)-sheaves over the ordinal that is the supremum of all ordinals in **Ord** — which obviously itself lives in a version of **Ord** in a larger Grothendieck universe. The upshot of step-indexing over all ordinals in the universe is that then the existential property holds for any set/type in the universe (see Appendix A).

The only downside of working with **Ord** is that it is not closed under suprema. That is, there are subsets $A \subseteq \mathbf{Ord}$ such that $\sup(A)$ does not exist (technically it does but it lives in a copy of **Ord** in a larger universe). To compensate for this issue, many of our definitions and constructions are parameterized by an arbitrary *downwards-closed* subset of ordinals instead of ordinals. This can intuitively be thought of as working with *the completion of* **Ord** as a lattice instead of **Ord** itself. This significantly simplifies the presentation, and more importantly mechanization of our results; see Section 5.

Equivalence of Categories of Sheaves Over Ordinals and Presheaves Over Ordinals As we will discuss in Section 2 the category of sheaves over ordinals, $\mathbf{Sh}(\mathbf{Ord})$, is equivalent (in fact adjoint equivalent) to the category of presheaves over **Ord**, $\mathbf{PSh}(\mathbf{Ord})$. Thus, technically speaking, the results of Birkedal *et al.* [9] subsume the results we present in this paper. That is, since Birkedal *et al.* [9] construct solutions to guarded domain equations over sheaves over complete Heyting algebras with a well-founded basis, and ordinals are a particular instance of such Heyting algebras, one can obtain solutions to equations over $\mathbf{PSh}(\mathbf{Ord})$ from solutions to equations over $\mathbf{Sh}(\mathbf{Ord})$. This is similar to how one obtains solutions to classical domain equations (over category **Dom** of domains) from those over the equivalent category **CUSL** of conditional upper semi-lattices with a least elements [47, chapter 4].

Contributions Notwithstanding the point above regarding the equivalence of categories of presheaves and sheaves over ordinals, the main contribution of this paper is simplifying and mechanizing the results of Birkedal *et al.* [9] to the setting of presheaves over ordinals and locally contractive functors on them which is much more amenable to mechanization. In fact, the aforementioned equivalence is the reason we were convinced that a simplification to the setting of presheaves, and a direct solution construction in that setting is achievable and mechanizable. In this paper we present this simplified version and its mechanization in the Rocq Prover. We also mechanize the symmetrization argument [16] to solve mixed-variance recursive domain equations and provide an example of solving a concrete mixed-variance equation using our framework. All results marked with 🐓 are mechanized [46] in the Rocq Prover. We only present a few high-level proofs in this paper which help the reader appreciate the results. For the rest we refer to our Rocq mechanization [46].

The Structure of the Rest of the Paper Section 2 introduces some basic constructions over the category of sheaves and presheaves over ordinals, including their equivalence. In Section 3 we present categories enriched over (the cartesian structure of) presheaves over ordinals, including the central concepts of enriched and locally contractive functors. We also present the concepts of ordinal-partial isomorphism and enriched-pointwiseness of limits, which play an important role in our construction of solutions of domain equations. Section 4 gives details of our construction of solutions to domain equations as well as their uniqueness. The technicalities involved in the mechanization of the results are discussed in Section 5. In Section 6 we discuss other works related to ours, and present our future work and concluding remarks in Sections 7 and 8 respectively. The Appendix A presents discusses the need for higher ordinals, while Appendices B–D include some details omitted from the main text.

▶ Remark 1 (Notation). We fix the following notational convention for the rest of the paper:

Notation	Convention / Meaning
$X := Y$	X is defined as Y
$\alpha, \beta, \gamma, \dots$	Ordinals
α^+	Successor of α
$\alpha \prec \beta, \alpha \preceq \beta$	Order of ordinals
$F, G, H, \dots$	(Pre)sheaves, functors
$A, B, C, \dots$	Objects, could be (pre)sheaves
$\eta, \xi, \zeta, \dots$	Natural transformation
$f, g, h, \dots$	Morphisms, could be natural transformations
Π^L_A	Projection from L onto $F(A)$ when L is the limit of the functor F
$F\|_A$	Restrict the domain of the functor (or function) F to A

Notation	Convention / Meaning
$\mathcal{Y}$	The Yoneda embedding
$A \simeq B$	Isomorphism
$f : A \xrightarrow{\simeq} B$	The morphism $f : A \to B$ has an inverse and A and B are isomorphic
$F_{\beta \preceq \alpha}$	The map $F(\beta) \to F(\alpha)$ induced by the (pre)sheaf F
$\lim_{\beta \prec \alpha} F(\beta)$	Limit of the diagram F whose domain is (restricted to) $\{\beta \| \beta \prec \alpha\}$; when obvious, we drop the $\beta \prec \alpha$ part
$\lim_{\beta} f_\beta$	The unique morphism from A to $\lim_{\beta} F(\beta)$ when $f_\beta : A \to F(\beta)$
$curry\,(f)$	The exponential transpose of f

2 On Sheaves and Presheaves Over Ordinals

We start by giving a few basic definitions in the category of presheaves over ordinals, $\mathbf{PSh}(\mathbf{Ord})$. In particular, there are two important endo-functors on $\mathbf{PSh}(\mathbf{Ord})$ called *later* ($\blacktriangleright : \mathbf{PSh}(\mathbf{Ord}) \to \mathbf{PSh}(\mathbf{Ord})$), and *earlier* ($\blacktriangleleft : \mathbf{PSh}(\mathbf{Ord}) \to \mathbf{PSh}(\mathbf{Ord})$). These functors are defined as follows:

$$\blacktriangleright F(\alpha) := \lim_{\beta \prec \alpha} F(\beta) \qquad \blacktriangleleft F(\alpha) := F(\alpha^+)$$
$$(\blacktriangleright F)_{\beta \preceq \alpha} := \lim_{\gamma \prec \beta} \Pi_\gamma^{\blacktriangleright F(\alpha)} \qquad (\blacktriangleleft F)_{\beta \preceq \alpha} := F_{\beta^+ \preceq \alpha^+}$$

The object map of $\blacktriangleright$, at each stage, takes the limit (in **Set**) of the diagram induced by the object (presheaf) it is mapping at all smaller stages. In particular, $\blacktriangleright F(0)$ is always the terminal (singleton) set, and $\blacktriangleright F(\alpha^+) \simeq F(\alpha)$ (see Lemma 41 in Appendix D). The morphism map of the functor $\blacktriangleright$, $(\blacktriangleright F)_{\beta \preceq \alpha}$ is defined as the amalgamation of projections $\Pi_\gamma^{\blacktriangleright F(\alpha)} : \blacktriangleright F(\alpha) \to F(\gamma)$ of the limit that is $(\blacktriangleright F)(\alpha)$. The functoriality of $\blacktriangleleft$ is trivial. The functoriality of $\blacktriangleright$, on the other hand, follows from properties of limits. It is well-known that these two functors, later and earlier, form an adjunction [9]: $\blacktriangleleft \dashv \blacktriangleright$.

There is an important natural transformation $\mathsf{Next} : \mathsf{id}_{\mathbf{PSh}(\mathbf{Ord})} \to \blacktriangleright$. The map (morphism in **Set**) $\mathsf{Next}_F(\alpha) : F(\alpha) \to \blacktriangleright F(\alpha)$ is constructed as follows: given an element

$x \in F(\alpha)$, $(\{*\}, \{F_{\beta\prec\alpha}\}_{\beta\prec\alpha})$ is a cone on the diagram $F|_{\{\beta|\beta\prec\alpha\}}$ in **Set** — the vertex of this cone, $\{*\}$, is the terminal object of **Set**. Since $\blacktriangleright F(\alpha)$ is the limit of this diagram, there is a unique map from the $\{*\}$ into $\blacktriangleright F(\alpha)$. We take the image this map to be the result of Next:

$$\mathsf{Next}_F(\alpha)(x) := \left(\lim_{\beta\prec\alpha} N^x_\beta\right)(*) \qquad \text{where} \qquad N^x_\beta(*) := F_{\beta\prec\alpha}(x)$$

That this construction is natural both in F and α, like most properties that are related to later, naturality follows from properties of limits.

2.1 Equivalence of the Category of Sheaves Over Ordinals and the Category of Presheaves Over Ordinals

Sheaves are presheaves that additionally satisfy the so-called “sheaf condition” [35]. In the particular case of ordinals (seen as a topological space) the sheaf condition boils down to the following: at any limit ordinal, including zero, the value of the sheaf must be the (categorical) limit of all the sets below it. That is, a presheaf $F : \mathbf{Ord}^{\mathsf{op}} \to \mathbf{Set}$ is a sheaf if and only if we have both that $F(0) \simeq \{*\}$ and that $F(\lambda) \simeq \lim_{\alpha\prec\lambda} F(\alpha)$ via mediating morphisms, for any limit ordinal λ.

As per the sheaf condition above, by construction, $\blacktriangleright F$ is always a sheaf, regardless of F. Thus, $\blacktriangleright$ is also a functor from the category of presheaves over ordinals the category of sheaves over ordinals. On the other hand, $\blacktriangleleft F$ need not be a sheaf, even if F is. When viewed as functors between the category of sheaves and presheaves, the earlier and later functors form not only an adjunction, as noted above, but an adjoint equivalence.[3] That is, the following isomorphisms hold and are both natural in F:

$$\blacktriangleleft(\blacktriangleright(F)) \simeq F \quad \text{for any } \textit{presheaf } F \qquad \text{and} \qquad \blacktriangleright(\blacktriangleleft(F)) \simeq F \quad \text{for any } \textit{sheaf } F$$

These isomorphisms and their naturality rely on Lemma 41 in Appendix D.

▶ Remark 2. The discussion above of the adjoint equivalence of $\mathbf{Sh}(\mathbf{Ord})$ and $\mathbf{PSh}(\mathbf{Ord})$ in fact holds generally for any limit ordinal λ, *i.e.*, for showing adjoint equivalence of $\mathbf{Sh}(\lambda)$ and $\mathbf{PSh}(\lambda)$.

2.2 Contractive Morphisms and Their Fixed Points

Here, we define contractive morphisms in the category of presheaves (natural transformations) and show that they always have unique fixed points — the construction and the proof are very similar to the classical Banach fixed point theorem. Fixed points of contractive morphism are useful in defining so-called guarded recursive predicates which are particularly useful when working in step-indexed logics [8]. In addition to this, we present contractive morphisms and construction of their fixed points not only to highlight the difference in the construction compared to Birkedal *et al.* [9], but also because they are used in proving uniqueness of solutions of domain equations — an important fact in our development; see Section 4.3.

We say a morphism in $\mathbf{PSh}(\mathbf{Ord})$, *i.e.*, a natural transformation, is contractive, if it factors through Next. We write $\mathsf{ContrMorph}(\eta)$ when η is contractive.

▶ **Definition 3.** *A natural transformation $\eta : F \to G$ is contractive, i.e.,* $\mathsf{ContrMorph}(\eta)$*, if there is a natural transformation $\eta' : \blacktriangleright F \to G$ such that $\eta = \eta' \circ \mathsf{Next}_F$. We call η' a* witness *of contractivity of η.*

[3] This equivalence was noticed in a discussion the second author had with Daniel Gratzer.

▶ **Lemma 4** (). *Let $\eta : F \to G$ be a contractive morphism, i.e.,* $\mathsf{ContrMorph}(\eta)$*, with* η' *a witness of contractivity. Then the following holds for any* $\xi : H \to F$ *and any* $\zeta : G \to H$*:*

$$\mathsf{ContrMorph}(\eta \circ \xi) \qquad \textit{witnessed by } \eta' \circ \blacktriangleright \xi$$
$$\mathsf{ContrMorph}(\zeta \circ \eta) \qquad \textit{witnessed by } \zeta \circ \eta'$$

▶ **Definition 5** (Fixed Points). *We define three notions of fixed points of morphisms:*
1. $\xi : B \to A$ *is a fixed point of* $\eta : \blacktriangleright A \times B \to A$ *if* $\eta \circ \langle \mathsf{Next}_A \circ \xi, \mathsf{id}_B \rangle = \xi$
2. $\xi : 1 \to A$ *is a fixed point of* $\eta : \blacktriangleright A \to A$ *if* $\eta \circ \mathsf{Next}_A \circ \xi = \xi$
3. $\xi : 1 \to A$ *is a fixed point of the contractive morphism* $\eta : A \to A$ *if* $\eta \circ \xi = \xi$

▶ Remark 6 (). Definition 5 defines three kinds of fixed points each weaker than the one before in that if (unique) solutions to one kind of fixed point exist, so do (unique) solutions to the next kind. Theorem 7 below immediately implies existence of unique fixed points of the first kind and thus existence of unique fixed points of all kinds. We will write $\mathsf{fix}(f)$ for the *unique* fixed point of map f for any of these kinds of fixed points.

In order to construct these fixed points we show that there is a general fixed point combinator $fix_A : A^{\blacktriangleright A} \to A$. Note that here the fixed point combinator fix_A is a natural transformation (a morphism in the category of presheaves) from the exponential object $A^{\blacktriangleright A}$ to A.

▶ **Theorem 7** (). *For any presheaf A, there is a natural transformation $fix_A : A^{\blacktriangleright A} \to A$ in the category of presheaves over ordinals that acts as the fixed point combinator constructing unique fixed points. That is, for any $\eta : \blacktriangleright A \times B \to A$ we have $fix_A \circ curry(\eta)$ is the unique natural transformation from B to A such that: $\eta \circ \langle \mathsf{Next}_A \circ fix_A \circ curry(\eta), \mathsf{id}_B \rangle = fix_A \circ curry(\eta)$ where $curry(\eta)$ is the exponential transpose of η.*

▶ Remark 8. The proof of Theorem 7 differs from the proof given by Birkedal *et al.* [9] in that working in the category of sheaves, the value of the fixed point presheaf is uniquely determined at 0 and all limit ordinals. In contrast, our construction applies η at every single stage of the construction including at 0 and limit ordinals. In other words, at 0 and limit ordinals, we apply η *"one more time"* after computing what one would compute in the case of sheaves. To see this, note that a natural transformation $fix_A : A^{\blacktriangleright A} \to A$ essentially amounts to maps (morphisms in **Set**) $\zeta_\alpha : (\mathcal{Y}_\alpha \times \blacktriangleright A \to A) \to A(\alpha)$ that are natural in α — each ζ_α is a map from the set of natural transformations $(\mathcal{Y}_\alpha \times \blacktriangleright A \to A)$ to the set $A(\alpha)$. Thus, at 0, we are given a function, say $f : \mathcal{Y}_\alpha(0) \times \blacktriangleright A(0) \to A(0)$ and need to produce an element of $A(0)$, for which we will use f. Intuitively, the function f here is the natural transformation η at stage 0, *i.e.*, the natural transformation we are taking the fixed point of.

3 Enrichment Over Categories of Presheaves Over Ordinals

Enrichment is often studied over monoidal categories [30]. Here, we work specifically with the monoidal structure of the cartesian closedness of the enriching category, *i.e.*, the category **PSh**(**Ord**). We briefly present the basic definitions here just to fix notation. Our notion of locally contractive functor is exactly that in Birkedal *et al.* [9].

3.1 Enriched Categories and Functors; Locally Contractive Functors

▶ **Definition 9** (Enriched Category). *We say a category $\mathcal{C}$ is enriched over a cartesian closed category $\mathcal{E}$ if we have the following:*

- *An internal hom object in $\mathcal{E}$, written $\mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{A,B}$, for any pair of objects A and B in $\mathcal{C}$*
- *A map $\lceil\cdot\rceil : Hom_\mathcal{C}(A, B) \to Hom_\mathcal{E}(1, \mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{A,B})$ embedding $\mathcal{C}$ morphisms into $\mathcal{E}$*
- *A map $\lfloor\cdot\rfloor : Hom_\mathcal{E}(1, \mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{A,B}) \to Hom_\mathcal{C}(A, B)$ projecting $\mathcal{C}$ morphisms out of $\mathcal{E}$*
- *The maps $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ are inverses of one another*
- *Internal composition morphisms: $\mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{A,B,C} : \mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{A,B} \times \mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{B,C} \to \mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{A,C}$*
- *Expressed in terms of equality of morphisms in $\mathcal{E}$, we have that $\mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{A,B,C}$ is in agreement with composition in $\mathcal{C}$, is associative and respects identity morphisms*

▶ **Definition 10** (Enriched Functor). *Let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathcal{E}$-enriched categories. We say a functor $F : \mathcal{C} \to \mathcal{D}$ is $\mathcal{E}$-enriched if there are morphisms $\mathbb{E}^{\mathsf{hm}_F}_{A,B} : \mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{A,B} \to \mathbb{E}^{\mathsf{hom}_\mathcal{D}}_{F(A),F(B)}$ in $\mathcal{E}$ that acts as the $\mathcal{E}$-internal functor action of F and, expressed in terms of equality of morphisms in $\mathcal{E}$, the morphisms $\mathbb{E}^{\mathsf{hm}_F}_{A,B}$ preserve identity and composition.*

Following Birkedal *et al.* [9] we define a locally contractive functor to be an enriched functor (over the category of presheaves) that also has a *contracted* internal functor action morphism. Intuitively, what we want is to say that a functor is locally contractive if its internal functor action is a contractive morphism in the sense of Definition 3. The definition below furthermore requires the witness of contractivity of the internal functor action to also act functorially in the sense that it must also preserve compositions and identities. This extra requirement, as also pointed out by Birkedal *et al.* [9], is the reason why we can develop the theory of ordinal-partial isomorphisms and enriched-pointwise limits as we present in this section. Birkedal *et al.* [9] present the theory of ordinal-partial isomorphisms but do not make enriched-pointwise limits formal — they only mention intuitively that "since limits are computed pointwise, . . . " when presenting their approach to constructing solutions to domain equations. Because we mechanize our solution to the domain equation problem we had to formalize and mechanize this intuitive line of argument.

▶ **Definition 11** (Locally Contractive Functors). *Let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathbf{PSh}(\mathbf{Ord})$-enriched categories. We say a $\mathbf{PSh}(\mathbf{Ord})$-enriched functor $F : \mathcal{C} \to \mathcal{D}$ is locally contractive if the internal action morphisms of F, $\mathbb{E}^{\mathsf{hm}_F}_{A,B}$, are contractive with the witness of contractivity being morphisms $\mathbb{E}^{\blacktriangleright\mathsf{hm}_F}_{A,B} : \blacktriangleright\mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{A,B} \to \mathbb{E}^{\mathsf{hom}_\mathcal{D}}_{F(A),F(B)}$. Furthermore, expressed in terms of equality of morphisms in $\mathbf{PSh}(\mathbf{Ord})$, the morphisms $\mathbb{E}^{\blacktriangleright\mathsf{hm}_F}_{A,B}$ must preserve identity and composition.*

▶ **Lemma 12** (Composition of Enriched and Locally Contractive Functors). *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{B}$ be three $\mathcal{E}$-enriched categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{B}$ be two $\mathcal{E}$-enriched functors. The composition $G \circ F$ is also an $\mathcal{E}$-enriched functor. Furthermore, $G \circ F$ is locally contractive if at least one of F or G is.*

Enriched functors are closed under many useful constructions: constant functors, the identity functor, products of functors, their sums, diagonal functors ($\Delta_n : A \mapsto A^n$), *etc.* In particular, this includes all finitary polynomial functors. Lemma 12 shows that there also exists a similarly large collection of locally contractive functors because the later functor, $\blacktriangleright : \mathbf{PSh}(\mathbf{Ord}) \to \mathbf{PSh}(\mathbf{Ord})$ is both enriched and locally contractive; see Appendix B where we also show that the earlier functor $\blacktriangleleft : \mathbf{PSh}(\mathbf{Ord}) \to \mathbf{PSh}(\mathbf{Ord})$ is not even enriched, let alone locally contractive.

3.2 Ordinal-Partial Isomorphisms

In this section, following Birkedal *et al.* [9], we define a notion of ordinal-partial isomorphisms, indexed over ordinals, for categories enriched over $\mathbf{PSh}(\mathbf{Ord})$ and prove a few useful lemmas about such morphisms that we will later use in solving domain equations.

▶ **Definition 13** (Ordinal-Partial Isomorphism). *Let $\mathcal{C}$ be a category enriched over* $\mathbf{PSh}(\mathbf{Ord})$ *and $f : A \to B$ be a morphism in $\mathcal{C}$. We say that f is an α-isomorphism if we have an element $x \in \mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{B,A}(\alpha)$ called partial inverse of f at stage α such that*

$$\mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{B,A,B}(\alpha)(x, \lceil f \rceil(\alpha)(*)) = \lceil \mathsf{id}_B \rceil(\alpha)(*) \qquad \text{(part-iso-left-id)}$$
$$\mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{A,B,A}(\alpha)(\lceil f \rceil(\alpha)(*), x) = \lceil \mathsf{id}_A \rceil(\alpha)(*) \qquad \text{(part-iso-right-id)}$$

By functoriality of $\mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{B,A}$ and naturality of $\mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{A,B,A}$ and $\mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{B,A,B}$, we know that if f is an α-isomorphism it is also β-isomorphism for any $\beta \preceq \alpha$. Given a downwards-closed subset of ordinals $A \subseteq \mathbf{Ord}$, we say a morphism f is an A-isomorphism if it is an α-isomorphism for any $\alpha \in A$. Whenever A has a maximal element γ being an A-isomorphism is equivalent to being a γ-isomorphism. However, A-isomorphisms are in general useful for working with morphisms that are A-isomorphisms for an unbounded downwards-closed subset A. Intuitively, f being an α-isomorphism means that it behaves like an isomorphism up to stage α, even though an inverse morphism may not even exist.

▶ Remark 14. Although we define ordinal-partial isomorphisms almost exactly as Birkedal *et al.* [9] do, due to the differences between sheaves and presheaves, in the setting of Birkedal *et al.* [9] every morphism is a 0-isomorphism, and also any morphism that is a $\{\alpha | \alpha \prec \lambda\}$-isomorphism for some limit ordinal λ is also a λ-isomorphism. This is not the case in our setting.

▶ **Lemma 15** (). *Let $\mathcal{C}$ be a* $\mathbf{PSh}(\mathbf{Ord})$*-enriched category and let $f : A \to B$ a morphism in $\mathcal{C}$. The morphism f is an isomorphism, i.e., there is a morphism $g : B \to A$ such that $f \circ g = \mathsf{id}_B$ and $g \circ f = \mathsf{id}_A$,* if and only if *$f$ is an α-isomorphism for all $\alpha \in$* $\mathbf{Ord}$.

▶ **Lemma 16** (). *Let $\mathcal{C}$ and $\mathcal{D}$ be two* $\mathbf{PSh}(\mathbf{Ord})$*-enriched categories and $F : \mathcal{C} \to \mathcal{D}$ an* $\mathbf{PSh}(\mathbf{Ord})$*-enriched functor. For any α-isomorphism $f : A \to B$ in $\mathcal{C}$, $F(f)$ is an α-isomorphism.*

▶ **Lemma 17** (). *Let $\mathcal{C}$ and $\mathcal{D}$ be two* $\mathbf{PSh}(\mathbf{Ord})$*-enriched categories and $F : \mathcal{C} \to \mathcal{D}$ a locally contractive functor. Furthermore, let $f : A \to B$ in $\mathcal{C}$ be a $\{\beta | \beta \prec \alpha\}$-isomorphism. The image of f under F, $F(f)$, is an α-isomorphism in $\mathcal{D}$.*

3.3 Enriched-Pointwise Limits

In this section we develop the theory of enriched-pointwise limits in categories enriched over $\mathbf{PSh}(\mathbf{Ord})$. This is an abstract way of representing the idea that limits are suitably "pointwise". In particular, when we consider the self-enrichment of $\mathbf{PSh}(\mathbf{Ord})$ and the enrichment of $(\mathbf{PSh}(\mathbf{Ord}))^{\mathsf{op}}$ over $\mathbf{PSh}(\mathbf{Ord})$ this notion directly corresponds to (co-)limits in $\mathbf{PSh}(\mathbf{Ord})$ being pointwise; see Section D.3. We will use enriched-pointwise limits to state and prove the important Lemma 21 and Corollary 22.

▶ **Definition 18** (Enriched-Pointwise Cones). *Let $\mathcal{C}$ be a category enriched over* $\mathbf{PSh}(\mathbf{Ord})$ *and $F : \mathcal{J} \to \mathcal{C}$ be a $\mathcal{J}$-shaped diagram in $\mathcal{C}$. Given an ordinal α, an enriched-pointwise cone $(V, \{x_j\}_{j \in J})$ at stage α over the diagram F consists of a vertex object V in $\mathcal{C}$ together with elements $x_j \in \mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{V,F(j)}(\alpha)$ such that for any morphism $f : j \to j'$ in $\mathcal{J}$ we have $\mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{V,F(j),F(j')}(\alpha)(x_j, \lceil F(f) \rceil(\alpha)(*)) = x_{j'}$.*

Since $F(j)$ is a functor (presheaf) and $\mathbb{E}^{\mathsf{comp}_F}_{V,F(j),F(j')}$ is natural, a cone at stage α is also a cone at stage $\beta \preceq \alpha$. In addition, given a cone $(V, \{S_j\}_{j \in \mathcal{J}})$ over a diagram $F : \mathcal{J} \to \mathcal{C}$ we obtain an enriched-pointwise cone $(V, \{\lceil S_j \rceil(\alpha)\}_{j \in \mathcal{J}})$ of diagram F at any stage α.

▶ **Definition 19** (Enriched-Pointwise Cone Homomorphisms). *Let $\mathcal{C}$ be a category enriched over* $\mathbf{PSh}(\mathbf{Ord})$ *and $F : \mathcal{J} \to \mathcal{C}$ be a $\mathcal{J}$-shaped diagram in $\mathcal{C}$. Moreover, let $(V, \{x_j\}_{j\in J})$ and $(V', \{x'_j\}_{j\in J})$ be two enriched-pointwise cones over F both at stage α. A cone homomorphism from V to V' is an element $h \in \mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{V,V'}(\alpha)$ such that $\mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{V,V',F(j)}(\alpha)(h, x'_j) = x_j$.*

▶ **Definition 20** (Enriched-Pointwise Limits). *Let $\mathcal{C}$ be a category enriched over* $\mathbf{PSh}(\mathbf{Ord})$ *and $F : \mathcal{J} \to \mathcal{C}$ be a $\mathcal{J}$-shaped diagram in $\mathcal{C}$. Furthermore, let $(V, \{S_j\}_{j\in\mathcal{J}})$ be a limit cone of diagram F in $\mathcal{C}$. We say this limit is enriched-pointwise if we have that for any enriched-pointwise cone $(W, \{x_j\}_{j\in\mathcal{J}})$ at stage α there is a unique enriched-pointwise cone homomorphism from $(W, \{x_j\}_{j\in\mathcal{J}})$ to the enriched-pointwise cone $(V, \{\lceil S_j \rceil (\alpha)\}_{j\in\mathcal{J}})$.*

▶ **Lemma 21** (). *Let $\mathcal{J}$ be a strongly connected preorder category, i.e., for any two objects $j, j' \in \mathcal{J}$, $Hom_\mathcal{J}(j, j') \cup Hom_\mathcal{J}(j', j) \neq \emptyset$. Furthermore, let $\mathcal{C}$ be a category enriched over* $\mathbf{PSh}(\mathbf{Ord})$ *and $F : \mathcal{J} \to \mathcal{C}$ be a diagram such that for any $f : j \to j'$ in $\mathcal{J}$, the morphism $F(f)$ is an α-isomorphism. Finally, let the limit of F be enriched-pointwise in the sense of Definition 20. Under these circumstances every projection of the limit of F is an α-isomorphism.*

▶ **Corollary 22** (). *Let $\mathcal{C}$ a category enriched over* $\mathbf{PSh}(\mathbf{Ord})$ *and $F : \{\beta|\beta \prec \alpha\} \to \mathcal{C}$ be a diagram whose limit is enriched-pointwise. Fix $\delta \prec \alpha$. Assume that for any $\delta \preceq \gamma \prec \alpha$ the morphism $F_{\delta\preceq\gamma}$ is a δ-isomorphism. The projection $\Pi_\delta : \lim F \to F(\delta)$ is a δ-isomorphism.*

4 Solving Domain Equations

In this section we show that for a $\mathbf{PSh}(\mathbf{Ord})$-enriched category $\mathcal{C}$, any locally contractive functor $F : \mathcal{C} \to \mathcal{C}$ has a unique solution up to isomorphism.

4.1 Uniqueness of Solutions up to Isomorphism

By a well-known result attributed to Lambek [44], the initial F-algebra for a functor F is an isomorphism, and hence a solution to the domain equation for F. The following theorem establishes the converse for locally contractive functors showing uniqueness of solutions.

▶ **Theorem 23** (). *Let $\mathcal{C}$ be a* $\mathbf{PSh}(\mathbf{Ord})$*-enriched category and $F : \mathcal{C} \to \mathcal{C}$ a locally contractive functor. The F-algebra (S, s) induced by a solution $s : F(S) \xrightarrow{\simeq} S$ is an initial algebra.*

Proof. The proof we have mechanized is the exact proof given by Birkedal *et al.* [9, 11]. Given an F-algebra (A, ϕ_A) we need to construct a unique F-algebra morphism from (S, s) to (A, ϕ_A). Observe that a morphism $h : S \to A$ is an F-algebra morphism if and only if we have $h = \phi_A \circ F(h) \circ s^{-1}$. A different way to look at this fact is that given a morphism $h : S \to A$, we can construct another morphism from S to A by taking $\phi_A \circ F(h) \circ s^{-1}$. This mapping induces the following morphism in $\mathbf{PSh}(\mathbf{Ord})$:

$$\mu := compL^{\phi_A}_A \circ compR^{s^{-1}}_S \circ \mathbb{E}^{\mathsf{hm}_F}_{S,A} : \mathbb{E}^{\mathsf{hom}_{\mathbf{PSh}(\mathbf{Ord})}}_{S,A} \to \mathbb{E}^{\mathsf{hom}_{\mathbf{PSh}(\mathbf{Ord})}}_{S,A}$$

By Lemma 4 the morphism μ is contractive because F is locally contractive. Thus, by Theorem 7 and Remark 8 μ has a unique fixed point for which $\lfloor \mathit{fix}(\mu) \rfloor = \phi_A \circ F(\lfloor \mathit{fix}(\mu) \rfloor) \circ s^{-1}$. Hence, $\lfloor \mathit{fix}(\mu) \rfloor$ is the unique algebra morphism we needed. ◻

4.2 Constructing the Solution

In Section 4.1 we discussed that the solution to the domain equation is an F-algebra (A, ϕ_A) where ϕ_A is an isomorphism. Accordingly, our strategy to solving domain equations is to find such an F-algebra (A, ϕ_A) where ϕ_A is an isomorphism. This approach differs from the approach of Birkedal *et al.* [9] in that Birkedal *et al.* [9] work directly in the category $\mathcal{C}$ instead of $\mathcal{A}lg(F)$; see Remark 31. Although this aspect of the difference is rather superficial, it does help simplify the construction in the sense that it breaks the construction into a few simpler concepts and lemmas which are ultimately nicer to mechanize; see (canonical) partial solutions, dominating cones, *etc.* as presented below. Nevertheless, the fact that finding solutions to domain equations can be reduced to finding (initial) algebras is common knowledge [44].

Let us assume for the rest of this section that we are given a locally contractive endofunctor $F : \mathcal{C} \to \mathcal{C}$ over a $\mathbf{PSh}(\mathbf{Ord})$-enriched category $\mathcal{C}$ which is complete, and for which all limits are enriched-pointwise. We start by defining a notion of a partial solution.

▶ **Definition 24** (Partial Solution). *Let $A \subseteq \mathbf{Ord}$ be a downwards-closed subset of ordinals. An A-partial solution is an A^{op}-shaped diagram P in the category of F-algebras such that:*
(PS-1) *For any $\alpha \in A$, $\phi_{\mathsf{P}(\alpha)} : F(\mathsf{P}(\alpha)) \to \mathsf{P}(\alpha)$ is an α-isomorphism.*
(PS-2) *For any $\beta \preceq \alpha \in A$, $\mathsf{P}_{\beta \preceq \alpha} : \mathsf{P}(\alpha) \to \mathsf{P}(\beta)$ is a β-isomorphism.*

The definition of partial solutions above is local in that the conditions (PS-1) and (PS-2) only refer to individual objects or individual morphisms. As a result, given an A-partial solution P, restricting P (as a diagram and hence a functor) to a downwards-closed subset $B \subseteq A$, written $\mathsf{P}|_B$, is again a B-partial solution.

▶ **Definition 25** (Dominating Cone). *Let $A \subseteq \mathbf{Ord}$ be a downwards-closed subset of ordinals and P an A-partial solution. We say that a cone $((D, \phi_D), \{S_\alpha\}_{\alpha \in A})$ over P dominates P if we have:*
(DA-1) *For any $\alpha \in A$, $S_\alpha : D \to \mathsf{P}(\alpha)$ is an α-isomorphism.*
(DA-2) *The map ϕ_D is an α-isomorphism for any α for which we have $\{\beta | \beta \prec \alpha\} \subseteq A$.*

Note that the condition (DA-2) above is equivalent to saying that ϕ_D is a $(\sup A)$-isomorphism in the event $\sup A$ does exist. In particular, the condition (DA-2) implies that if P is a $\mathbf{Ord}$-partial solution, then a cone dominating it is a solution to the domain equation; see the proof of Theorem 30.

▶ **Lemma 26** (). *Let $A \subseteq \mathbf{Ord}$ be a downwards-closed subset of ordinals and P an A-partial solution. By Remark 44 (Appendix D) the functor F applied to the limiting cone of P is also a cone on P. We will write $DCone(\mathsf{P})$ for this cone. The cone $DCone(\mathsf{P})$ dominates P.*

Proof. Let us write $((L, \phi_L), \{\Pi^L_\alpha\}_{\alpha \in A})$ for the cone that is the limit of P.

First we show that ϕ_L is an A-isomorphism by showing that it is an α-isomorphism for any $\alpha \in A$. Observe that by (PS-2) we know that Corollary 22 applies and thus Π^L_α is an α-isomorphism, and by Lemma 16 so is $F(\Pi^L_\alpha)$. Furthermore, as Π^L_α is a morphism in the category of F-algebras, which means that the following diagram commutes for any $\alpha \in A$:

$$\begin{array}{ccc} F(L) & \xrightarrow{\phi_L} & A \\ {\scriptstyle F(\Pi^L_\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle \Pi^L_\alpha} \\ F(\mathsf{P}(\alpha)) & \xrightarrow{\phi_{\mathsf{P}(\alpha)}} & \mathsf{P}(\alpha) \end{array}$$

Also, by condition (PS-1), $\phi_{\mathsf{P}(\alpha)}$ is an α-isomorphism. Thus, by Remark 40 (Appendix C), ϕ_L is an α-isomorphism since the three other sides of the diagram above are α-isomorphisms.

The cone $DCone(\mathsf{P})$ that we wish to show dominates P is the following:

$$DCone(\mathsf{P}) = \Big((F(L), F(\phi_L)), \{\phi_{\mathsf{P}(\alpha)} \circ F(\Pi^L_\alpha)\}_{\alpha \in A}\Big)$$

We already established (DA-1) when we argued that the diagram above consists of α-isomorphisms. For (DA-2), let us assume we are given an ordinal α for which we have $\{\beta|\beta \prec \alpha\} \subseteq A$. We just observed that ϕ_L is an A-isomorphism and thus also a $\{\beta|\beta \prec \alpha\}$-isomorphism. Hence, by Lemma 17 $F(\phi_L)$ is an α-isomorphism. ☐

Next we define what we call canonical partial solutions and show how to patch canonical partial solutions together in order to construct larger ones. The latter is used in Theorem 30 for constructing partial solutions by well-founded induction on ordinals.

▶ **Definition 27** (Canonical Partial Solutions). *Let $A \subseteq \mathbf{Ord}$ be a downwards-closed subset of ordinals and P an A-partial solution. We say P is a canonical partial solution if it is constructed at all stages via the construction in Lemma 26. That is, if for any $\alpha \in A$ we have*

$$\Big(\mathsf{P}(\alpha), \{\mathsf{P}_{\beta \preceq \alpha}\}_{\beta \preceq \alpha}\Big) = DCone\Big(\mathsf{P}|_{\{\beta|\beta \prec \alpha\}}\Big)$$

are equal cones of the diagram $\mathsf{P}|_{\{\beta|\beta \prec \alpha\}}$.

▶ **Lemma 28** (). *Let P and Q be two canonical A-partial solutions. We have $\mathsf{P} = \mathsf{Q}$ (as diagrams, i.e., functors).*

On paper, the Lemma 28 above is proven through a simple argument by transfinite induction. However, as we will discuss in Section 5, it is far from obvious to mechanize.

▶ **Lemma 29** (). *Let $\{\mathsf{P}^\alpha\}_{\alpha \in A}$ be a collection of canonical partial solutions indexed by some downwards-closed subset of ordinals A such that P^α is a canonical $\{\beta|\beta \preceq \alpha\}$-partial solution. We can construct a canonical A-partial solution Q by patching the partial solutions $\{\mathsf{P}^\alpha\}_{\alpha \in A}$ together. That is, we take $\mathsf{Q}(\alpha) := \mathsf{P}^\alpha(\alpha)$ and take $\mathsf{Q}_{\beta \preceq \alpha} := \mathsf{P}^\alpha_{\beta \preceq \alpha}$.*

Note that the proof, and even the well-formedness of the statement of Lemma 29 above depends on Lemma 28. In particular, note that $\mathsf{Q}_{\beta \preceq \alpha}$ must be a morphism from $\mathsf{Q}(\alpha)$ to $\mathsf{Q}(\beta)$, or equivalently from $\mathsf{P}^\alpha(\alpha)$ to $\mathsf{P}^\beta(\beta)$, whereas the morphism $\mathsf{P}^\alpha_{\beta \preceq \alpha}$ is a morphism from $\mathsf{P}^\alpha(\alpha)$ to $\mathsf{P}^\alpha(\beta)$. Thus, one would need to prove $\mathsf{P}^\alpha(\beta) = \mathsf{P}^\beta(\beta)$ for it to even make sense to take $\mathsf{Q}_{\beta \preceq \alpha} := \mathsf{P}^\alpha_{\beta \preceq \alpha}$. This is the case because by Lemma 28 $\mathsf{P}^\alpha|_{\{\gamma|\gamma \preceq \beta\}} = \mathsf{P}^\beta$. However, in type theory, in our Rocq mechanization, one needs to work up to the equality $\mathsf{P}^\alpha(\beta) = \mathsf{P}^\beta(\beta)$ (transport along this equality) when defining Q, which also includes establishing its functoriality, that it is a partial solution, and its canonicity. We will discuss these subtleties in Section 5.

▶ **Theorem 30** (). *The locally contractive functor F has a solution.*

Proof. We first construct a canonical $\mathbf{Ord}$-partial solution P. By Lemma 29 it suffices to construct canonical $\{\beta|\beta \preceq \alpha\}$-partial solutions for all $\alpha \in \mathbf{Ord}$. We do so by well-founded induction on α. Thus, let us assume that we have canonical $\{\gamma|\gamma \preceq \beta\}$-partial solutions for all $\beta \prec \alpha$. We use Lemma 29 to construct a canonical $\{\beta|\beta \preceq \alpha\}$-partial solution as required. The solution is the F-algebra of $DCone(\mathsf{P})$. We only need to show that the map $\phi_{DCone(\mathsf{P})}$ is an isomorphism. By Lemma 15 it suffices to show that $\phi_{DCone(\mathsf{P})}$ is an α-isomorphism for all $\alpha \in \mathbf{Ord}$. However, by Lemma 26 $DCone(\mathsf{P})$ dominates P. Hence, by the property (DA-2) of dominating cones we only need to show that $\{\beta|\beta \prec \alpha\} \subseteq \mathbf{Ord}$, which holds trivially. ☐

▶ Remark 31. In addition to the difference of working with the category of F-algebras as opposed to working directly in $\mathcal{C}$, our approach to solving domain equations differs from that presented by Birkedal *et al.* [9] in how we treat zero and limit ordinals. Working with sheaves, Birkedal *et al.* [9] at zero and limit ordinals simply take the limit of the construction at stages below. By contrast, we apply F to the obtained cone of the limit at every single stage of the construction and not just at successor ordinals. Another way to look at this difference is if we look at the sequence of objects constructed in these two approaches (in our case the carrier objects of the algebras we compute). Up to isomorphism, what we compute is the sequence X below while what Birkedal *et al.* [9] compute is the sequence Y:

$$\begin{array}{llllll} X_0 := F(1); & X_1 := F(F(1)); & X_2 := F(F(F(1))); & \cdots & X_\omega := F(\lim_{\alpha \prec \omega} X_\alpha); & X_{\omega^+} := F(X_\omega); \quad \cdots \\ Y_0 := 1; & Y_1 := F(1); & Y_2 := F(F(1)); & \cdots & Y_\omega := \lim_{\alpha \prec \omega} Y_\alpha; & Y_{\omega^+} := F(Y_\omega); \quad \cdots \end{array}$$

4.3 Mixed-Variance Domain Equations

In addition to covariant functors of the form $\mathcal{C} \to \mathcal{C}$, we in general need to [9] solve domain equations for mixed-variance functors of the form $\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$.

▶ **Example 32 ( ).** As a simple minimal example we have used our development to solve the domain equation for the following mixed-variance functor which is a simplified version of the functor used by Frumin *et al.* [17]:

$$F(X, Y) := \Delta(\mathbb{N}) + \blacktriangleright(Y^X) + \blacktriangleright Y$$

where $\Delta(A)$ is the constant presheaf mapping all ordinals to the set A.

The following lemma shows that *mixed-variance* locally contractive functors $F : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$ like the one in Example 32 have unique solutions.

▶ **Lemma 33 ( ).** *Let $\mathcal{C}$ be a* $\mathbf{PSh}(\mathbf{Ord})$*-enriched category and $F : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$ be a locally contractive functor (if $\mathcal{C}$ is enriched, so is $\mathcal{C}^{\mathsf{op}}$ and also their product). Furthermore, assume that $\mathcal{C}$ is complete and co-complete with enriched-pointwise limits and co-limits. The functor F has a unique solution.*

Proof. Define the functor $\tilde{F}(A, B) := (F(B, A), F(A, B))$ from $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$ to $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$. The functor $\tilde{F}$ is also locally contractive and hence, by Theorem 30, has a solution, say (X, Y) for which $\tilde{F}(X, Y) \simeq (X, Y)$. In that case, by symmetry of $\tilde{F}$, we have $\tilde{F}(Y, X) \simeq (Y, X)$. Thus, by Theorem 23, $(X, Y) \simeq (Y, X)$ which implies that $X \simeq Y$, and hence, $F(X, X) \simeq X$. ☐

5 Rocq Mechanization

All results that have been marked by a symbol throughout the paper and the appendices are mechanized [46] in the Rocq Prover. For this mechanization we have used the step-indexing interface of Spies *et al.* [45] which abstracts a step-indexing structure that Spies *et al.* [45] instantiate twice: once with natural numbers in Rocq (for step-indexing over ω), and once with ordinal numbers. Spies *et al.* [45] use the mechanization of ordinal numbers by Kirst *et al.* [31]. We use the following axioms in our mechanization: axiom of choice (and its consequence excluded middle), propositional extensionality (and its consequences proof irrelevance and uniqueness of identity proofs (UIP)), and functional extensionality. The first two of these axioms are already assumed by Kirst *et al.* [31] to construct a model of set theory in Rocq. The inclusion of functional extensionality, on the other hand, is necessary for formalization of category theory, at least the way we have; we will discuss this below.

The Notion of Equality of Homorphisms in Type Theory There are many efforts mechanizing category theory in type-theory-based proof assistants [24, 40, 25, 52, 21, 49, 50, 2, 36]. Hu and Carette [24] give an extensive survey comparing existing type-theory-based category theory mechanizations across multiple different axes. One important design point in mechanizing category theory in type theory is representing equality of morphisms. Roughly speaking, this design decision divides mechanizations into two camps: those using setoids, also known as Bishop sets [12], for representing morphisms [24, 40, 25, 52] and those using equality [21, 49, 50, 2, 36] (including those working in HoTT [50] settings where equality plays a key role).

There are multiple advantages to using setoids as pointed out by Hu and Carette [24]. Hu and Carette [24] work in Agda and state "Our principal theoretical contribution is to show that *setoid-based proof-relevant category theory* works just as well as various other 'flavours' of category theory by supporting a large number of definitions and theorems." One of the main advantages of using setoids is avoiding axioms such as functional extensionality (for proving equality of functors, natural transformations, *etc.*) and classical axioms (for constructing quotient types, *e.g.*, for co-limits in **Set** or presheaf categories) — of course, works based in HoTT admit these as theorems. Indeed, our development also started with implementing the necessary basic concepts in category theory *using setoids*. However, we discovered an issue that fundamentally precludes the use of setoids for morphisms in our mechanization. This problem arises in the proof of Lemma 28. For this lemma, we need to show that two F-algebras are equal which are constructed based on the limits of two diagrams that are equal (by our transfinite induction hypothesis). However, the natural notion of equality of functors in setoid settings is to ask that the morphism maps of functors map setoid-equal morphisms into setoid-equal morphisms. Consequently, the best one can prove is that equal functors in this sense produce isomorphic limits. Thus, instead of Lemma 28, one could prove that canonical partial solutions are naturally isomorphic. However, as remarked after Lemma 29 in Section 4.2, this is not even sufficient for the statement of Lemma 29 to be well-formed. This is why we chose to change our mechanization to use the equality notion from Rocq's standard library instead of setoid equality. It appears that existing mechanizations of category theory that formalize collections of morphisms as setoids have never attempted formalizing a construction such as ours that involves defining a functor on ordinals by transfinite induction where the construction at each stage involves taking the limit of the construction up to below that stage.

Working With All Ordinals in The Universe As we discussed in the Introduction, we work with the category of presheaves over all ordinals in the universe, *i.e.*, the set **Ord** which is *not* closed under suprema. Thus, many of our definitions are parameterized by a downwards-closed subset of ordinals. The collection of downwards-closed subsets of ordinals, ordered by the subset relation, can be thought of as the ideal completion of **Ord**. In particular, the total set **Ord** is the maximal element of this order which represents the supremum of the entire set **Ord**. This means that the key lemma of our formalization, Lemma 29, is applicable to both proper downwards-closed subsets of ordinals (which represent ordinals that do happen to be in **Ord**) as well as the aforementioned supremum of the entire set **Ord** (which itself is not in **Ord**). This is why we can apply Lemma 29 twice in the proof of Theorem 30, once for constructing $\{\beta|\beta \preceq \alpha\}$-partial solutions for all $\alpha \in$ **Ord** by transfinite induction on α, and once to put all those together to construct a **Ord**-partial solution. This is key in significantly reducing the size of the mechanization as otherwise a mechanization working with all ordinals in the universe would have to prove two different versions of Lemma 29 for the two different use cases in Theorem 30; once for individual ordinals, and once for all ordinals. Spies *et al.* [45] also work with the collection of ordinals in the universe; as mentioned above we took their notion of step-indexing and ordinals verbatim in our mechanization. And, they indeed duplicate Lemma 29 as we just explained. This requires them to repeat multiple definitions and lemmas for these two versions of our single Lemma 29.

Well-Behaved Subset Types in Rocq Working with downwards-closed subsets of ordinals, we need to formalize them in our Rocq mechanization. In our mechanization, we define downwards-closed subsets of ordinals as *subset types* which we represent as a record that packages an ordinal together with a *proof* that said ordinal is in the downwards-closed subset. We first define downwards-closed predicates over ordinals, `downset_pred`, as a record type consisting of a predicate together with a proof that this predicate is downwards-closed (ignore the decidability part for now, we will get back to it):

```
Polymorphic Record downset_pred (SI : indexT) := MkDownSetPred {
  dsp_pred :> SI → Prop;
  dsp_pred_dec : ∀ α, Decision (dsp_pred α);
  dsp_pred_downwards : ∀ α β, α ⪯ β → dsp_pred β → dsp_pred α; }.
```

Here, `indexT` is the generalized type exposed by the step-indexing interface of Spies *et al.* [45]; the type of all ordinals in the universe being an instance of this structure. Based on the downwards-closed predicates defined above, we would like to define downwards-closed subsets essentially as a record type that packages together an ordinal, together with a proof that it belongs to the provided downwards-closed predicate. However, a naïve encoding as such a record type leads to a problem: given a downwards-closed predicate over ordinals, two elements of such a type with the same ordinals but different proofs would not be *definitionally* equal — of course, they are *propositionally* equal as we assume proof irrelevance. This problem is especially noticeable when we look at two downwards-closed subsets where one is included in the other. We define the inclusion relation between two downwards-closed predicates `dsp` and `dsp'` as one would expect: `∀ α, dsp α → dsp' α`. This allows us to define a simple function that lifts ordinals from the smaller downwards-closed subset to the larger one. We use this function in our Rocq mechanization to define the restriction operation in Section 4.2 on presheaves over downwards-closed subsets of ordinals.

We solve the issue discussed above by defining the record type `downset` as follows:

```
#[projections(primitive = yes)]
Record downset {SI} (dsp : downset_pred SI) := MkDS {
  ds_idx :> SI;
  ds_in_dsp : squashed (dsp ds_idx); }.
```

where the type `squashed` is exactly as defined in Gilbert *et al.* [20]:

```
Inductive squashed (P : Prop) : SProp := squash : P → squashed P.
```

The idea here is that since `squashed` is in the universe `SProp` of definitionally proof-irrelevant propositions, and the fact that type `downset` is defined as a record with primitive projections (and hence it is subject to the η conversion law for records), two terms of the type `downset dsp` are *definitionally* equal as soon as their underlying ordinal, the projection `ds_idx`, are *definitionally* equal. Now, the problem is that when working with elements of `downset dsp`, we need to have a proof that the underlying ordinal is indeed in `dsp`, *i.e.*, we need something of type `dsp ds_idx`, whereas we are *only* given a proof of `squashed (dsp ds_idx)`. Importantly, the type `squashed` above cannot be eliminated to produce a term of a type that is outside the universe `SProp` — in technical terms, this is because the first argument of the constructor `squash` (the argument with type `P`) is *non-forced*, and is also not in `SProp` [20]. Nevertheless, inspired by the constant map from identity proofs to identity proofs in the proof of Hedberg's theorem [23], we prove the following lemma which allows us to recover a proof of `dsp ds_idx` from an element of `squashed (dsp ds_idx)`:[4]

```
Lemma unsquash {P : Prop} `{!Decision P} (s : squashed P) : P.
```

The lemma above is the reason why we included a proof of decidability of the subset predicate in the definition of `downset_pred` above.

6 Related Works

Domain Theory We have already discussed works on domain theory that are most closely related to ours in the Introduction, including the most closely related work to ours [9] which our work is closely based on, and which we have compared our work to throughout the paper.

Fixed Points in Type Theory When working with inductive and co-inductive types and proofs in type theory, it is required to follow restrictive syntactic checks (*e.g.*, productivity and guardedness for co-induction). These overly strong syntactic conditions protect mechanizations against inconsistencies, but reject many valid definitions. Motivated alleviate this situation, Di Gianantonio and Miculan [18] introduce complete ordered families of equivalences (COFEs) as a unifying theory for mixed-variance recursive definitions that support construction of fixed points. They define COFEs over an arbitrary well-founded order and prove a generalized fixed point theorem for contractive endofunctions over these COFEs. In a subsequent work, Di Gianantonio and Miculan [19] generalize this result to sheaf categories over topologies with a well-founded basis — this is very close to the setting of Birkedal *et al.* [9] upon which we have based our work. The main difference between the works of Di Gianantonio and Miculan [18, 19] and Birkedal *et al.* [9], and thereby also our work, is that the former only constructs fixed points of morphisms (similar to our results in Section 2.2) whereas the latter also constructs fixed points of functors.

Mechanizations of Solutions to Domain Equations Benton *et al.* [6] mechanize solution to domain equations over directed-complete partial orders (DCPOs) in the Rocq Prover based on the mechanization of DCPOs by Paulin-Mohrig [38]. Huffman [26] constructs a universal domain into which all bifinite domains can be embedded. Dockins [15] mechanizes solutions of domain equations over the category of profinite domains [22] in Rocq. All these works are based on classical domain theory, and as also pointed out by Sieczkowski *et al.* [43], unlike our guarded domains, do not appear to be suitable for modeling higher-order program logics like Iris [29].

The most closely related works to us are Rocq mechanizations of the domain equation solver of the ModuRes library [43], the domain equation solver of the Iris program logic [28] which is a nicer reimplementation of the domain equation solver of the ModuRes library, and the domain equation solver of transfinite Iris [45]. The former two mechanizations work with the category of COFEs (these are COFEs over ω and not over an arbitrary ordered set like Di Gianantonio and Miculan [18]), a representation of the category of complete bisected bounded

[4] Gilbert *et al.* [20] use the name `unsquash` for the eliminator of their `squashed` type (which they call `squash`, and its constructor `sq`) that only eliminates into other `SProp` types.

ultra metric spaces (CBUlt) [11] that is particularly amenable to mechanizations [43]. These works only support step-indexing up to ω. Transfinite Iris, inspired by Birkedal *et al.* [9], extend the definition of OFEs (COFEs without completeness requirement) and COFEs over ω to those over **Ord**. However, Transfinite Iris, unlike the ModuRes library and Iris, only solves domain equations for functors of the form $\mathrm{OFE}^{\mathsf{op}} \times \mathrm{OFE} \to \mathrm{COFE}$ and not $\mathrm{COFE}^{\mathsf{op}} \times \mathrm{COFE} \to \mathrm{COFE}$. An example of a functor that is not supported by transfinite Iris as a result of this limitation is our Example 32.

Mechanizations of Category Theory We mentioned the existing mechanizations of category theory in Section 5. We refer to Hu and Carette [24] who give an extensive survey comparing these mechanizations. Regarding our mechanization of category theory, we only mention that its span is not significant compared to the works cited, compared, and contrasted by Hu and Carette [24]. We have only mechanized what was necessary for formalizing our main results: Theorem 23, Theorem 30 and Lemma 33.

7 Future Work

Our main future goal is to build a step-indexed (program) logic similar to the Iris framework [29] based on our development. We hope to use such a step-indexed logic to study weak bisimulation of guarded interaction trees, i.e., objects similar to the one shown in Example 32. This requires transfinite step-indexing because we need to allow either side of the bisimulation relation to take finitely many silent steps (τ-steps). However, as we discussed in Section 6, the existing work on transfinite step-indexing does not support equations like that in Example 32. However, there is a significant amount of nontrivial technical work that needs to be done before we can put our mechanization to use for constructing a user-friendly step-indexed (program) logic framework, *e.g.*, providing an interactive proof mode similar to that of the Iris framework [33, 32]. In principle, the main limiting factor is the complexity of working with presheaves compared to COFEs in proof assistants; using our solution forces one to always use categories and categorical constructions. For instance, maps between COFEs are non-expansive functions (Rocq functions with a side-condition), while maps between presheaves are natural transformations (families of functions with a naturality side-condition relating these families). Thus, while presheaves are more amenable to mechanization than sheaves, there remains a substantial amount of work required to build a user-friendly (program) logic framework on top of our category theoretic development. This makes developing a user-friendly system on top of our development very challenging, which we leave as an important future work.

8 Conclusion

After motivating the need for solving domain equations over the category of presheaves over ordinals, we presented the theory of solving such domain equations and discussed its mechanization as well as the challenges we faced mechanizing this theory. As demonstrated by our Example 32, this domain equation solver can be used to solve domain equations stated as mixed-variance functors like those that are needed for guarded interaction trees [17] or program logics [45].

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A The Need for Step-Indexing Over Higher Ordinals

As we discussed in the Introduction, the main issue is that when working with step-indexing over ω the existential property fails to hold. That is, if we know that $\models \exists x : A.\ \phi(x)$ holds, we do necessarily get that there exists an $a \in A$ such that $\models \phi(a)$ holds. In this appendix, we first discuss why the existential property fails when step-indexing over ω. We then present a proof for a general criterion of when the existential property does hold, and based on this criterion motivate our choice (and that of Spies *et al.* [45]) to use step-indexing over all ordinals in a Grothendieck universe.

A.1 Failure of the Existential Property

Recall that in step-indexing over ω the set of truth values is the Heyting algebra of the downwards-closed subsets of ω. In this setting, the interpretation of ϕ, $[\![\phi]\!]$, is a function from A into this Heyting algebra, and we have $[\![\exists x : A.\ \phi(x)]\!] = \bigcup_{x \in A} [\![\phi]\!](x)$. Thus, $\models \exists x : A.\ \phi(x)$ is equivalent to saying that $\bigcup_{x \in A} [\![\phi]\!](x) = \omega$, *i.e.*, that the interpretation of $\exists x : A.\ \phi(x)$ is the truth value $\top$, which in our Heyting algebra is the entire set ω. Now, take $A := \mathbb{N}$ and $\phi(n) := \triangleright^n \bot$. Readers not familiar with step-indexed logics can ignore the exact definition of ϕ. (These readers are kindly referred to Jung *et al.* [29] for a detailed explanation of the syntax and semantics of the step-indexed logic Iris). What is important for us is that $[\![\triangleright^n \bot]\!] = \{k | k < n\}$. Based on this interpretation one can easily see that $\bigcup_{n \in \mathbb{N}} [\![\triangleright^n \bot]\!] = \omega$ but there is no $n \in \mathbb{N}$ such that $[\![\triangleright^n \bot]\!] = \omega$.

A.2 A General Criterion for the Existential Property to Hold

Here, we present a general criterion of when the universal property holds. This section is based on a blog post by the second author [48].

▶ **Theorem 34** (). *If the step-indexing is a regular ordinal and the cardinality of the set quantified over is strictly smaller than that of the step-indexing ordinal, then the existential property holds.*

▶ Remark 35. The Rocq mechanization of Theorem 34 does not use ordinals or a step-indexed logic. It follows very closely the ideas in the blog post [48]. It first defines an analogue of regularity for an arbitrary Rocq type A with a relation $R \subseteq A \times A$ on it. Then, it shows that $\forall a : A.\ \exists b : B.\ P(a, b)$ implies $\exists b : B.\ \forall a : A.\ P(a, b)$ whenever the type A is regular, the cardinality of B is strictly smaller than that of A, and furthermore P is downwards-closed with respect to R, *i.e.*, if $\forall a, a' \in A, b \in B.\ R(a, a') \wedge P(a', b) \implies P(a, b)$.

Below, we first give the definition of regular ordinals (which can be found in most standard textbooks on set theory [27]) and then give the proof of Theorem 34.

▶ **Definition 36.** *We say an ordinal γ is regular, if the supremum of any sequence of ordinals strictly smaller than γ, indexed by an ordinal strictly smaller than γ, is also strictly smaller than γ. In other words, for any sequence of ordinals $\{\beta_\alpha\}_{\alpha \preceq \delta}$ indexed by an ordinal δ, if we have both that $\delta \prec \gamma$, and that $\forall \alpha \preceq \delta.\ \beta_\alpha \prec \gamma$, then $\bigcup_{\alpha \preceq \delta} \beta_\alpha \prec \gamma$.*

Proof of Theorem 34. Let us assume that we are working with a logic step-indexed over a regular ordinal γ — thus, the set of truth values is the Heyting algebra of downwards-closed subsets of γ. Furthermore, let us assume we are given a set A whose cardinality is strictly smaller than that of γ. (More specifically, let us assume that $A = \{a_\alpha | \alpha \preceq \delta\}$ for some $\delta \prec \gamma$.) Finally, assume we are given a predicate ϕ over A such that $[\![\exists x : A.\ \phi(x)]\!] = \bigcup_{x \in A} [\![\phi]\!](x) = \gamma$. We show, using proof by contradiction, that there exists an element $a \in A$ such that $[\![\phi]\!](a) = \gamma$.

Assume, to the contrary that there is no element $a \in A$ such that $[\![\phi]\!](a) = \gamma$. In other words, for any $a \in A$ there is an ordinal $\beta \prec \gamma$ such that $\beta \notin [\![\phi]\!](a)$. Now, since $A = \{a_\alpha | \alpha \preceq \delta\}$, this forms a sequence of ordinals $\{\beta_\alpha\}_{\alpha \preceq \delta}$ such that $\forall \alpha \preceq \delta.\ \beta_\alpha \prec \gamma$ and that $\forall \alpha \preceq \delta.\ \beta_\alpha \notin [\![\phi]\!](a_\alpha)$. Thus, by regularity of γ, we have that $\bigcup_{\alpha \preceq \delta} \beta_\alpha \prec \gamma$, and by the fact that for any a the set $[\![\phi]\!](a)$ is downwards-closed, we have that $\forall \alpha \preceq \delta.\ \bigcup_{\alpha \preceq \delta} \beta_\alpha \notin [\![\phi]\!](a_\alpha)$. Hence, it must be the case that $\bigcup_{\alpha \preceq \delta} \beta_\alpha \notin \bigcup_{\alpha \preceq \delta} [\![\phi]\!](a_\alpha) = \bigcup_{x \in A} [\![\phi]\!](x) = [\![\exists x : A.\ \phi(x)]\!]$, which is a contradiction. ◻

We remark that the set of all ordinals in the universe acts basically as a regular ordinal — indeed, this must be understood as step-indexing over the supremum of that set which is regular. In other words, we have by definition that for any function $A \to \mathbf{Ord}$ where A is a set/type in the universe, the supremum of the image of the function is again an ordinal in $\mathbf{Ord}$. Thus, when step-indexing over all ordinals in the universe, by Theorem 34, we get that the existential property holds for quantification over any set in the universe.

B Later is Locally Contractive, Earlier is not Even Enriched

▶ **Theorem 37** (Later is Enriched and Locally Contractive). *The functor* $\blacktriangleright : \mathbf{PSh}(\mathbf{Ord}) \to \mathbf{PSh}(\mathbf{Ord})$ *is an enriched and locally contractive functor.*

▶ Remark 38 (Earlier is *not* Enriched). The category $\mathbf{PSh}(\mathbf{Ord})$ is enriched over itself. Hence, we can ask whether the functor $\blacktriangleleft : \mathbf{PSh}(\mathbf{Ord}) \to \mathbf{PSh}(\mathbf{Ord})$ is an enriched functor? The answer is negative. Here we give an intuitive explanation as to why this is not the case. We will formally prove this negative answer in Lemma 39.

To understand why earlier is not enriched note that we need to produce a natural transformation for the internal action of the earlier functor. That is, a natural transformation $G^F \to \blacktriangleleft G^{\blacktriangleleft F}$. By adjunction of exponentiation it is the same as requiring a natural transformation $G^F \times \blacktriangleleft F \to \blacktriangleleft G$. Let us look at an arbitrary component of this natural transformation at stage α. That is, a function (morphism in $\mathbf{Set}$) $(\mathcal{Y}_\alpha \times F \to G) \times F(\alpha^+) \to G(\alpha^+)$. Such a map, given a natural transformation $\eta : \mathcal{Y}_\alpha \times F \to G$ and an element $x \in F(\alpha^+)$ must produce a an element of $G(\alpha^+)$. The only possibility here is to use $\eta_{\alpha^+} : \mathcal{Y}_\alpha(\alpha^+) \times F(\alpha^+) \to G(\alpha^+)$. However, set $\mathcal{Y}_\alpha(\alpha^+) = \{* | \alpha^+ \preceq \alpha\}$ is empty.

As a simple corollary of uniqueness of solutions for locally contractive functors we prove that the functor earlier cannot be an enriched functor.

▶ **Lemma 39.** *The functor* $\blacktriangleleft : \mathbf{PSh}(\mathbf{Ord}) \to \mathbf{PSh}(\mathbf{Ord})$ *is not an enriched functor for the self-enrichment of the category* $\mathbf{PSh}(\mathbf{Ord})$.

Proof. Assume that $\blacktriangleleft$ is an enriched functor. In that case, by Lemma 12 the functor $\blacktriangleleft \circ \blacktriangleright$ would be a locally contractive functor as $\blacktriangleright$ is by Lemma 37. However, the functor $\blacktriangleleft \circ \blacktriangleright$ is naturally isomorphic to $\mathsf{id}_{\mathbf{PSh}(\mathbf{Ord})}$. Recall that the co-unit of the adjunction $\blacktriangleleft \dashv \blacktriangleright$ is a natural isomorphism. Hence, any presheaf is a solution for the locally contractive functor $\blacktriangleleft \circ \blacktriangleright$, *i.e.*, for any presheaf F we have $\blacktriangleleft(\blacktriangleright F) \simeq F$. Consequently, by Lemma 23 all presheaves over ordinals must be isomorphic which is a contradiction. ◻

An alternative proof could be given through violating Lemma 16. Consider the unique morphism $f : \blacktriangleright 0 \to 1$ (0 and 1 being the initial and terminal presheaf respectively). This morphism is a 0-isomorphism while $\blacktriangleleft f : 0 \to 1$ (note that $\blacktriangleleft \blacktriangleright 0 = 0$ and $\blacktriangleleft 1 = 1$) is not a 0-isomorphism.

C Omitted Properties of Ordinal-Partial Isomorphisms

▶ Remark 40 (). Ordinal-partial isomorphisms satisfy many properties that ordinary isomorphisms do. In particular, we remark the properties listed below which are all easy to show:

(PIso-1) Identity morphisms id_A are α-isomorphisms for any α.

(PIso-2) For any α-isomorphism $f : A \to B$ and any $g, h \in \mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{B,C}(\alpha)$:

$$\mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{A,B,C}(\alpha)(\lceil f \rceil(\alpha)(*), g) = \mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{A,B,C}(\alpha)(\lceil f \rceil(\alpha)(*), h) \implies g = h$$

(PIso-3) For any α-isomorphism $f : A \to B$ and any $g, h \in \mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{C,A}(\alpha)$:

$$\mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{C,A,B}(\alpha)(g, \lceil f \rceil(\alpha)(*)) = \mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{C,A,B}(\alpha)(h, \lceil f \rceil(\alpha)(*)) \implies g = h$$

(PIso-4) For any α-isomorphism $f : A \to B$ where $x \in \mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{B,A}(\alpha)$ is f's partial inverse and any $g, h \in \mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{A,C}(\alpha)$:

$$\mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{B,A,C}(\alpha)(x, g) = \mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{B,A,C}(\alpha)(x, h) \implies g = h$$

(PIso-5) For any α-isomorphism $f : A \to B$ where $x \in \mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{B,A}(\alpha)$ is f's partial inverse and any $g, h \in \mathbb{E}^{\mathsf{hom}_\mathcal{C}}_{C,B}(\alpha)$

$$\mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{C,B,A}(\alpha)(g, x) = \mathbb{E}^{\mathsf{comp}_\mathcal{C}}_{C,B,A}(\alpha)(h, x) \implies g = h$$

(PIso-6) If $f : A \to B$ and $g : B \to C$ are both α-isomorphisms then so is $g \circ f$.

(PIso-7) For any $f : A \to B$ and $g : B \to C$, if $g \circ f$ is an α-isomorphism, then f is an α-isomorphism if and only if g is an α-isomorphism.

D Some Categorical Definitions and Constructions

Here we present a few basic and well-known category theoretic facts and constructions that are nevertheless worth presenting here so that we can refer to them in the main text.

D.1 Some Properties of Presheaves Over Ordinals

▶ **Lemma 41** (). *Let F be a presheaf over ordinals and $\gamma \preceq \alpha$ be two ordinal numbers. The set $F(\alpha)$ is the limit of the diagram $F|_{\{\beta|\gamma\preceq\beta\preceq\alpha\}}$ being the functor F where the domain is restricted to the set of ordinals $\{\beta|\gamma \preceq \beta \preceq \alpha\}$.*

D.2 Extending Partial Ordinal-Shaped Diagrams

Here, by a partial ordinal-shaped diagram we mean diagram $F : \{\beta|\beta \prec \alpha\}^{\mathsf{op}} \to \mathcal{C}$ whose index category is ordinals strictly below a certain ordinal α. We show that given a cone $(V, \{S_\gamma\}_{\gamma\in\{\beta|\beta\prec\alpha\}})$ on F, we can extend the diagram into a diagram whose index is $\{\beta|\beta \preceq \alpha\}^{\mathsf{op}}$. We write $F^{\mathsf{Ext}(V)}$ for this extended diagram.

$$F^{\mathsf{Ext}(V)}(\gamma) = \begin{cases} V & \text{if } \gamma = \alpha \\ F(\gamma) & \text{otherwise} \end{cases}$$

$$F^{\mathsf{Ext}(V)}_{\delta\preceq\gamma} = \begin{cases} \mathsf{id}_V & \text{if } \gamma = \alpha \text{ and } \delta = \alpha \\ S_\delta & \text{if } \gamma = \alpha \text{ and } \delta \prec \alpha \\ F_{\delta\preceq\gamma} & \text{otherwise} \end{cases}$$

The fact that $(V, \{S_\gamma\}_{\gamma\in\{\beta|\beta\prec\alpha\}})$ is a cone on F ensures that $F^{\mathsf{Ext}(V)}$ is a functor and hence a $\{\beta|\beta \preceq \alpha\}^{\mathsf{op}}$-shaped diagram.

D.3 (Co-)Limits in Functor Categories are Pointwise

Consider the category of functors from $\mathcal{C}$ to $\mathcal{D}$ and natural transformation between them. It is well known that the category $\mathcal{D}^\mathcal{C}$ is complete whenever $\mathcal{D}$ is. Furthermore, in that case limits are pointwise. We state this fact formally here. Consider a diagram $F : \mathcal{J} \to \mathcal{D}^\mathcal{C}$. Given an object A of $\mathcal{C}$, we define the pointwise diagram $F^A : \mathcal{J} \to \mathcal{D}$ as the functor defined as follows:

$$F^A(J) := F(J)(A)$$

$$F^A(h) := F(h)(\mathsf{id}_A) \qquad \text{for any morphism } h : J \to J'$$

▶ **Theorem 42** (Limits in Functor Categories). *Given a diagram $F : \mathcal{J} \to \mathcal{D}^\mathcal{C}$, a functor $L : \mathcal{C} \to \mathcal{D}$ is the limit of the diagram F if and only if for any object A of $\mathcal{C}$, $L(A)$ is the limit of the diagram F^A.*

Similarly, co-limits in functor categories are pointwise — the functor category is co-complete whenever the co-domain category is.

D.4 On Algebras of Endo-Functors and their Categories

Recall that given an endo-functor $T : \mathcal{C} \to \mathcal{C}$, a T-algebra is a pair (A, ϕ_A) of an object A of $\mathcal{C}$ together with a morphism $\phi_A : T(A) \to A$. Furthermore, an algebra morphism from (A, ϕ_A) to (B, ϕ_B) is a morphism $h : A \to B$ such that the following diagram commutes:

$$\begin{array}{ccc} T(A) & \xrightarrow{\phi_A} & A \\ {\scriptstyle T(h)}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ T(B) & \xrightarrow{\phi_B} & B \end{array} \tag{alg-hom}$$

For any endo-functor T, T-algebras together with T-algebra morphisms form a category $\mathcal{A}lg(T)$. We write $\mathcal{U}_{\mathcal{A}lg(T)} : \mathcal{A}lg(T) \to \mathcal{C}$ for the forgetful functor of $\mathcal{A}lg(T)$.

▶ **Theorem 43** (). *Let T be an endo-functor on $\mathcal{C}$. The category $\mathcal{A}lg(T)$ is complete whenever $\mathcal{C}$ is.*

Proof. Let $F : \mathcal{J} \to \mathcal{A}lg(T)$ be a diagram of T-algebras. We endow the limit of the $\mathcal{C}$ diagram $\mathcal{U}_{\mathcal{A}lg(T)} \circ F$ with an algebra structure. Let L be the limit of this diagram with projections $\Pi^L_J : L \to F(J)$. The morphisms $\phi_{F(J)} \circ T(\Pi^L_J) : T(L) \to F(J)$ form a cone on the diagram $\mathcal{U}_{\mathcal{A}lg(T)} \circ F$. We define $\phi_L : T(L) \to L$ as the unique morphism into the limit L from this cone. That is,

$$\phi_L := \lim_{J \in \mathcal{J}} \left(\phi_{F(J)} \circ T(\Pi^L_J)\right)$$

It remains to show that the projections of the limit $\Pi^L_J : L \to F(J)$ are algebra homomorphisms, and that given any cone over F in the category $\mathcal{A}lg(T)$ there is a unique T-algebra homomorphism from that cone to (L, ϕ_L). We leave the latter as a simple exercise. As for the former, we need to show that the following diagram commutes

$$\begin{array}{ccc} T(L) & \xrightarrow{\phi_L} & A \\ {\scriptstyle T(\Pi^L_J)}\big\downarrow & & \big\downarrow{\scriptstyle \Pi^L_J} \\ T(F(J)) & \xrightarrow{\phi_{F(J)}} & F(J) \end{array}$$

which simply holds by the definition of ϕ_L above. ◻

▶ Remark 44 (). For any T-algebra (A, ϕ_A), $(T(A), T(\phi_A))$ is also a T-algebra. Furthermore, if h is a T-algebra morphism, so is $T(h)$. Thus, T forms an endo-functor on the category of T-algebras. Consequently, the image of any commutative diagram in $\mathcal{A}lg(T)$ under T is also a commutative diagram. Hence, for a diagram $F : \mathcal{J} \to \mathcal{A}lg(T)$ of T-algebras and a cone $\Big((A, \phi_A), \{S_j : A \to F(J)\}_{j \in J}\Big)$ on diagram F, the cone below is also a cone on diagram F:

$$\Big((T(A), T(\phi_A)), \{\phi_{F(J)} \circ T(S_J) : T(A) \to F(J)\}_{j \in J}\Big)$$