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Preprint
August
1995

IMS

Institute of Mathematical Statistics
University of Copenhagen

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July 26, 1995

Abstract

The likelihood ratio test for cointegration is analysed for partial systems in Gaussian vector autoregressive error correction models. Under the assumption of weak exogeneity of the conditioning variables for the cointegrating parameters the test is derived and the asymptotic distribution is given.

1 Introduction

For the statistical analysis of vector autoregressive models it is in general optimal to analyse the full system of all variables. However, in the situation where some of the variables are (weakly) exogenous, it may be desirable to perform the analysis conditional on these. By the conditional or the partial analysis the intention is to obtain a reduction of the dimension of the system analysed. It is demonstrated here that by assuming that the variables conditioned upon are

*We would like to thank D. F. Hendry, Nuffield College, Oxford, for good comments.

weakly exogenous for the cointegrating parameters, it is indeed possible to obtain a reduction of the system analysed for cointegration. It is shown in Johansen [2] that the cointegrating vectors of the partial system can be estimated by reduced rank regression. The asymptotic distribution was described with and without the assumption of weak exogeneity of the variables conditioned upon.

In the case of weak exogeneity the maximum likelihood estimator of the cointegrating relations and the likelihood ratio test statistic for the number of these can be determined from the partial system. It turns out that the modelling of deterministic terms is crucial for the asymptotic distribution of the likelihood ratio test. In a model allowing for a linear trend in all linear combinations of the process, but not a quadratic trend, the test will be asymptotically similar and furthermore invariant to the drift terms. The focus will mainly be on such a model. However, it will also be demonstrated that asymptotic inference will depend on a nuisance parameter, when the differences of the process are modelled by an unrestricted constant.

If the variables conditioned upon can not be modelled by a vector autoregression as assumed here, there is no reason to believe that the results about inference for cointegrating rank should hold. Often the reason for performing partial analysis and not model the full system is indeed that the stochastic behaviour of the variables conditioned upon is difficult to model, e.g. the oilprice. However, it may happen that not all data of the series conditioned upon are equally important, but only some specific interventions or shocks are of importance, e.g. the oil crisis. In such a situation a partial analysis using the tables of this paper should probably not be performed, but instead the exogeneous series should be replaced with intervention dummies. In this case the asymptotic distributions are without nuissance parameters but require special tables. These may be simulated by the program DisCo, cf. Johansen and Nielsen [7].

In section 2 of the paper the main model and the cointegration hypotheses are presented. The statistical analysis follows in sections 3. In section 4 the asymptotic distribution of the likelihood ratio test is presented, whereas the proof is located in the Appendix. In section 5 a slightly different model is discussed and it is shown how the asymptotic inference in that case involves a nuisance parameter depending on the parameters in the model for the variables conditioned upon. Section 6 contains a discussion of how to test for weak exogeneity and in section 7 the ideas of the paper are illustrated by an example. Tables of the asymptotic distributions of the tests are enclosed.

2 Model and Cointegration Hypothesis

Consider initially the p-dimensional autoregressive model with a linear drift, given by, the equation

$$\Delta X_t = \Pi X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \mu_1 + \mu_2 t + \varepsilon_t\,, \tag{2.1}$$

where $t = 1, \ldots, T$ while $X_{-k+1}, \ldots, X_0$ are fixed and $\{\varepsilon_t\}_{t=1,\ldots,T}$ are independently mean zero Gaussian distributed with variance Ω.

Let the process, X_t be divided into Y_t of dimension p_Y and Z_t of dimension p_Z, where $p = p_Y + p_Z$. For the partial analysis it will be assumed that the increments of the process, Z_t, contain no information about the parameters of the level of the process. This is equivalent to assume the impact matrix Π has p_Z zero rows corresponding to the Z_t variables. With p_Z zero rows in Π, the hypothesis of at most r cointegrating relations in (2.1) is given by reduced rank of the remaining p_Y rows of Π, Π_Y say. That is, rank$(\Pi_Y) \leq r$ or equivalently $\Pi_Y = \alpha_Y \beta'$, where $\alpha_Y \in \mathbf{R}^{p_Y \times r}$, $\beta \in \mathbf{R}^{p \times r}$. From Johansen [4] it follows that with $\Pi_Z = 0$, then Z_t is weakly exogeneous with respect to the cointegrating vectors, β. With the asumption of weak exogeneity and cointegration imposed, equation (2.1) is given by,

$$\Delta X_t = \begin{pmatrix} \alpha_Y \beta' \\ 0 \end{pmatrix} X_{t-1} + \sum_{i=1}^{k-1} \begin{pmatrix} \Gamma_i^Y \\ \Gamma_i^Z \end{pmatrix} \Delta X_{t-i} + \begin{pmatrix} \mu_1^Y \\ \mu_1^Z \end{pmatrix} + \begin{pmatrix} \mu_2^Y \\ \mu_2^Z \end{pmatrix} t + \begin{pmatrix} \varepsilon_t^Y \\ \varepsilon_t^Z \end{pmatrix}\,. \tag{2.2}$$

The Granger Representation Theorem, cf. Johansen [4] states that if ΔX_t is stationary and rank$(\Pi_Y) = r$, then the cointegrating relations $\beta' X_{t-1}$ are stationary with a linear trend. Furthermore, the process X_t is $I(1)$ and will in general have a quadratic trend. However, if the linear drift term, μ_2 is proportional to the adjustment coefficient, α, then the process will have only a linear trend. Hence, the model allowing for a linear trend in all linear combinations of X_t, with at most r cointegrating relations and weak exogeneity of Z_t with respect to the cointegrating relations is formulated in terms of restrictions on both the impact matrix and the deterministic terms,

$$H_2(r) : \Pi = \binom{\alpha_Y \beta'}{0}, \quad \mu_2 = \binom{\alpha_Y \beta_2'}{0}. \tag{2.3}$$

It is given by the equation

$$\Delta X_t = \begin{pmatrix} \alpha_Y \left(\beta' X_{t-1} + \beta_2' t\right) \\ 0 \end{pmatrix} + \sum_{i=1}^{k-1} \begin{pmatrix} \Gamma_i^Y \\ \Gamma_i^Z \end{pmatrix} \Delta X_{t-i} + \begin{pmatrix} \mu_1^Y \\ \mu_1^Z \end{pmatrix} + \begin{pmatrix} \varepsilon_t^Y \\ \varepsilon_t^Z \end{pmatrix}. \tag{2.4}$$

The Gaussian error term has marginal variances Ω_{YY} and Ω_{ZZ} respectively and covariance Ω_{YZ}. The **partial** or the conditional model is then given by the model for ΔY_t conditional on ΔZ_t and the past, or equivalently,

$$\Delta Y_t = \alpha_Y \left(\beta' X_{t-1} + \beta_2' t\right) + \sum_{i=1}^{k-1} \Gamma_i^{Y \cdot Z} \Delta X_{t-i} + \mu_1^{Y \cdot Z} + \omega \Delta Z_t + \varepsilon_t^{Y \cdot Z} \tag{2.5}$$

and the **marginal** model by

$$\Delta Z_t = \sum_{i=1}^{k-1} \Gamma_i^Z \Delta X_{t-i} + \mu_1^Z + \varepsilon_t^Z. \tag{2.6}$$

Here $\varepsilon_t^{Y \cdot Z}$ and ε_t^Z are independently mean zero Gaussian distributed with variances $\Omega_{YY \cdot Z} = \Omega_{YY} - \Omega_{YZ}\Omega_{ZZ}^{-1}\Omega_{ZY}$ and Ω_{ZZ} respectively, and $\omega = \Omega_{YZ}\Omega_{ZZ}^{-1}$. The parameters $\Gamma_i^{Y \cdot Z}$ and $\mu_i^{Y \cdot Z}$ are short hand notation for $\Gamma_i^Y - \omega \Gamma_i^Z$ and $\mu_i^Y - \omega \mu_i^Z$ respectively.

It should be stressed that the assumption of weak exogeneity of Z_t implies that the parameters of the partial and the marginal models vary freely, so that the likelihood function is maximized with respect to $(\beta', \beta_2')'$ by maximizing solely the partial likelihood. Likewise the likelihood ratio test statistic for the hypothesis of cointegration is derived by analysing only the partial model.

Two submodels of (2.3) are of interest. The first one is the model allowing for an intercept in all linear combinations of the process but no linear trend and is given by

$$H_1(r) : \Pi = \binom{\alpha_Y \beta'}{0}, \quad \mu_1 = \binom{\alpha_Y \beta_1'}{0}, \quad \mu_2 = 0. \tag{2.7}$$

The second model is without deterministic terms at all,

$$H_0(r) : \Pi = \binom{\alpha_Y \beta'}{0}, \quad \mu_1 = \mu_2 = 0. \tag{2.8}$$

These submodels are nested in the main model as follows

$$H_0(r) \subset H_1(r) \subset H_2(r). \tag{2.9}$$

It turns out that the hypotheses, $H_i(r)$ above can be tested by asymptotic similar inference. After having determined the cointegrating rank, inference for restrictions on the cointegrating vector will be asymptotic χ^2 and e.g. the hypothesis of no linear trend in the cointegrating relation is verified in model $H_2(r)$ by testing $\beta_2 = 0$. The likelihood ratio test for this hypothesis is given in (3.5) in section 3.

Remark: For a process without short term dynamics, i.e. $\Gamma_1 = \ldots = \Gamma_{k-1} = 0$, the weakly exogenous marginal process, Z_t is a random walk and hence nonstationary. However, this is not necessarily the case for a process *with* short term dynamics, that is, some or all of the components of Z_t may be stationary. An example is the two dimensional process, $X_t = (Y_t, Z_t)'$, given by the equation

$$\Delta X_t = \Pi X_{t-1} + \Gamma \Delta X_{t-1} + \varepsilon_t\,, \tag{2.10}$$

where

$$\alpha = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \Gamma = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}, \tag{2.11}$$

and $\{\varepsilon_t\}_{t=1,\ldots,T}$ are independently $N_2(0, I_2)$ distributed. In this case the vectors α and β are orthogonal and the process Z_t is stationary.

3 Statistical Analysis

As already mentioned the weak exogeneity assumption implies that all parameters in the partial model and in the marginal model vary freely. Therefore the likelihood function is maximized with respect to $(\beta', \beta_2')'$ by maximizing solely the partial likelihood, and likewise the likelihood ratio test for cointegrating rank is derived by analysing only the partial model.

Now, under $H_2(r)$, the partial model is given by (2.5),

$$\Delta Y_t = \alpha_Y \begin{pmatrix} \beta \\ \beta_2 \end{pmatrix}' \begin{pmatrix} X_{t-1} \\ t \end{pmatrix} + \sum_{i=1}^{k-1} \Gamma_i^{Y \cdot Z} \Delta X_{t-i} + \mu_1^{Y \cdot Z} + \omega \Delta Z_t + \varepsilon_t^{Y \cdot Z},$$

with Z_t weakly exogenous for the parameter $(\beta', \beta_2')'$. By regression of ΔY_t and $(X_{t-1}', t)'$ on the lagged differences, $(\Delta X_{t-i})_{i=1,\ldots,k-1}$, the constant and ΔZ_t, the likelihood function is concentrated with respect to the parameters $(\Gamma_i^{Y\cdot Z})_{i=1,\ldots,k-1}$, $\mu^{Y\cdot Z}$ and ω. From the initial regressions one obtains the residuals $R_t^{Y\cdot Z}$, $(p_Y \times 1)$ and $R_t^{X\cdot Z}$, $((p+1) \times 1)$, in terms of which the concentrated partial likelihood function is given by

$$L_{max}^{-2/T} = |\Omega_{YY\cdot Z}| \exp\left\{ \frac{1}{T} \sum_{t=1}^{T} (R_t^{Y\cdot Z} - \alpha_Y \begin{pmatrix} \beta \\ \beta_2 \end{pmatrix}' R_t^{X\cdot Z})' \Omega_{YY\cdot Z}^{-1} (R_t^{Y\cdot Z} - \alpha_Y \begin{pmatrix} \beta \\ \beta_2 \end{pmatrix}' R_t^{X\cdot Z}) \right\}.$$

For fixed (β', β_2'), the maximum likelihood estimators of α_Y and $\Omega_{YY\cdot Z}$ are then found by ordinary regression, leading to the definition of the residual product moment matrices,

$$S_{ij\cdot Z} = \tfrac{1}{T} \textstyle\sum_{t=1}^{T} R_t^{i\cdot Z} R_t^{j\cdot Z\prime} \quad (i,j = X, Y). \tag{3.1}$$

Finally reduced rank regression of ΔY_t on (X_{t-1}', t) corrected for lagged differences, a constant and ΔZ_t is performed by solving the $(p+1)$-dimensional eigenvalue problem

$$\left| \lambda S_{XX\cdot Z} - S_{XY\cdot Z} S_{YY\cdot Z}^{-1} S_{YX\cdot Z} \right| = 0, \tag{3.2}$$

for eigenvalues $1 > \hat\lambda_1 > \ldots > \hat\lambda_{p_Y} > 0$ and $\hat\lambda_{p_Y+1} = \ldots = \hat\lambda_{p+1} = 0$ and for eigenvectors $\hat V = (\hat v_1, \ldots, \hat v_{p+1})$. The $(p+1-p_Y)$ smallest eigenvalues in (3.2) are equal to zero, since $S_{YY\cdot Z}$ has rank p_Y. Then $(\beta', \beta_2')'$ is estimated by $(\hat v_1, \ldots, \hat v_r)$. The likelihood ratio test of at most r cointegrating relations is given by,

$$\mathrm{LR}(H_2(r)|H_2(p_Y)) = -T \sum_{i=r+1}^{p_Y} \ln(1 - \hat\lambda_i). \tag{3.3}$$

Instead of the eigenvalue problem (3.2), the dual p_Y-dimensional eigenvalue problem

$$\left| \lambda S_{YY\cdot Z} - S_{YX\cdot Z} S_{XX\cdot Z}^{-1} S_{XY\cdot Z} \right| = 0\,, \tag{3.4}$$

might be solved for eigenvalues $(\hat\lambda_1^d, \ldots, \hat\lambda_{p_Y}^d)$ and eigenvectors $\hat V^d = (\hat v_1^d, \ldots, \hat v_{p_Y}^d)$. Note that $\hat\lambda_i^d = \hat\lambda_i$ and $\hat v_i = \frac{1}{\sqrt{\hat\lambda_i}} S_{XX\cdot Z}^{-1} S_{XY\cdot Z} \hat v_i^d$ for $i = 1, \ldots, p_Y$.

In $H_2(r)$ the hypothesis of no linear trend in the cointegrating relations is given by $\beta_2 = 0$ as mentioned. With $\beta^* = (\beta', \beta_2')'$ the hypothesis is of the form $\beta^* = H\beta$, with the matrix H being of dimension $((p+1) \times p)$ and is similar to the hypothesis tested in Johansen [4], Theorem 3.2. Thus the likelihood ratio test which is asymptotically $\chi^2(r)$ distributed is given by,

$$T \sum_{i=1}^{r} \ln\{(1-\hat{\lambda}_i^{\beta_2})/(1-\hat{\lambda}_i)\}\,. \tag{3.5}$$

The $\hat{\lambda}_i$ are the r largest eigenvalues solving the eigenvalue problem in (3.2) and likewise the eigenvalues $\hat{\lambda}_i^{\beta_2}$ are the r largest eigenvalues solving the eigenvalue problem corresponding to the reduced rank regression of ΔY_t on $(X'_{t-1}, t)H = X'_{t-1}$ correcting for lagged differences, a constant and ΔZ_t.

The sub-model $H_1(r)$ with restricted drift term is analysed by reduced rank regression of ΔY_t on $(X'_{t-1}, 1)'$, correcting for the lagged differences and ΔZ_t. And likewise the sub-model $H_0(r)$ without any drift term at all is analysed by reduced rank regression of ΔY_t on X_{t-1}, correcting for the lagged differences and ΔZ_t.

4 Distributions of the Cointegration Tests

The asymptotic theory for the rank tests given in section 3 is derived using the Granger Representation Theorem in Johansen [4]. Define for any $(n \times m)$ matrix b of rank $m \leq n$, the matrix $b_\perp$ to be a $(n \times (n-m))$ matrix orthogonal to b such that $(b, b_\perp)$ spans $\mathbf{R}^n$. Note, that

$$\alpha = \begin{pmatrix} \alpha_Y \\ 0 \end{pmatrix}, \quad \alpha_\perp \begin{pmatrix} \alpha_{Y\perp} & 0 \\ 0 & I_{pz} \end{pmatrix}. \tag{4.1}$$

With these definitions the assumptions under which the Representation Theorem hold are:

- The characteristic polynomium of the process, X_t has roots outside the unit circle or at 1.
- The matrices α_Y and β have rank equal to r, the cointegrating rank.
- The matrix $\alpha'_\perp(I - \sum_{i=1}^{k-1} \Gamma_i)\beta_\perp$ has full rank $p - r$.

It is important to stress that these assumptions concern the joint process, $X_t = (Y_t', Z_t')'$ as given by the models $H_i(r)$. That is, the full system is vector autoregressive with a marginal process, ΔZ_t being weakly exogeneous for the cointegrating parameters. With these assumptions imposed, the process can be given the following representations,

$$H_2(r): \quad X_t = C\sum_{i=1}^{t}\varepsilon_i + \tau_1 t + \tau_0 + \text{stationary process}_t + \text{initial value} \tag{4.2}$$

$$H_1(r): \quad X_t = C\sum_{i=1}^{t}\varepsilon_i + \tau_2 + \text{stationary process}_t + \text{initial value} \tag{4.3}$$

$$H_0(r): \quad X_t = C\sum_{i=1}^{t}\varepsilon_i + \text{stationary process}_t + \text{initial value} \tag{4.4}$$

Moreover, $\beta'\tau_1 = -\beta_2$ and it follows that under $H_2(r)$, $\beta' X_t$ is indeed trend stationary. Likewise $\beta' X_t$ is stationary with or without an intercept under $H_1(r)$ or $H_0(r)$ respectively.

As is the case in general for autoregressive analysis of cointegrating rank, the asymptotic distributions are given by the trace of stochastic matrices involving Brownian motions. However, the Brownian motions involved in the partial analysis are of different dimensions, when compared with the full system analysis, merely reflecting the conditioning.

Theorem 4.1 *Under $H_i(r)$ it holds that, if the cointegrating rank is r and the additional assumptions of the Granger Representation Theorem hold,*

$$LR(H_i(r)|H_i(p_Y)) \xrightarrow{w} tr\left\{\int_0^1 dGF'(\int_0^1 FF'du)^{-1}\int_0^1 FdG'\right\}, \tag{4.5}$$

as $T \to \infty$. The functions F and G are expressed in terms of a $(p-r)$ dimensional standard Brownian motion $W(u)$ and the $(p_Y - r)$ dimensional process G is given by,

$$G_i(u) = W_i(u) \quad i=1,\ldots,p_Y\text{-}r. \tag{4.6}$$

When testing $H_2(r)$ in $H_2(p_Y)$ the $(p-r+1)$ dimensional process F is given by,

$$F_i(u) = \begin{cases} W_i(u) - \overline{W}_i & i=1,\ldots,p\text{-}r \\ u - \frac{1}{2} & i=p\text{-}r\text{+}1 \end{cases} \tag{4.7}$$

and the distribution (4.5) is tabulated in Table 3.

When testing $H_1(r)$ in $H_1(p_Y)$ the $(p-r+1)$ dimensional process F is given by,

$$F_i(u) = \begin{cases} W_i(u) & i=1, \ldots, p-r \\ 1 & i=p-r+1 \end{cases} \tag{4.8}$$

and the distribution (4.5) is tabulated in Table 4.

When testing $H_0(r)$ in $H_0(p_Y)$ the $(p-r)$ dimensional process F is given by,

$$F_i(u) = \left\{ W_i(u) \quad i=1, \ldots, p-r \right. \tag{4.9}$$

and the distribution (4.5) is tabulated in Table 5.

The *proof* of the Theorem 4.1 is given in the Appendix A.

The enclosed Tables of quantiles have been produced by simulation. The number of simulations were set to 10,000 and the number of observations were set to 400.

5 On the model with unrestricted trend

In the models $H_i(r)$ the cointegration hypotheses are formulated in terms of restrictions on both the impact matrix and the deterministic terms. The model $H_2(r)$ allows for linear trends in all linear combinations of the process X_t and thus essentially trend stationarity is tested against nonstationarity with a linear trend. Consider now the model where the drift parameter is left unrestricted rather than proportional to the loadings. It is given by

$$\tilde{H}_1(r) : \Pi = \begin{pmatrix} \alpha_Y \beta' \\ 0 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} \mu_1^Y \\ \mu_1^Z \end{pmatrix}, \quad \mu_2 = 0 \tag{5.1}$$

or equivalently by the equation,

$$\Delta X_t = \begin{pmatrix} \alpha_Y \\ 0 \end{pmatrix} \beta' X_{t-1} + \sum_{i=1}^{k-1} \begin{pmatrix} \Gamma_i^Y \\ \Gamma_i^Z \end{pmatrix} \Delta X_{t-i} + \begin{pmatrix} \mu_1^Y \\ \mu_1^Z \end{pmatrix} + \begin{pmatrix} \varepsilon_t^Y \\ \varepsilon_t^Z \end{pmatrix}. \tag{5.2}$$

This model is nested with previous mentioned models as follows,

$$H_0(r) \subset H_1(r) \subset \tilde{H}_1(r) \subset H_2(r). \tag{5.3}$$

The statistical analysis presented in section 3 applies here and leads to a reduced rank regression of ΔY_t on X_{t-1} correcting for ΔZ_t, levels and lagged differences. However, as stated in Theorem 5.1 below, the asymptotic distribution of the likelihood ratio test depends on a nuisance parameter c, which makes the inference for cointegration difficult.

Hence, instead of analysing model $\tilde{H}_1(r)$ it is suggested to analyse $H_2(r)$ in order to determine the cointegrating rank r. Having determined the rank using Table 3, it is as previously mentioned possible to test for staionarity against trend stationarity by a $\chi^2(r)$ test of $\beta_2 = 0$, cf. (3.5). The difference between the $H_i(r)$ and the $\tilde{H}_1(r)$ modelling is demonstrated in the example in section 7.

Theorem 5.1 *Under $\tilde{H}_1(r)$ it holds that, if the cointegrating rank is r and the additional assumptions of the Granger Representation Theorem hold,*

$$LR(\tilde{H}_1(r)|\tilde{H}_1(p_Y)) \xrightarrow{w} tr\left\{\int_0^1 dGF'(\int_0^1 FF'du)^{-1}\int_0^1 FdG'\right\}, \qquad (5.4)$$

as $T \to \infty$. The functions F and G are expressed in terms of a $(p-r)$ dimensional Brownian motion $W(u)$ and a nuisance parameter, c, given by,

$$c = \sqrt{\frac{\mu_1^{Y\cdot Z\prime}\alpha_{Y\perp}(\alpha'_{Y\perp}\Omega_{YY\cdot Z}\alpha_{Y\perp})^{-1}\alpha'_{Y\perp}\mu_1^{Y\cdot Z}}{\mu_1^{Z\prime}\Omega_{ZZ}^{-1}\mu_1^{Z}}}. \qquad (5.5)$$

Here it is assumed that either $\alpha'_{Y\perp}\mu_1^{Y\cdot Z} \neq 0$ or $\mu_1^Z \neq 0$. The $(p_Y - r)$ dimensional process G and the $(p-r)$ dimensional process F are given by,

$$G_i(u) = W_i(u) \quad i=1,\ldots,p_Y-r, \qquad (5.6)$$

$$F_i(u) = \begin{cases} u - \frac{1}{2} & i=1, \\ W_i(u) - \overline{W}_i & i=2,\ldots,p-r-1, \\ W_c(u) - \overline{W}_c & i=p-r, \end{cases} \qquad (5.7)$$

where

$$W_c(u) = \frac{1}{\sqrt{1+c^2}}W_1(u) + \frac{c}{\sqrt{1+c^2}}W_{p-r}(u). \qquad (5.8)$$

Here $\overline{W}_i = \int_0^1 W_i(u)du$ and $u \in [0,1]$.

For the case $c = 0$ the distribution is tabulated in Table 6, and for the case $c = \infty$ in Table 7.

Theorem 5.1 does not include the case where $\alpha'_{Y\perp}\mu_1^{Y\cdot Z} = 0$ and $\mu_1^Z = 0$. In this case the limit distribution differs and has, we believe, larger tails. This case is also treated in model $H_1(r)$.

The *proof* of Theorem 5.1 follows closely the ideas of the proof of Theorem 4.1 and is given in Appendix B.

The limit distribution of $\tilde{H}_1(r)$ in $\tilde{H}_1(p_Y)$ when $c = 0$ is also obtained in other cointegration tests. When $p_Z = 1$ the distribution corresponds to that of Table 5 in Johansen [5]. When $p_Z > 1$ one obtains the same distribution as when testing $H_2(r)$ in $H_2(p_Y)$ with $p_Z - 1$ exogeneous variables. However the distributions have been simulated twice and appear as slightly different results in Table 3 and Table 6, due to simulation inaccuracy.

The nuisance parameter c is equal to the ratio of the length of the drift parameters in the partial system and in the marginal system, standardised by the covariances of the respective systems. Thus an estimate of c is needed in order to use the distribution for inference, and may be obtained using the definition in (5.5). The asymptotic distribution under $\tilde{H}_1(r)$ has been studied for different values of the nuisance parameter, c. The limit distribution is a continuous function of c for $0 \leq c \leq \infty$, and a simple empirical relation seems to exist between the distributions as a function of c. That is, for each c, the asymptotic distribution of $\mathrm{LR}(\tilde{H}_1(r)|\tilde{H}_1(p_Y))$, here denoted $\mathcal{L}_c(p_Y - r, p_Z)$ can be approximated by

$$\frac{1}{1+c^2}\mathcal{L}_0(p_Y - r, p_Z) + \frac{c^2}{1+c^2}\mathcal{L}_\infty(p_Y - r, p_Z). \tag{5.9}$$

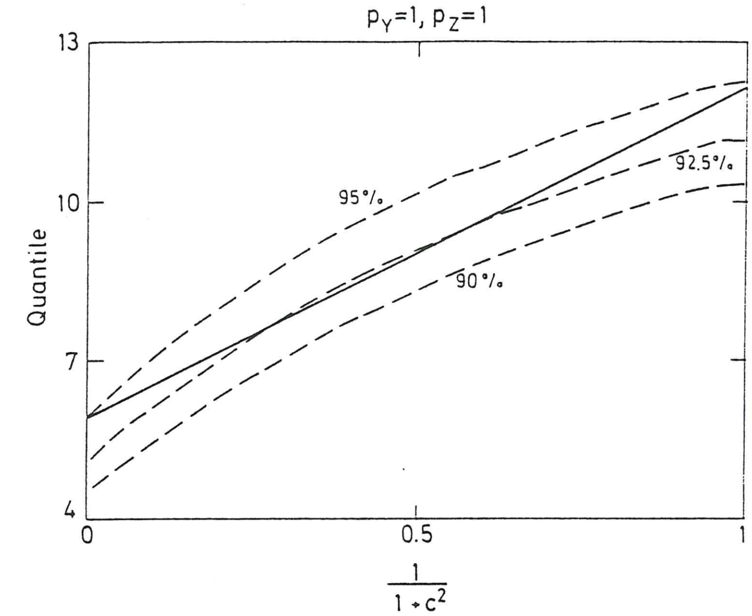
This relationship has been simulated for all possible values of $(p_Y - r, p_Z)$ where $p_Y - r + p_Z \leq 7$, for 20 choices of c and 10,000 repetitions of processes with 400 observations.

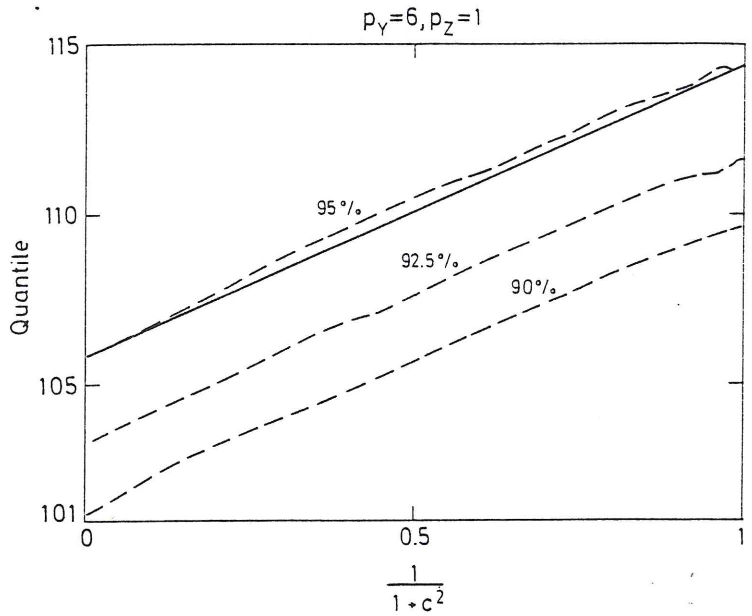
The absolute error between the distribution $\mathcal{L}_c(p_Y - r, p_Z)$ and the approximation (5.9) is between -0.07 and 0.03 for the mean of the distributions and between -0.2 and 1.2 for the 95 % quantiles which are small errors as compared to Table 6 and 7.

As measured in p-values the worst of the simulated cases is when $p_Y - r = 1$, $p_Z = 1$. Then the approximation may lead to the asymptotic 92 % quantile instead of the 95 % quantile, cf. Figure 1. For another case $p_Y - r = 6$, $p_Z = 1$ the approximation is actually very good, cf. Figure 1 as well.

Figure 1

The asymptotic distribution of $LR(\tilde{H}_1(r)|\tilde{H}_1(p_Y))$ depends on a nuisance parameter c given in Theorem 5.1. This dependency is illustrated below for two choices of dimensions.





The dotted lines (– –) show simulated quantiles, q_c as a function of $\frac{1}{1+c^2}$, wheras the solid line (–) shows the approximation of the 95% quantile given by (5.10).

To summarise, when testing for the cointegrating rank in $\tilde{H}_1(r)$ the appropriate quantile, $q_c(p_Y - r, p_Z)$ may be found by the approximation,

$$\frac{1}{1+c^2} q_0(p_Y - r, p_Z) + \frac{c^2}{1+c^2} q_\infty(p_Y - r, p_Z), \tag{5.10}$$

where q_0 and q_∞ are found in Table 6 and 7 respectively.

By following the suggestion to test for cointegration in model $H_2(r)$ the question whether the cointegrating relation has a linear trend is to be imposed after determination of rank. The feature of no linear trend in the stationary directions may also be imposed before testing for cointegration by analysing model $\tilde{H}_1(r)$. Thus a strategy for analysing $\tilde{H}_1(r)$ is suggested, where the question of whether $\mu_1^Z = 0$ is first settled. The hypothesis $\mu_1^Z = 0$ is tested in the marginal model of ΔZ given lagged differences of ΔY and ΔZ. Only the differences are needed since ΔZ is assumed weakly exogeneous for the coefficients to the levels. The asymptotic distribution of the test is $\chi^2(p_Z)$. Contingent on this outcome one proceeds as follows.

Case I: $\mu_1^Z = 0$.

In this case the test of $\tilde{H}_1(r, \mu_1^Z = 0)$ in $\tilde{H}_1(p_Y, \mu_1^Z = 0)$ has two different limit distributions. If $\alpha'_{Y\perp}\mu_1^{Y\cdot Z} \neq 0$ one finds (5.4) with $c = \infty$, and the relevant quantiles are found in Table 7. If, however, $\alpha'_{Y\perp}\mu_1^{Y\cdot Z} = 0$ then the relevant asymptotic distribution as noted has larger quantiles and therefore the asymptotic test is not similar. In order to control the asymptotic size of the test the quantiles are needed for the case $\alpha'_{Y\perp}\mu_1^{Y\cdot Z} = 0$, and it seems unreasonable that this particular value should dominate the inference. This particular case is studied by model $H_1(r)$. Therefore the idea of Pantula [9] used in Johansen [3] is suggested to be used here. To present the idea, suppose that $\tilde{H}_1(r-1, \mu_1^Z = 0)$ is rejected. The possible models are then nested like

$$\begin{array}{ccc} H_1(r) & \subset & H_1(p_Y) \\ \cap & & \| \\ \tilde{H}_1(r, \mu_1^Z = 0) & \subset & \tilde{H}_1(p_Y, \mu_1^Z = 0). \end{array} \tag{5.11}$$

The rank is determined by first testing model $H_1(r)$ in $H_1(p_Y)$. If this is rejected according to the quantiles in Table 4 then $\tilde{H}_1(r, \mu_1^Z = 0)$ is tested using Table 7. In full system analysis the testing strategy suggested here leads to an asymptotic consistent test for the cointegrating rank, cf. Johansen [3].

Case II: $\mu_1^Z \neq 0$.

The limit distribution of $LR(\tilde{H}_1(r)|\tilde{H}_1(p_Y))$ is now given by Theorem 5.1 with nuisance parameter, c. In order to apply the tables an estimate of c is needed and it is easily estimated from (5.5). Note, however, that the marginal analysis of ΔZ_t given lagged ΔZ_t and ΔY_t is needed in order to estimate both μ_1^Z and Ω_{ZZ}. Note also that it is necessary to re-estimate c for each value of r, since $\alpha_{Y\perp}$ depends on the value chosen for r. With the estimate of c an approximation for the relevant quantile can be calculated from (5.10). In order to avoid the estimate of c one can instead calculate

$$sup_c P_c\left\{LR(\tilde{H}_1(r)|\tilde{H}_1(p_Y)) > q\right\} = P_0\left\{LR(\tilde{H}_1(r)|\tilde{H}_1(p_Y)) > q\right\} \qquad (5.12)$$

and apply Table 6. Another possibility is to replace Z_t by a detrended process $Z_t^+ = Z_t - at$, so that $\Delta Z^+ = \Delta Z_t - a$ for a suitable estimate of a. This procedure for achieving similarity of the test has been suggested in another context by Kiviet and Phillips [8] and investigated for cointegration in full systems by Rahbek [10].

6 Test for Weak Exogeneity

The assumption of weak exogeneity is important for the asymptotic distribution of the test. Indeed when it fails inference for the partial model becomes very complicated, cf. Johansen [2].

In the unrestricted VAR model given by (2.1) a likelihood ratio test for weak exogeneity can be performed in two steps. First a cointegration analysis of the full system will determine the number of cointegrating relations. This is next combined with a test for linear restrictions on α given by $\alpha = (\alpha_Y', 0)'$. However, in this case the partial cointegration analysis is redundant, and the idea of partial analysis violated.

Another approach is simply after analysing model $H_i(r)$ to insert the estimator of the cointegrating parameter, e.g. $(\hat{\beta}', \hat{\beta}_2')'$, from the partial analysis in the marginal model for ΔZ_t. Regression analysis of the extended marginal model results in an asymptotic $\chi^2(p_Z)$-distributed test for the hypothesis that the loadings in the marginal equation, denoted α_Z, equal zero. This approach is demonstrated in the example which follows.

7 Example

In Hendry and Ericsson [1] the demand for money is investigated for the United Kingdom. The basic variables analysed are,

m: nominal money (M1)

y: income; constant price Total Final Expenditure (TFE) at 1985 prices

p: prices; implicit deflator of TFE

R^*: opportunity cost of holding money; that is the 3 month local authority interest rate less the learning adjusted retail sight-deposit interest rate.

The variables are quarterly reported in the period 1963.1 to 1989.2 and (m, y, p) are in log-levels. Hendry and Ericsson searched e.g. for homogeneity of the money and income variables,

$$m_t - p_t = \delta y_t + \gamma R_t^*. \tag{7.1}$$

Among others Johansen [6] reanalysed the data. The variables m, y, R^* were found to be I(1), whereas p is I(2). By introducing the variable $m_t - p_t$ one can eliminate the I(2) component and then work in an I(1) model using the inflation, Δp_t instead of p_t. Thus the full system considered consists of the variables $(m_t - p_t, \Delta p_t, y_t, R_t^*)$ which are analysed by the model

$$\Delta X_t = \Pi X_{t-1} + \sum_{i=1}^{4} \Gamma_i \Delta X_{t-i} + \mu + \varepsilon_t. \tag{7.2}$$

In order to illustrate the ideas of the partial analysis, consider first the full system analysis[1]. Cointegration analysis gives evidence for one cointegrating relation as reported by Johansen [6]. Next the hypothesis that the variables, $(\Delta p_t, y_t, R_t^*)$ are weakly exogeneous is tested. A likelihood ratio test of this hypothesis is performed by imposing restrictions on α. The test statistic has value 4.5, which should be compared with 7.81, the 95% quantile of the $\chi^2(3)$-distribution, and

[1]The statistical analysis has been performed using the program RATS, ver. 4.10 and the macro package CATS created by Hansen and Juselius, Univ. of Copenhagen.

the hypothesis is accepted. The full system maximum likelihood estimators of the parameters α and β are given by

$$\begin{aligned} \hat{\beta}' &= \begin{pmatrix} 1 & 7.49 & -1.10 & 7.28 \end{pmatrix} \\ \hat{\alpha}' &= \begin{pmatrix} -0.147 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{7.3}$$

Next for comparison the cointegrating rank is determined by an analysis of the partial system of $m_t - p_t$ conditional on $(\Delta p_t, y_t, R_t^*)$, where the latter variables are assumed to be weakly exogenous. This is done in model $H_2(r)$ where a linear trend is included in the cointegrating vector. From Table 1 one concludes that the cointegrating rank is one. The maximum likelihood estimator of the cointegrating

Table 1:

$\hat{\lambda}_i$	*trace*	r	p_Z	$p_Y - r$	Table	95%	*Accept*
0.36	44.9	0	3	1	4	20.7	*no*

parameter, β and the loading, α_Y is then

$$\begin{aligned} \hat{\beta}' &= \begin{pmatrix} 1 & 8.22 & -1.52 & 6.82 & 0.003 \end{pmatrix} \\ \hat{\alpha_Y}' &= \begin{pmatrix} -0.152 \end{pmatrix}. \end{aligned} \tag{7.4}$$

The likelihood ratio test for a zero coefficient of the linear trend, cf. (3.5) has value 0.60 as compared to 3.84, the critical value of the asymptotic $\chi^2(1)$ distribution. The maximum likelihood estimator of the cointegrating parameter, β is the same as in (7.3), and the estimate of the loading α_Y in the partial model equals the first entry of $\hat{\alpha}$.

Following the ideas presented in section 6, the assumption of weak exogeneity of $(\Delta p_t, y_t, R_t^*)$ is tested by regression of these variables on $\hat{\beta}' X_{t-1}$, the lagged differences of all variables and the constant. The least squares estimate of α_Z is

$$\begin{array}{cccc} \hat{\alpha}'_Z = & (\; 0.022 & -0.00043 & 0.033 \;) \\ [std] & [0.013] & [0.023] & [0.023] \end{array} \tag{7.5}$$

An asymptotic $\chi^2(3)$-test for $\alpha_Z = 0$ has value 3.62 as compared to 7.81. Hence, the results of the partial analysis confirm the results of the full system analysis.

This analysis could also be performed directly in model $\tilde{H}_1(r)$ with no linear trend in the cointegrating relation. The first step is then to test whether $\mu_Z = 0$, cf. section 5. The least squares estimate of μ_Z is

$$\underset{[std]}{\hat{\mu}'_Z} = \left(\underset{[0.0013]}{-0.0022} \quad \underset{[0.0022]}{0.010} \quad \underset{[0.0022]}{-0.0041} \right) \tag{7.6}$$

An asymptotic $\chi^2(3)$-test for $\mu_Z = 0$ has value 25.3 as compared to 7.81, and $\mu_Z = 0$ is therefore rejected. Next under $\tilde{H}_1(0)$ the nuisance parameter is $\hat{c} = 0.016$ and the quantile given in Table 2 below is calculated using the approximation in (5.10). From Table 2 one can again conclude that the cointegrating rank is 1. The maximum likelihood estimators of the cointegrating parameter and of the loadings are as before.

Table 2:

$\hat{\lambda}_i$	*trace*	r	p_Z	$p_Y - r$	c	Table	95%	*Accept*
0.36	44.3	0	3	1	0	6	18.0	*no*
					0.02	*Approx.*	17.9	*no*
					∞	7	9.49	*no*

Table 3:

Asymptotic quantiles of $LR(H_2(r)|H_2(p_Y))$.
p_Z: is the number of exogenous variables. p_Y: is the number of variables in the partial system.
r: is the number of cointegrating relations.

Dimension		*Quantiles*					*Moments*	
p_Z	$p_Y - r$	50%	80%	90%	95%	97.5%	*Mean*	*Variance*
1	1	7.50	11.0	13.2	15.2	17.4	8.15	14.6
	2	19.4	24.7	27.8	30.5	33.3	20.1	33.7
	3	35.3	41.9	45.9	49.6	52.4	35.9	57.6
	4	54.8	63.0	67.9	71.7	75.2	55.5	87.6
	5	78.4	88.1	93.5	98.0	102	78.9	123
	6	105	116	123	127	132	106	161
2	1	9.40	13.3	15.7	17.9	20.2	10.0	18.4
	2	23.2	29.1	32.5	35.5	38.5	23.9	42.0
	3	41.1	48.4	52.6	56.3	59.3	41.6	69.6
	4	62.6	71.4	76.2	80.9	84.5	63.1	104
	5	87.7	98.1	104	108	113	88.2	142
3	1	11.3	15.6	18.2	20.7	23.1	12.0	22.5
	2	27.2	33.3	36.9	39.9	43.1	27.7	48.7
	3	46.6	54.6	59.0	62.9	66.6	47.2	80.4
	4	69.9	79.6	85.1	89.2	93.2	70.6	118
4	1	13.2	17.7	20.5	22.9	25.4	13.8	24.8
	2	30.7	37.4	41.3	44.5	47.5	31.4	54.6
	3	52.5	60.9	65.6	69.7	73.1	53.0	92.2
5	1	15.1	19.9	22.8	25.3	27.7	15.7	28.5
	2	34.7	41.8	45.8	49.3	52.8	35.3	63.2
6	1	17.0	22.0	25.0	27.7	30.3	17.6	31.6

Table 4:

Asymptotic quantiles of $LR(H_1(r)|H_1(p_Y))$.

p_Z: is the number of exogenous variables. p_Y: is the number of variables in the partial system.

r: is the number of cointegrating relations.

Dimension		*Quantiles*					*Moments*	
p_Z	$p_Y - r$	50%	80%	90%	95%	97.5%	*Mean*	*Variance*
1	1	5.32	8.32	10.4	12.3	14.0	5.97	10.7
	2	15.2	20.0	22.9	25.5	27.6	15.9	27.0
	3	29.0	36.6	38.5	41.7	44.9	29.6	47.5
	4	46.6	54.2	58.5	62.4	65.7	47.2	75.1
	5	68.0	77.2	82.2	86.7	90.6	68.6	107
	6	93.3	104	110	115	119	93.9	147
2	1	7.29	10.8	13.0	15.1	17.0	7.92	14.6
	2	19.1	24.2	27.3	30.0	32.8	19.7	34.3
	3	34.9	41.7	45.6	49.0	52.1	35.4	59.6
	4	54.4	62.7	67.5	71.5	74.8	54.9	90.0
	5	77.8	87.5	93.0	97.9	102	78.3	127
3	1	9.23	13.1	15.6	17.8	19.9	9.87	18.1
	2	23.1	28.8	32.2	35.4	38.0	23.6	42.0
	3	40.5	48.0	52.3	55.9	58.9	41.1	70.4
	4	61.9	71.0	76.2	80.5	84.3	62.6	105
4	1	11.2	15.4	18.2	20.6	22.5	11.8	21.9
	2	26.9	33.2	36.6	40.0	42.8	27.4	49.1
	3	46.3	54.4	58.9	62.9	66.2	46.9	82.5
5	1	13.1	17.7	20.4	23.0	25.3	13.7	25.7
	2	30.7	37.5	41.2	44.6	47.8	31.3	56.6
6	1	15.0	19.9	22.8	25.4	28.2	15.6	29.2

Table 5:

Asymptotic quantiles of $LR(H_0(r)|H_0(p_Y))$.
p_Z: is the number of exogeneous variables. p_Y: is the number of endogenous variables.
r: is the number of cointegrating relations.

Dimension		*Quantiles*				*Moments*	
p_Z	$p_Y - r$	50%	90%	95%	97.5%	*Mean*	*Variance*
1	1	2.39	6.38	7.91	9.64	3.02	6.53
	2	9.31	15.9	18.1	20.4	9.96	19.3
	3	20.1	28.9	31.9	34.5	20.7	37.5
	4	34.8	45.9	49.4	52.6	35.4	61.9
2	1	4.32	9.30	11.2	13.1	4.97	10.8
	2	13.2	20.8	23.5	25.9	13.8	27.4
	3	25.9	35.9	39.3	42.1	26.6	49.9
3	1	6.38	12.1	14.2	16.1	6.95	14.8
	2	17.1	25.7	28.4	31.1	17.7	35.5
4	1	8.22	14.6	16.7	18.9	8.83	18.7

Table 6:

Asymptotic quantiles of $LR(\tilde{H}_1(r)|\tilde{H}_1(p_Y))$.
Nuisance parameter: $c = 0$, or equivalently, no drift in the partial system.
p_Z: is the number of exogenous variables. p_Y: is the number of variables in the partial system.
r: is the number of cointegrating relations.

Dimension		*Quantiles*					*Moments*	
p_Z	$p_Y - r$	50%	80%	90%	95%	97.5%	*Mean*	*Variance*
1	1	5.61	8.49	10.4	12.2	14.1	6.21	10.2
	2	15.6	20.2	23.1	25.5	28.1	16.3	26.0
	3	29.6	35.7	39.2	42.2	45.1	30.2	46.1
	4	47.3	54.8	59.1	62.9	66.2	47.9	72.2
	5	68.9	77.7	82.6	87.0	90.5	69.4	103
	6	93.8	104	110	114	119	94.4	137
2	1	7.52	11.0	13.2	15.3	17.3	8.15	14.5
	2	19.5	24.8	28.0	30.7	33.0	20.1	34.0
	3	35.3	42.0	46.0	49.5	52.4	35.9	58.3
	4	54.9	63.0	67.5	71.7	75.5	55.4	86.9
	5	78.0	87.9	93.4	97.8	102	78.7	123
3	1	9.42	13.4	15.7	18.0	20.2	10.1	18.6
	2	23.3	29.1	32.3	35.3	38.2	23.9	41.2
	3	40.9	48.3	52.4	56.2	59.1	41.5	68.2
	4	62.4	71.5	76.4	80.6	84.1	63.0	102
4	1	11.3	15.6	18.1	20.4	22.5	11.9	21.2
	2	27.0	33.2	36.8	39.6	42.4	27.6	47.5
	3	46.7	54.7	59.1	63.0	66.2	47.3	80.5
	4	70.1	79.4	84.7	89.1	93.4	70.6	119
5	1	13.2	17.8	20.4	22.9	25.3	13.8	25.3
	2	30.8	37.6	41.3	44.5	47.8	31.5	55.0
6	1	15.2	19.8	22.7	25.2	27.5	15.7	27.8

Table 7:

Asymptotic quantiles of $LR(\tilde{H}_1(r)|\tilde{H}_1(p_Y))$.
Nuisance parameter: $c = \infty$, or equivalently drift in the partial system.
p_Z: is the number of exogenous variables. p_Y: is the number of variables in the partial system.
r: is the number of cointegrating relations.

Dimension		*Quantiles*					*Moments*	
p_Z	$p_Y - r$	50%	80%	90%	95%	97.5%	*Mean*	*Variance*
1	1	1.39	3.22	4.61	5.99	7.38	χ^2_2	
	2	10.5	14.5	17.1	19.4	21.6	11.1	20.2
	3	23.5	29.1	32.4	35.3	38.2	24.1	40.1
	4	40.1	47.4	51.6	55.4	58.5	40.8	67.2
	5	60.5	69.3	74.3	78.5	82.4	61.3	97.2
	6	85.1	95.3	101	106	111	85.7	138
2	1	2.37	4.64	6.25	7.81	9.35	χ^2_3	
	2	13.5	18.0	20.9	23.4	25.8	14.0	26.0
	3	28.1	34.6	38.2	41.2	44.5	28.8	50.4
	4	46.8	54.8	59.1	63.0	66.7	47.4	79.0
	5	69.4	78.8	84.3	89.1	93.3	70.0	118
3	1	3.36	5.99	7.78	9.49	11.1	χ^2_4	
	2	16.2	21.3	24.4	26.8	29.6	16.9	31.4
	3	32.9	39.7	43.7	47.0	50.1	33.6	59.3
	4	53.5	62.2	67.0	71.2	74.9	54.2	93.8
4	1	4.35	7.29	9.24	11.1	12.8	χ^2_5	
	2	19.1	24.5	27.7	30.6	33.6	19.8	37.4
	3	37.7	45.2	49.6	53.6	56.6	38.4	70.5
	4	60.3	69.3	74.4	78.7	82.3	60.9	106
5	1	5.35	8.56	10.6	12.6	14.4	χ^2_6	
	2	22.0	27.9	31.3	34.2	36.8	22.7	42.3
6	1	6.35	9.80	12.0	14.1	16.0	χ^2_7	

Appendix

A *Proof of Theorem 4.1*

The proof of the results for the models $H_0(r)$, $H_1(r)$, and $H_2(r)$ are similar and follows the outline of the proof of Theorem 2.2 in Johansen [4]. Thus only the latter case is studied and without loss of generality the number of lags, k is set to 2. The process is then given by the equation,

$$\Delta X_t = \begin{pmatrix} \alpha_Y\,(\beta' X_{t-1} + \beta_2' t) \\ 0 \end{pmatrix} + \begin{pmatrix} \Gamma^Y \\ \Gamma^Z \end{pmatrix} \Delta X_{t-1} + \begin{pmatrix} \mu_1^Y \\ \mu_1^Z \end{pmatrix} + \begin{pmatrix} \varepsilon_t^Y \\ \varepsilon_t^Z \end{pmatrix}, \tag{A.1}$$

or equivalently by the equations,

$$\Delta Y_t = \alpha_Y \begin{pmatrix} \beta \\ \beta_2 \end{pmatrix}' \begin{pmatrix} X_{t-1} \\ t \end{pmatrix} + \Gamma^{Y\cdot Z} \Delta X_{t-1} + \mu_1^{Y\cdot Z} + \varepsilon_t^{Y\cdot Z}, \tag{A.2}$$

$$\Delta Z_t = \Gamma^Z \Delta X_{t-1} + \mu_1^Z + \varepsilon_t^Z. \tag{A.3}$$

As mentioned in section 2, then under the assumptions of Granger's Representation Theorem, in particular the rank of α_Y and β equals r, and the additional assumption of full rank of

$$\alpha_\perp' \Psi \beta_\perp = \alpha_\perp' (I - \begin{pmatrix} \Gamma^Y \\ \Gamma^Z \end{pmatrix}) \beta_\perp ,$$

Granger's Representation Theorem implies that the process $X_t' = (Y_t', Z_t')$ has the representation,

$$X_t = C \sum_{i=1}^{t} \varepsilon_i + \tau_1 t + \tau_0 + V_t . \tag{A.4}$$

Here $C = \beta_\perp (\alpha_\perp' \Psi \beta_\perp)^{-1} \alpha_\perp'$ and V_t is some stationary process. The linear trend parameter τ_1 satifies $\beta' \tau_1 = -\beta_2'$, so that $\beta' X_t$ is trend stationary with a trend given by β_2. The process X_t is p-dimensional and for the study of the asymptotic behaviour the representation leads to a choice of coordinate system in $\mathbf{R}^p$ given by

$$(\beta, \Psi' \alpha_\perp). \tag{A.5}$$

Clearly β and $\Psi' \alpha_\perp$ are not orthogonal but together they span $\mathbf{R}^p$, which is seen by premultiplying in (A.5) by the orthogonal basis $(\beta, \beta_\perp)'$ and using the full rank assumption of $\alpha_\perp' \Psi \beta_\perp$. In order to study the asymptotic behaviour of the $(p+1)$-dimensional process $X_t^* \equiv (X_t', t)'$ the coordinate system in (A.5) is extended to span $\mathbf{R}^{p+1}$,

$$(\beta^*, B_T^*) \equiv \left(\begin{pmatrix} \beta \\ \beta_2 \end{pmatrix}, \begin{pmatrix} \Psi'\alpha_\perp & 0 \\ -\tau_1'\Psi'\alpha_\perp & \frac{1}{\sqrt{T}} \end{pmatrix} \right). \tag{A.6}$$

The representation in (A.4) shows that the β^* direction of the process X_t^* is trend stationary, whereas in the remaining direction given by B_T^*, the process is nonstationary. Thus the asymptotic behaviour differs in the two directions. Next from the trend stationarity of $\beta' X_t$ and the stationarity of ΔX_t, define the variance conditional on the past by,

$$\operatorname{Var}\left(\begin{array}{c|c} \beta' X_{t-1} & \\ \Delta Y_t & \Delta X_{t-1} \\ \Delta Z_t & \end{array} \right) = \begin{pmatrix} \Sigma_{\beta\beta} & \Sigma_{\beta Y} & 0 \\ \Sigma_{Y\beta} & \Sigma_{YY} & \Sigma_{YZ} \\ 0 & \Sigma_{ZY} & \Sigma_{ZZ} \end{pmatrix} = \begin{pmatrix} \Sigma_{\beta\beta} & \Sigma_{\beta X} \\ \Sigma_{X\beta} & \Sigma_{XX} \end{pmatrix}.$$

That $\operatorname{Cov}(\beta' X_{t-1}, \Delta Z_t | \Delta X_{t-1}) = 0$ follows by ΔZ_t being weakly exogenous and the fact that ε_t^Z and $\beta' X_{t-1}$ are independent. In an obvious notation the variance of $\beta' X_{t-1}$ and ΔY_t conditional on the past and ΔZ_t is defined by,

$$\operatorname{Var}\left(\begin{array}{c|c} \beta' X_{t-1} & \Delta Z_t, \Delta X_{t-1} \\ \Delta Y_t & \end{array} \right) = \begin{pmatrix} \Sigma_{\beta\beta} & \Sigma_{\beta Y \cdot Z} \\ \Sigma_{Y\beta\cdot Z} & \Sigma_{YY\cdot Z} \end{pmatrix}.$$

Now since $\operatorname{Var}(\Delta X_t | \beta' X_{t-1}, \Delta X_{t-1}) = \operatorname{Var}(\varepsilon_t) = \Omega$, the following identity holds,

$$\Omega = \begin{pmatrix} \Sigma_{YY} - \Sigma_{Y\beta}\Sigma_{\beta\beta}^{-1}\Sigma_{\beta Y} & \Sigma_{YZ} \\ \Sigma_{ZY} & \Sigma_{ZZ} \end{pmatrix}.$$

Finally the following identity is useful.

Lemma A.1 *Under the assumption that* $\alpha_Z = 0$,

$$\alpha_{Y\perp}(\alpha_{Y\perp}'\Omega_{YY\cdot Z}\alpha_{Y\perp})^{-1}\alpha_{Y\perp}' = \Sigma_{YY\cdot Z}^{-1} - \Sigma_{YY\cdot Z}^{-1}\Sigma_{Y\beta\cdot Z}[\Sigma_{\beta Y\cdot Z}\Sigma_{YY\cdot Z}^{-1}\Sigma_{Y\beta\cdot Z}]^{-1}\Sigma_{\beta Y\cdot Z}\Sigma_{YY\cdot Z}^{-1} \tag{A.7}$$

Proof: By Johansen [4],

$$\alpha_\perp(\alpha_\perp'\Omega\alpha_\perp)^{-1}\alpha_\perp' = \Sigma_{XX}^{-1} - \Sigma_{XX}^{-1}\Sigma_{X\beta}[\Sigma_{\beta X}\Sigma_{XX}^{-1}\Sigma_{X\beta}]^{-1}\Sigma_{\beta X}\Sigma_{XX}^{-1}, \tag{A.8}$$

and the desired identity is the upper left entry of this identity. □

As for the asymptotics in the nonstationary directions B_T^* it follows by (A.4), that

$$B_T^{*\prime} X_t^* = \begin{pmatrix} \alpha_\perp' \Psi(C \sum_1^t \varepsilon_i + \tau_0 + V_t) \\ \frac{1}{\sqrt{T}} t \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_{Y\perp}' \sum_1^t \varepsilon_i^Y + \alpha_{Y\perp}' \tau_0^Y + \alpha_{Y\perp}' \tilde{V}_t^Y \\ \sum_1^t \varepsilon_i^Z + \tau_0^Z + \alpha_{Y\perp}' \tilde{V}_t^Z \\ \frac{1}{\sqrt{T}} t \end{pmatrix}. \tag{A.9}$$

Then, mimicking the statistical analysis, conditioning of the first equation on the second is obtained by a transformation given by a matrix M_1. Furthermore the random walks involved are normalized by multiplying with a matrix M_2,

$$M_2 M_1 B_T^{*\prime} X_t^* = \begin{pmatrix} (\alpha_{Y\perp}' \Omega_{YY \cdot Z} \alpha_{Y\perp})^{-\frac{1}{2}} & 0 & 0 \\ 0 & \Omega_{ZZ}^{-\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_{p_Y - r} & -\alpha_{Y\perp}' \omega & 0 \\ 0 & I_{p_Z} & 0 \\ 0 & 0 & 1 \end{pmatrix} B_T^{*\prime} X_t^*$$

$$= \begin{pmatrix} (\alpha_{Y\perp}' \Omega_{YY \cdot Z} \alpha_{Y\perp})^{-\frac{1}{2}} \sum_1^t \varepsilon_i^{Y \cdot Z} \\ \Omega_{ZZ}^{-\frac{1}{2}} \sum_1^t \varepsilon_i^Z \\ \frac{1}{\sqrt{T}} t \end{pmatrix} + O_P(1). \tag{A.10}$$

Note that the 'conditional' as well as the 'marginal' process consists of a random walk and that the linear trend has been isolated by the choice of B_T^*.

Lemma A.2 *With $C_T^{*\prime} = \frac{1}{\sqrt{T}} M_2 M_1 B_T^{*\prime}$ and $u \in [0,1]$, then as $T \to \infty$,*

$$C_T^{*\prime} X_{[Tu]}^* \xrightarrow{w} \begin{pmatrix} W^{Y \cdot Z} \\ W^Z \\ u \end{pmatrix} \equiv F(u), \tag{A.11}$$

where $W^{Y \cdot Z}$ and W^Z are independent standard Brownian motions of dimension p_Y and p_Z respectively.

It is now possible to formulate a Lemma which is similar to Lemma A.4 in Johansen [4].

Lemma A.3 *With C_T^* given in* Lemma A.2,

$$C_T^{*\prime} S_{XX \cdot Z} C_T^* \xrightarrow{w} \int_0^1 (F - \bar{F})(F - \bar{F})' \, du \tag{A.12}$$

$$\sqrt{T} C_T^{*\prime} (S_{XY \cdot Z} - S_{XX \cdot Z} \beta \alpha_Y') \xrightarrow{w} \int_0^1 (F - \bar{F}) \, dB_{\Omega_{YY \cdot Z}}', \tag{A.13}$$

with $B_{\Omega_{YY\cdot Z}}$ a Brownian motion given by the weak limit of $T^{-1/2}\sum_1^{[T\cdot]}\varepsilon_i^{Y\cdot Z}$, which has covariance $\Omega_{YY\cdot Z}$.

Furthermore $\beta^{*\prime}S_{XX\cdot Z}\beta^{*} \xrightarrow{P} \Sigma_{\beta\beta}$, $\beta^{*\prime}S_{XY\cdot Z} \xrightarrow{P} \Sigma_{\beta Y\cdot Z}$ and $S_{YY\cdot Z} \xrightarrow{P} \Sigma_{YY\cdot Z}$.

Now, define,

$$S(\lambda) = \lambda S_{XX\cdot Z} - S_{XY\cdot Z}S_{YY\cdot Z}^{-1}S_{YX\cdot Z}. \tag{A.14}$$

It then follows that with $A_T^{*} = (\beta^{*}, C_T^{*})$,

$$|A_T^{*\prime}S(\lambda)A_T^{*}| \xrightarrow{w} |\lambda\Sigma_{\beta\beta\cdot Z} - \Sigma_{\beta Y\cdot Z}\Sigma_{YY\cdot Z}^{-1}\Sigma_{Y\beta\cdot Z}||\lambda\int_0^1 (F-\bar{F})(F-\bar{F})'\,du|, \tag{A.15}$$

which has (p+1-r) zero roots and r positive roots.

With the (p+1-r) smallest eigenvalues of $S(\lambda)$ approaching zero at the rate of T, define $\rho = \lambda T$,

$$|(\beta^{*}, \sqrt{T}C_T^{*})'S(\frac{\rho}{T})(\beta^{*}, \sqrt{T}C_T^{*})| =$$
$$|\beta^{*\prime}S(\frac{\rho}{T})\beta^{*}||TC_T^{*\prime}\{S(\frac{\rho}{T}) - S(\frac{\rho}{T})\beta^{*}(\beta^{*\prime}S(\frac{\rho}{T})\beta^{*})^{-1}\beta^{*\prime}S(\frac{\rho}{T})\}C_T^{*}|. \tag{A.16}$$

When T increases the first term on the right hand side tends to

$$-\Sigma_{\beta Y\cdot Z}\Sigma_{YY\cdot Z}^{-1}\Sigma_{\beta Y\cdot Z}$$

by Lemma A.3, showing that in the limit the first term has no roots. It follows by Lemma A.3 too that the second term of (A.16) is,

$$\rho C_T^{*\prime}S_{XX\cdot Z}C_T^{*} - \sqrt{T}C_T^{*\prime}S_{XY\cdot Z}N\sqrt{T}S_{YX\cdot Z}C_T^{*} + o_P(1), \tag{A.17}$$

where

$$N = \Sigma_{YY\cdot Z}^{-1} - \Sigma_{YY\cdot Z}^{-1}\Sigma_{Y\beta\cdot Z}(\Sigma_{\beta Y\cdot Z}\Sigma_{YY\cdot Z}^{-1}\Sigma_{Y\beta\cdot Z})^{-1}\Sigma_{\beta Y\cdot Z}\Sigma_{YY\cdot Z}^{-1}, \tag{A.18}$$

which by Lemma A.1 equals $\alpha_{Y\perp}(\alpha_{Y\perp}'\Omega_{YY\cdot Z}\alpha_{Y\perp})^{-1}\alpha_{Y\perp}'$. Hence Lemma A.3 gives the limit of $\sqrt{T}C_T^{*\prime}S_{XY\cdot Z}\alpha_{Y\perp}$. And the $(p+1-r)$ smallest eigenvalues normalized by T converge to those of the equation

$$0 = |\rho\int_0^1 (F-\bar{F})(F-\bar{F})'\,du -$$
$$\int_0^1 (F-\bar{F})\,dB_{\Omega_{YY\cdot Z}}'\alpha_{Y\perp}(\alpha_{Y\perp}'\Omega_{YY\cdot Z}\alpha_{Y\perp})^{-1}\alpha_{Y\perp}'\int_0^1 dB_{\Omega_{YY\cdot Z}}(F-\bar{F})'|,$$

with F given in Lemma A.2. By definition

$$(\alpha'_{Y\perp}\Omega_{YY\cdot Z}\alpha_{Y\perp})^{-\frac{1}{2}}\alpha'_{Y\perp}B_{\Omega_{YY\cdot Z}} = W^{Y\cdot Z} \tag{A.19}$$

and the result in Theorem 4.1 follows by the expansion

$$\begin{aligned} LR(H_2(r)|H_2(p_Y)) &= T\sum_{i=r+1}^{p_Y}\hat{\lambda}_i + o_P(1) \xrightarrow{w} \sum_{i=r+1}^{p_Y}\hat{\rho}_i \\ &= \operatorname{tr}\{\int_0^1 dW^{Y\cdot Z}F'(\int_0^1 FF')^{-1}\int_0^1 FdW^{Y\cdot Z\prime}\}. \end{aligned} \tag{A.20}$$

□

B Proof of Theorem 5.1

The proof mimicks that of Theorem 4.1 and here it will only be shown how Lemma A.2 is changed. When the number of lags k is set to two then model $\tilde{H}_1(r)$ for $X'_t = (Y'_t, Z'_t)$ is given by the equations,

$$\Delta Y_t = \alpha_Y\beta' X_{t-1} + \Gamma^{Y\cdot Z}\Delta X_{t-1} + \mu_1^{Y\cdot Z} + \varepsilon_t^{Y\cdot Z}, \tag{B.1}$$

$$\Delta Z_t = \Gamma^Z \Delta X_{t-1} + \mu_1^Z + \varepsilon_t^Z. \tag{B.2}$$

and under the assumptions af Grangers Representation Theorem the process behaves like

$$X_t = C\sum_{i=1}^{t}\varepsilon_i + C\mu_1 t + V_t\,. \tag{B.3}$$

For the study of the asymptotic behaviour of X_t the coordinate system, $(\beta, \Psi'\alpha_\perp)$ will be used, cf. (A.5). The representation (B.3) shows that $\beta' X_t$ is stationary, whereas $\alpha'_\perp \Psi X_t$ is nonstationary and given by

$$\alpha'_\perp \Psi X_t = \begin{pmatrix} \alpha'_{Y\perp}\sum_1^t \varepsilon_i^Y + \alpha'_{Y\perp}\mu_1^Y t + \alpha'_{Y\perp}\tilde{V}_t^Y \\ \sum_1^t \varepsilon_i^Z + \mu_1^Z t + \tilde{V}_t^Z \end{pmatrix}. \tag{B.4}$$

Conditioning of the first equation on the second and normalizing the random walks involved is performed by multiplying with a full rank matrix, M, cf. (A.10)

$$M\alpha'_\perp\Psi X_t = \begin{pmatrix} (\alpha'_{Y\perp}\Omega_{YY\cdot Z}\alpha_{Y\perp})^{-\frac{1}{2}}\alpha'_{Y\perp}\sum_1^t \varepsilon_i^{Y\cdot Z} + \tau^{Y\cdot Z}t \\ \Omega_{ZZ}^{-\frac{1}{2}}\sum_1^t \varepsilon_i^Z + \tau^Z t \end{pmatrix} + O_P(1), \tag{B.5}$$

where

$$
\begin{aligned}
\tau^{Y\cdot Z} &= (\alpha'_{Y\perp}\Omega_{YY\cdot Z}\alpha_{Y\perp})^{-\frac{1}{2}}\alpha'_{Y\perp}\mu_{Y\cdot Z}, && \text{(B.6)}\\
\tau^{Z} &= \Omega_{ZZ}^{-\frac{1}{2}}\mu_{Z}. && \text{(B.7)}
\end{aligned}
$$

The processes, the 'conditional' as well as the 'marginal', consist of a random walk and a linear trend, and thus the asymptotic behaviour is dominated by a linear trend. In order to transform the processes such that only one direction is present with a trend and the remaining p_Y+p_Z-r-1 are without, three different cases need to be discussed. These are,

- $\tau^{Y\cdot Z} \neq 0$ and $\tau^Z \neq 0$, corresponding to the nuisance parameter $(0 < c < \infty)$.
- $\tau^{Y\cdot Z} = 0$ and $\tau^Z \neq 0$, corresponding to the nuisance parameter $c = 0$.
- $\tau^{Y\cdot Z} \neq 0$ and $\tau^Z = 0$, corresponding to the nuisance parameter $c = \infty$.

The remaining combination, that $\tau^{Y\cdot Z} = 0$ and $\tau^Z = 0$ is not treated here, cf. the remarks in Section 5 after the Theorem 5.1.

Assume first that $\tau^{Y\cdot Z} \neq 0$ and $\tau^Z \neq 0$. Define $\gamma^{Y\cdot Z}$ as a (p_Y-r)$\times$(p_Y-r-1) matrix of length 1, orthogonal to $\tau^{Y\cdot Z}$ and such that $(\tau^{Y\cdot Z}, \gamma^{Y\cdot Z})$ spans $\mathbf{R}^{p_Y - r}$. Similarly define γ^Z as a $p_Z \times$ (p_Z-1) matrix orthogonal to τ^Z. Further define $\bar{\tau}^{Y\cdot Z} = \tau^{Y\cdot Z}(\tau^{Y\cdot Z\prime}\tau^{Y\cdot Z})^{-1}$.

The next transformation may be viewed as isolating the trend in both processes by a matrix M_a given by

$$
M_a M\alpha'_{\perp}\Psi X_t = \begin{pmatrix} \bar{\tau}^{Y\cdot Z\prime} & 0 \\ \gamma^{Y\cdot Z\prime} & 0 \\ 0 & \gamma^{Z\prime} \\ 0 & \bar{\tau}^{Z\prime} \end{pmatrix} M\alpha'_{\perp}\Psi X_t
$$

$$
= \begin{pmatrix} \bar{\tau}^{Y\cdot Z\prime}(\alpha'_{Y\perp}\Omega_{YY\cdot Z}\alpha_{Y\perp})^{-\frac{1}{2}}\alpha'_{Y\perp}\sum_1^t \varepsilon_i^{Y\cdot Z} + t \\ (\alpha'_{Y\perp}\Omega_{YY\cdot Z}\alpha_{Y\perp})^{-\frac{1}{2}}\alpha'_{Y\perp}\sum_1^t \varepsilon_i^{Y\cdot Z} \\ \Omega_{ZZ}^{-\frac{1}{2}}\sum_1^t \varepsilon_i^Z \\ \bar{\tau}^{Z\prime}\Omega_{ZZ}^{-\frac{1}{2}}\sum_1^t \varepsilon_i^Z + t \end{pmatrix} + O_P(1).
$$

Subtraction of the two trending processes and normalization of the processes by their convergence rates is performed by a matrix M_T.

$$M_T M_a M \alpha'_\perp \Psi X_t = T^{-\frac{1}{2}} \begin{pmatrix} T^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & I_{p_Y-r-1} & 0 & 0 \\ 0 & 0 & I_{p_Z-1} & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} M_a M \alpha'_\perp \Psi X_t$$

$$= \begin{pmatrix} T^{-1}t \\ \gamma^{Y\cdot Z\prime} T^{-\frac{1}{2}} (\alpha'_{Y\perp} \Omega_{YY\cdot Z} \alpha_{Y\perp})^{-\frac{1}{2}} \alpha'_{Y\perp} \sum_1^t \varepsilon_i^{Y\cdot Z} \\ \gamma^{Z\prime} T^{-\frac{1}{2}} \Omega_{ZZ}^{-\frac{1}{2}} \sum_1^t \varepsilon_i^Z \\ T^{-\frac{1}{2}} (\bar{\tau}^{Y\cdot Z\prime} (\alpha'_{Y\perp} \Omega_{YY\cdot Z} \alpha_{Y\perp})^{-\frac{1}{2}} \alpha'_{Y\perp} \sum_1^t \varepsilon_i^{Y\cdot Z} - \bar{\tau}'_Z \Omega_{ZZ}^{-\frac{1}{2}} \sum_1^t \varepsilon_i^Z) \end{pmatrix} + o_P(T^{-\frac{1}{2}}).$$

This shows that in only one direction a linear trend is left. Corresponding to the $(p_Y + p_Z - r)$-dimensional limit process F in Theorem 5.1, the asymptotic behaviour of the transformed process is given by,

Lemma B.1 *For the case where $\tau^{Y\cdot Z} \neq 0$ and $\tau^Z \neq 0$, with $B'_T = M_T M_a M \alpha'_\perp \Psi$ and $u \in [0,1]$, then as $T \to \infty$,*

$$B'_T X_{[Tu]} \xrightarrow{w} \begin{pmatrix} u \\ W_2 \\ \vdots \\ W_{p_Y+p_Z-r-1} \\ (\tau^{Y\cdot Z\prime} \tau^{Y\cdot Z})^{-\frac{1}{2}} W_1 - (\tau^{Z\prime} \tau^Z)^{-\frac{1}{2}} W_{p_Y+p_Z-r} \end{pmatrix} \tag{B.8}$$

where $W' = (W_1, \ldots, W_{p_Y+p_Z-r})$ is a standard Brownian motion.

Finally $\beta' S_{XX\cdot Z} \beta \xrightarrow{P} \Sigma_{\beta\beta}$, $\beta' S_{XY\cdot Z} \xrightarrow{P} \Sigma_{\beta Y\cdot Z}$, and $S_{YY\cdot Z} \xrightarrow{P} \Sigma_{YY\cdot Z}$.

Next in the case when $\tau^{Y\cdot Z} \neq 0$ but $\tau^Z = 0$, only the direction $\gamma^{Y\cdot Z}$ is needed in order to isolate the trend. Thus replace $M_T M_a$ with

$$M_T^{Y\cdot Z} = \begin{pmatrix} T^{-1} \bar{\tau}^{Y\cdot Z\prime} & 0 \\ T^{-\frac{1}{2}} \gamma^{Y\cdot Z\prime} & 0 \\ 0 & T^{-\frac{1}{2}} I_{p_Z} \end{pmatrix}, \tag{B.9}$$

which immediately gives,

Lemma B.2 *For the case where* $\tau^{Y\cdot Z} \neq 0$ *and* $\tau^{Z} = 0$, *with* $B_T^{Y\cdot Z\prime} = M_T^{Y\cdot Z} M \alpha_\perp' \Psi$ *and* $u \in [0,1]$, *then as* $T \to \infty$,

$$B_T^{Y\cdot Z\prime} X_{[Tu]} \xrightarrow{w} \begin{pmatrix} u \\ W_2 \\ \vdots \\ W_{p_Y+p_Z-r} \end{pmatrix} \tag{B.10}$$

where $W' = (W_1, \ldots, W_{p_Y+p_Z-r})$ *is a standard Brownian motion.*

Finally $\beta' S_{XX\cdot Z}\beta \xrightarrow{P} \Sigma_{\beta\beta}$, $\beta' S_{XY\cdot Z} \xrightarrow{P} \Sigma_{\beta Y\cdot Z}$, *and* $S_{YY\cdot Z} \xrightarrow{P} \Sigma_{YY\cdot Z}$.

A similar result is obtained for the final case, using a transformation M_T^Z corresponding to $M_T^{Y\cdot Z}$.

Lemma B.3 *For the case where* $\tau^{Y\cdot Z} = 0$ *and* $\tau^{Z} \neq 0$, *with* $B_T^{Z\prime} = M_T^{Z} M \alpha_\perp' \Psi$ *and* $u \in [0,1]$, *then as* $T \to \infty$,

$$B_T^{Z\prime} X_{[Tu]} \xrightarrow{w} \begin{pmatrix} u \\ W_2 \\ \vdots \\ W_{p_Y+p_Z-r-1} \\ W_1 \end{pmatrix} \tag{B.11}$$

where $W' = (W_1, \ldots, W_{p_Y+p_Z-r})$ *is a standard Brownian motion.*

Finally $\beta' S_{XX\cdot Z}\beta \xrightarrow{P} \Sigma_{\beta\beta}$, $\beta' S_{XY\cdot Z} \xrightarrow{P} \Sigma_{\beta Y\cdot Z}$, *and* $S_{YY\cdot Z} \xrightarrow{P} \Sigma_{YY\cdot Z}$.

□

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