The osculating plane of a space curve - synthetic formulations

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Introduction

The classical synthetic descriptions of the osculating plane of a space curve k, at a point $P \in k$, are: 1) it is the plane given by the tangent at P and a neighbour point on k; or 2) it is the plane given by P and a neighbour tangent; or 3) it is the plane given by P and two consecutive neighbours of P on k; or, finally, 4) it is the (common) tangent plane of the surface, swept out by the tangents of the space curve, taken at any point of the tangent through P. Only the last description matches perfectly with the analytic formalism, by means of which the theory is usually made rigourous. We shall present a formalism which a little more directly fits with (or vindicates) the three last of these descriptions; and which is rigourous. It does not vindicate the first description; for, in our formalism, the curve tangent through P contains already all neighbour points of P on the curve.

The analytic method in geometry is so strong and comprehensive that almost all geometric theory is formulated in terms of it, nowadays.Only little is formulated in synthetic (incidence theoretic) terms, like the theory of finite projective planes, say, which in this form is considered part of combinatorics.

The synthetic method in differential geometry is even less well established as a rigourous method (although some theory does exist, cf. e.g. [1], and the work of the topos theoretic school, see below), but it flourishes in many heuristic considerations of differential geometric nature. Through the work of the topos theoretic school, by Lawvere and his collaborators (see e.g. [4]), it was embedded in a coordinate context, where some of the synthetic notions can be formulated.

The existence of such coordinate models for the synthetic notions of differential geometry has lead to a sharpening of these notions, to an extent that the time by now is ripe for a formulation that does not presuppose any basic coordinate ring. In fact, the language has been sharpened enough so that the question of combinatorial models can be raised (- in a similar way as the sharpening of the incidence theoretic language of projective geometry in general was the prerequisite for projective geometry partly turning into a branch of combinatorics). I shall present a version of this language/formalism. As a kind of experimental project, I shall formulate the descriptions of the osculating plane in terms of this language, demonstrating a closer match between the formalism and three of the descriptions of the osculating planes quoted above; and also, a proof of the equivalence of the three descriptions is given, a proof which does not need to go outside the formalism. (This was already attempted in [4] I.21, but not completed.) This is the content of Section 1; in Section 3, we demonstrate how the pure synthetic notions can be interpreted in any of the coordinate models of Synthetic Differential Geometry. But my point is that the notions don't *have* to be thus interpreted, and by themselves form a very direct vehicle for geometric definition and reasoning.

The content of the present note is not exactly the same as the one I presented in Perugia in May 1997; that talk rather dealt with some related, but metric, aspects of Synthetic Differential Geometry ("Geometric Construction of the Levi-Civita Parallelism"), cf. [5]. I want to thank the organizers warmly for inviting me to speak at the meeting.

1 Synthetic Theory

Since the geometric notions that will enter in our formalism here all belong to just projective geometry (not to, say, affine or metric geometry), it is clear that the notions considered, and the results proved, belong to Projective Differential Geometry, and as such are invariant under the Projective General Linear Group.

Characteristic of the pure synthetic language I shall present, is that intensive quantities, real valued functions, say, do not play a role, in particular, coordinates are not considered; hence, also there is no occurrence of "Quantities vanishing to the first, or to the second, or ... order". Rather, the pure synthetic language deals with points, pairs of neighbour points, etc., as well as with lines, planes, curves, ... - they are all examples of extensive quantities. So in this sense, the language is one-sided, in a way that cannot be recommended in general.

So the vocabulary has words familiar from 3-dimensional projective geometry: point, line, plane, space, and the usual incidence relations. Also, besides equality of points, and of lines, planes, etc., one needs an apartness[1] relation for points, $P\sharp Q$, as well as a neighbour relation, $Q \sim R$ for points P, Q, R. The apartness relation is needed in order to formulate the most basic incidence theoretic property: *through two points which are apart, there passes a unique line.*

The neighbour relation is assumed to be reflexive and symmetric, but is not assumed transitive (transitivity would destroy the whole theory; so the neighbour relation is definitely not to be thought of as the relation of "being infinitesimally close", in the sense of Non Standard Analysis). The apartness relation should satisfy: $x\sharp y$ and $y \sim z$ implies $x\sharp z$.

The set of neighbours of a point P will be denoted $M(P)$ (M for 'monad', a term borrowed from Non Standard Analysis; in [4], it is denoted $\mathcal{M}_1(P)$, "first-order monad", to distinguish it from the second and higher order monads, which are larger.) We shall assume the following "linear sufficiency principle" for the monads: if h_1 and h_2 are linear subsets (lines, or planes) through P, then if $h_1 \cap M(P) \subseteq h_2 \cap M(P)$, we may conclude $h_1 \subseteq h_2$.

We assume now given a smooth space curve k; this means a subset of space, with the property that for any $P \in k$, there is a (necessarily unique) line l with the property that

$$l \cap M(P) = k \cap M(P).$$

This line l is the *tangent line* of k at P, denoted $t_P(k)$ or just t_P, since k will be fixed throughout.

We shall also consider smooth surfaces G; this means a subset G of space, with the property that for any $Q \in G$, there is a (necessarily unique) plane π with the property that

$$\pi \cap M(Q) = G \cap M(Q).$$

[1] As demonstrated in [3], it is possible to take $P\sharp Q$ to mean just $\neg(P = Q)$; but we do not, even in this case, have that $\neg(P\sharp Q)$ implies $P = Q$.

This plane is the *tangent plane* T_Q of G at Q. A line l with $l \cap M(Q) \subseteq G \cap M(Q)$ is called a *tangent line* to G at Q. Tangent lines to G at Q lie in the tangent plane T_Q, by "linear sufficiency".

We shall assume that the tangent lines of k sweep out a smooth surface G; more precisely, for each $P \in k$, let us denote by $t_P^\sharp$ the set of points on t_P apart from P. The set of $t_P^\sharp$'s, as P ranges over k, should be disjoint, and together make up a smooth surface G. (The geometric picture is that G consists of two sheets that meet along k, which is a sharp edge of the closure of G. One sheet is formed by the positive half tangents, the other by the negative ones; but this refinement of the picture is not utilized in the following. The 'disjointness' means: if $t_P^\sharp$ and $t_{P'}^\sharp$ have some point in common, then $P = P'$.) Since $t_P^\sharp \subseteq G$, it follows that for any $Q \in t_P^\sharp$, t_P is a tangent line to G at Q, and hence $t_P \subseteq T_Q$.

We shall also assume that if $Q \sim Q'$ are points on G, then $P \sim P'$, where P and P' are the unique points on k such that $Q \in t_P$ and $Q' \in t_{P'}$; and conversely, if $Q \in t_P$ and $P \sim P'$, then there is at least one point $Q' \in t_{P'}$ with $Q \sim Q'$. These requirements should be thought of as continuity requirements.

Now we can do an argument:

Proposition 1.1 *Let $Q \in G$, say $Q \in t_P$. Then for $P' \in k$ with $P' \sim P$, $t_{P'} \subseteq T_Q$. Verbally, if $Q \in t_P$, then any neighbour tangent to t_P is contained in the tangent plane at Q.*

Proof. By continuity, pick $Q' \in t_{P'}$, $Q' \sim Q$. Then $Q' \sharp P'$. Since $Q' \in G \cap M(Q)$, $Q' \in T_Q$. Since $P \in k \cap M(P')$, $P \in t_{P'}$. Since $P' \sharp Q'$ and $P \sim P'$, $P \sharp Q'$. So the unique line containing P and Q' is $t_{P'}$. Now $P \in t_P \subseteq T_Q$ and $Q' \in T_Q$. From this follows that the whole line these two points determine is contained in T_Q. So $t_{P'} \subseteq T_Q$.

Proposition 1.2 *Let Q_1 and Q_2 be points on G on the same t_P. Then $T_{Q_1} = T_{Q_2}$.*

Proof. By symmetry, it suffices to prove $T_{Q_1} \subseteq T_{Q_2}$, and for this, it suffices to prove $T_{Q_1} \cap M(Q_1) \subseteq T_{Q_2}$. But $T_{Q_1} \cap M(Q_1) = G \cap M(Q_1)$. So let $Q' \in G \cap M(Q_1)$. Then by continuity, $Q' \in t_{P'}$ for some $P' \sim P$, as in the previous proof. By the Proposition just proved (with $Q = Q_2$), we get that $t_{P'} \subseteq T_{Q_2}$, and since $Q' \in t_{P'}$, $Q' \in T_{Q_2}$, as desired.

The last Proposition shows that T_Q only depends on the P for which $Q \in t_P$; we may call the plane thus defined *the osculating plane of k at P*. Proposition 1.1 thus expresses verbally: *The osculating plane at P contains all neighbour tangents $t_{P'}$ of t_P.*

This in fact characterizes the osculating plane. For, let $P \in k$, and let π be a plane that contains all $t_{P'}$ (for all neighbour points P' of P on the curve). Then we prove that $\pi = T_Q$, for any $Q \in t_P^\sharp \subseteq G$. It suffices, by standard dimension argument, to prove $T_Q \subseteq \pi$, and for this, it suffices to prove that the inclusion holds after intersecting with $M(Q)$. But for a $Q' \in T_Q \cap M(Q)$, we have $Q' \in t_{P'}$, for some neighbour point P' on the curve (as above), and by assumption, $t_{P'} \subseteq \pi$. Thus

Proposition 1.3 *The osculating plane at $P \in k$ is the unique plane containing t_P and all its neighbour tangents.*

Of course, we may equivalently, with slight redundancy, state this: the osculating plane at P is the unique plane containing P and all the neighbour tangents $t_{P'}$ (t_P being itself among them.) This comes close to the "classical" synthetic formulation 2) given in the introduction; note, however, that we talk about *all* neighbour tangents collectively, where the classical formulation talks about *one* of them (this makes sense, if the "one" it talks about, is "the *generic* one", but this would require some explanation of what that is supposed to mean).

We have an analogous relation between the formulations of the "three-point" synthetic characterization of the osculating plane; we have, in our context:

Proposition 1.4 *Let $P \in k$. For arbitrary $P' \sim P$ and $P'' \sim P'$ (with P' and P'' on k), we have that $P, P' and P''$ are contained in the osculating plane at P; and this property characterizes the osculating plane.*

Proof. For P' and P'' as in the statement, we have $P' \in t_P$ and $P'' \in t_{P'}$, but both t_P and $t_{P'}$ are contained in the osculating plane at P, by Proposition 1.1. Conversely, if a plane π has the property stated, we can prove that it contains $t_{P'}$ for any $P' \sim P$ on the curve. It suffices to see that $t_{P'} \cap M(P') \subseteq \pi \cap M(P')$. The set on the left equals $k \cap M(P')$, but a P'' in this set is a point on the curve which is $\sim P'$, thus such P'' is in π by assumption.

It is clear that the first synthetic description given in the introduction is not vindicated in our formalism; for all the neighbour points in question already belong to t_P, and so, even collectively, do not determine a plane.

2 Linear Algebra over a Local Ring

We now turn to the question of existence of models. It will be no surprise that we turn to coordinate models, over a suitable commutative ring R, although, as stated in the introduction, one could conceivably have purely combinatorial models.

It will also be no surprise that R will be not a ring in the category of sets, but a ring object in a suitable chosen topos. In fact, we turn to the models provided by the topos theoretic school, cf. e.g. [4], [7], and the references in there. This means that the logic we employ has to be constructive (no law of excluded middle); and also, for the basic coordinate ring we cannot assume algebraic properties which are too strong, since the models don't provide us with such. However, all interesting models of SDG give that at least the coordinate ring is a local ring.

This requires us to establish a certain amount of linear algebra and matrix theory for local ring objects in a topos. This is partly taken from [2] and [3].

We consider a local ring R in a topos E; this means that R is a commutative ring object which satisfies $\neg(0 = 1)$ and

$$x + y \text{ invertible } \Rightarrow (x \text{ invertible } \vee y \text{ invertible }).$$

We also consider R-modules V, whose elements we call *vectors*. An R-module V is called *finite dimensional* if, for some natural number n, it satisfies "there exists an isomorphism of R-modules $V \cong R^n$". [2]

An n-tuple of vectors $v_1, \dots, v_n \in V$ is called a *basis* for V if the map $R^n \to V$ given by $(t_1, \dots, t_n) \mapsto \sum t_i v_i$ is an isomorphism of R-modules. Clearly V is finite dimensional iff there exists a basis for V.

[2] The phrase "there exists ..." is to be read according to standard sheaf semantics, as "there exists locally ...". Thus, in the topos of sheaves over a topological space, any (locally trivial) n-dimensional vector bundle will define an n-dimensional R-module, where R is the sheaf of germs of real valued functions on the space. - Similarly for all other similar uses of "there exists ...".

The notion of *linear independent set* ramifies, compared to standard linear algebra over a field in the topos of sets, and one of the ramifications we call *prebasis*; a prebasis in V is a k-tuple of vectors $v_1, \ldots, v_k$ that may be expanded to a basis $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$ for V [3].

For the case where $V = R^n$, this can be expressed matrix-theoretically. Given an $n \times k$ matrix A ($n \geq k$). Then A is called *regular* (or of *maximal rank*) if at least one of its $k \times k$ submatrices has invertible determinant. And it is called *singular* if all its $k \times k$ submatrices have determinant zero. From [2], Proposition 3.1 and Proposition 2.1, we quote

Proposition 2.1 *The k columns of an $n \times k$ matrix A ($k \leq n$) is a prebasis for R^n iff A is regular.*

Proposition 2.2 *Given a regular $n \times k$ matrix A ($k < n$), and given $z \in R^n$. Then the $n \times (k+1)$ matrix $[A \mid z]$ is singular iff z belongs to*[4] *the column space of A.*

We also shall prove a form of the Steinitz exchange theorem, in a stronger form than the one of [2]:

Proposition 2.3 *Let B be a regular $n \times l$ matrix, and let A be a regular $n \times k$ matrix ($l \leq k \leq n$). Assume that each of the l column vectors of B belong to the column space of A. Then there exists a regular $n \times k$ matrix, whose columns are the l columns of B together with $k - l$ columns from A, and whose column space is the same as the column space of A.*

Before we prove it, we need some preparations. First, a vector v in R^n is called *regular* if it is regular viewed as an $n \times 1$ matrix, i.e. if at least one of its entries is invertible. If $A = \{a_{ij}\}$ is a matrix, and $A \cdot v$ is a regular vector, then v is regular. For, by assumption, one of the coordinates of $A \cdot v$ is invertible, say the ith. But this entry is $\sum a_{ij} v_j$. By localness of R, one of the terms is invertible, say $a_{ij} v_j$, and this implies that v_j is invertible.

If A is an *invertible* $n \times n$ matrix, then it now follows that $A \cdot v$ is regular iff v is. From this, in turn follows that the notion of regularity of vectors

[3] Note that to be a prebasis is again an existential property: "there exist $v_{k+1}, \ldots, v_n$".

[4] The last clause is again an existential statement: "there exists a linear combination of the column vectors of A with z as value".

may be defined for vectors in arbitrary finite dimensional R-modules V, by transport along an isomorphism $R^n \to V$, but independent of the choice of the isomorphism.

In a similar way, it is easy to see that if a linear combination $\sum r_i a_i$ of vectors a_i is regular, then at least one of the coefficients r_i is invertible.

We shall in particular be interested in the exterior powers $\Lambda^k(R^n)$; these exist, and are finite dimensional (cf. [2]); and an $n \times k$ matrix (with $k \leq n$) is regular iff the wedge product w of its columns is a regular vector in $\Lambda^k(R^n)$ (for, with one of the standard choices of a basis for $\Lambda^k(R^n)$, the coordinates of w, with respect to this basis, is precisely the $\begin{pmatrix} n \\ k \end{pmatrix}$-tuple of $k \times k$ subdeterminants of the matrix).

We now prove the Steinitz exchange. From Proposition 2.1 follows that a subset of the set of columns of a regular matrix form a regular matrix; so in particular, since B is assumed regular, its first column vector b_1 is regular. By assumption b_1 is a linear combination of the columns of A, $a_1, \ldots, a_k$,

$$b_1 = r_1 a_1 + \ldots + r_k a_k,$$

and since b_1 is regular, at least one of the r_i's is invertible; for simplicity, assume it is r_1. Multiplying the equation by the scalar r_1^{-1} and rearranging, we get a_1 expressed as a linear combination of $b_1, a_2, \ldots, a_k$, and it is clear that this set of vectors has the same span as $a_1, \ldots, a_k$. In particular, b_2 can be expressed as a linear combination

$$b_2 = s_1 b_1 + s_2 a_2 + \ldots + s_k a_k. \tag{1}$$

Taking wedge product of this equation with the vector b_1 and using $b_1 \wedge b_1 = 0$, we get

$$b_1 \wedge b_2 = s_2(b_1 \wedge a_2) + \ldots + s_k(b_1 \wedge a_k).$$

Since the two columns b_1, b_2 form a regular matrix, their wedge product $b_1 \wedge b_2$ is a regular vector, and therefore, one of the coefficients $s_2, \ldots, s_k$ in the linear combination above is invertible; for simplicity, assume it is s_2. Then in (1), we divide by s_2 and rearrange; then we see that a_2 is in the span of $b_1, b_2, a_3 \ldots a_k$, and so also the span of these equals the span of the original $a_1, \ldots, a_k$. (Continue in a similar fashion if $l \geq 3$). The equality of the spans implies the existence of a $k \times k$ matrix C so that $A' \cdot C = A$ where A' is the matrix with columns $b_1, b_2, a_3 \dots a_k$, and since A is regular, it follows that A' is regular; for, the $k \times k$ minor determinants of A come about from the $k \times k$ minor determinants of A' by multiplication by the determinant of C. This proves the Proposition (for $l = 2$).

3 The coordinate model

We describe the incidence geometry, as well as the apartness- and neighbour relation, for a suitable variant (see [3]) of projective 3-space over R (where R is a local ring object in a topos; ultimately, R will be assumed to be a model of SDG, as in [4] or [7]).

The *points* of this projective 3-space are the equivalence classes of regular vectors in R^4, under the equivalence relation given by multiplication by invertible elements in R. (We view vectors as column vectors, i.e. as 4×1 matrices.)

The *lines* of projective 3-space are the equivalence classes of regular 4×2 matrices with entries from R, under the equivalence relation given by right multiplication by invertible 2×2 matrices; and the *planes* are similarly equivalence classes of regular 4×3 matrices, under the equivalence relation given by right multiplication by invertible 3×3 matrices.

A point is said to be *incident with* a line if the 4×3 matrix obtained by taking representative matrices of the point and the line, and writing them next to each other, is singular. Similarly, incidence of a point with a plane is defined in terms of singularity of the 4×4 matrix obtained from representative matrices of the point and the plane.

Incidence of a line with a plane is defined by singularity of the *two* 4×4 matrices obtained by taking each of the two columns of a representative matrix for the line and placing it next to a representative matrix for the plane.

Two points are defined to be *apart* if the 2×4 matrix, obtained by placing representative matrices of the points next to each other, is regular. This regular matrix then also will witness that through two points, which are apart, there passes a unique line.

All these notions are independent of choice of representatives. They could have been defined more abstractly in terms of the exterior algebra of an R-module, cf. [2] (where it is also argued that the duality principle of projective geometry holds for the model thus constructed).

Finally, to define the neighbour relation for points, let P and P' be points, represented by $(x_1, \ldots, x_4)$ and $(x'_1, \ldots, x'_4)$, respectively. A necessary condition that these points qualify as neighbours is that for each $i = 1, \ldots, 4$, x_i is invertible iff x'_i is. If so, assume for instance that x_4 and hence x'_4 is invertible. Then by dividing by x_4, respectively by x'_4, we get representatives for P and P' of form $(z_1, z_2, z_3, 1)$ and $(z'_1, z'_2, z'_3, 1)$, respectively. Then P and P' are declared *neighbours* if (z_1, z_2, z_3) and (z'_1, z'_2, z'_3) are neighbours in affine 3-space R^3, in the usual sense of [4] (meaning that $(z_i - z'_i) \cdot (z_j - z'_j) = 0$ for all $i, j = 1, 2, 3$, in particular $(z_i - z'_i)^2 = 0$).

In other words, we define the neighborhood relation by passing to the affine charts R^3 that cover projective 3-space.

We also shall indicate what trace the incidence combinatorics leaves on such an affine piece, say the one given by $x_4 \sharp 0$. Assume that a line l, represented by a certain regular 4×2 matrix A, has a point P in common with this affine piece. Let P be represented by the vector $u = (u_1, u_2, u_3, u_4)$, say, with u_4 invertible. Then P may equally be represented by a vector $x = (x_1, x_2, x_3, 1)$. Since P is assumed to be incident with the line, the 4×3 matrix $[x, A]$ is singular, and then it follows from the Steinitz Exchange (Proposition 2.3) that the line l may be represented by a 4×2 matrix B with x as its first column. Performing now an elementary column operation on B (which amounts to right multiplication by a suitable invertible 2×2 matrix), we finally conclude that l may be represented by a matrix of form

$$\begin{bmatrix} x_1 & r_1 \\ x_2 & r_2 \\ x_3 & r_3 \\ 1 & 0 \end{bmatrix}.$$

Let $y = (y_1, \ldots, y_4)$ represent a point Q incident with l. Then by Proposition 2.2, y belongs to the column space of the matrix; if further y is in the affine piece $x_4 \sharp 0$, we may, as before, assume that y_4 is actually 1, and then y belongs to the column space of the matrix iff for some s (necessarily unique)

$$(y_1, y_2, y_3) = (x_1, x_2, x_3) + s \cdot (r_1, r_2, r_3). \qquad (2)$$

This proves that a line l, which meets the affine piece R^3, may be exhibited in the parametric form (2), where (x_1, x_2, x_3) is an arbitrary point on the meet of the line with the affine piece R^3, and where (r_1, r_2, r_3) is a regular vector in R^3 (regular, because $(r_1, r_2, r_3, 0)$ is a regular vector, being a column of the regular matrix above). Of course, (r_1, r_2, r_3) is to be thought of as an (affine) *direction vector* for the line l.

Similarly, any plane π, which meets the affine piece $x_4 \sharp 0$, may be exhibited in parametric form with two parameters, using two "direction vectors," which together form a regular 3×2 matrix, (and with an arbitrary point, on the meet of π with the affine piece, as "base point").

We now prove that the monads are "linearly sufficient", which is of course where part of the basic axioms of SDG come in. We do it for the case of a line l and a plane π, the other cases (line/line and plane/plane being similar). So let P be a point on l and on π, such that $M(P) \cap l \subseteq \pi$. Without loss of generality, we may assume that P is in the affine piece R^3 given by $x_4 \sharp 0$. For simplicity, we may even assume that P is the point $(0, 0, 0) \in R^3$, so P is represented by the regular vector $(0, 0, 0, 1)$. Since $P \in l$ and $P \in \pi$, it follows from Steinitz Exchange that l and π may be represented by the regular matrices

$$\begin{bmatrix} 0 & r_1 \\ 0 & r_2 \\ 0 & r_3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & u_1 & v_1 \\ 0 & u_2 & v_2 \\ 0 & u_3 & v_3 \\ 1 & 0 & 0 \end{bmatrix},$$

respectively. Let $d \in R$ be any element with $d^2 = 0$. Then the point represented by $(dr_1, dr_2, dr_3, 1)$ is in $M(P) \cap l$ (in the affine piece R^3, it is just the point $(dr_1, dr_2, dr_3) \sim (0, 0, 0)$). Hence by assumption, it is in π. This implies that the 4×4 matrix

$$\begin{bmatrix} 0 & u_1 & v_1 & dr_1 \\ 0 & u_2 & v_2 & dr_2 \\ 0 & u_3 & v_3 & dr_3 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

is singular, i.e. has determinant 0. But this determinant is clearly (plus or minus) d times the determinant Δ consisting of the u's, v's and r's. A fundamental axiom, valid in the models of SDG, is that if $d \cdot k = 0$ for all $d \in R$ with $d^2 = 0$, then k is 0. So applying this principle, we see that Δ is 0, which in turn implies that the 4×4 matrix above, with d replaced by 1, is singular. From this immediately follows singularity of the two 4×4 matrices made up of the matrix for π and each of the two columns of the matrix for l.

Concerning the continuity assumptions from Section 1: the law, which to a point $Q \in G$ associates the unique $P \in k$ such that $Q \in t_P$, defines a function (in the topos) from G to k; and all functions defined in the topos models preserve the neighbourhood relation. So it follows that neighbour Q's define neighbour P's. For the other continuity requirement, we shall be sketchy only. Tangents t_P and $t_{P'}$ at neighbour points on k are themselves neighbours in the Grassmann manifold $G_{2,4}$ of lines in projective space. For any two neighbour lines l and l' in $G_{2,4}$, and any $Q \in l$, we may find a plane π which intersects l transversally in Q. It will then also intersect l' transversally; we thus get a map, in the topos, from an open subset of $G_{2,4}$ to π, namely: take a line $\in G_{2,4}$ to its intersection (assumed transversal) with π. Since l and l' are neighbours, their intersections with π are again neighbours. Thus the intersection of l' with π is the desired Q'.

To get a model for the reasoning of Section 1, we should finally exhibit some "curves" k, and here, at present, we have to assume a little more than was actually stated as assumptions; namely that the curve locally may be exhibited by a parametrization $r(t)$, with $r : R \to R^3$ a function (weaker assumptions would require that some implicit function theorem were available in the topos model, and this is not the case for many of the otherwise good models). If r then is sufficiently regular, the tangent line at a given point $r(t)$ is of course given in parametrized form $r(t) + s \cdot r'(t)$, and viewing both t and s here as parameters provides a parametrization of the surface G, provided r', r'' is a regular pair of vectors, in the sense of Section 2. We leave the details, since they are no different from those of any standard analytic formulation of the construction of the "tangent surface" G and the osculating planes of the parametrized curve k.

References

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