Square-densities, and volume forms: Heron's formula

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Introduction

The Greek geometers (Heron et al.) discovered a remarkable formula, expressing the area of a triangle in terms of the lengths of the three sides. Here, length and area are seen as non-negative numbers, which involves, in modern terms, formation of *absolute value* and *square root*. To express the notions and results involved without these non-smooth constructions, one can express the Heron Theorem in terms of the squares of the quantities in question: if g(A, B) denotes the square of the length of the line segment given by A and B, the Heron formula says that the square of the area of the triangle ABC may be calculated by a simple algebraic formula out the three numbers q(A, B), q(A, C), and q(B,C). Explicitly, the formula appears in (1) below. In modern terms, the formula is (except for a combinatorial constant -16^{-1}) the determinant of a certain symmetric 4×4 matrix constructed out of three numbers; see (2) below. This determinant, called the Cayley-Menger determinant, generalizes to simplices of higher dimesions, so that e.g. the square of the volume of a tetrahedron (3-simplex) (ABCD)in space is given (except for a combinatorial constant) by the determinant of a certain 5×5 matrix constructed out of the six square lengths of the edges of the tetrahedron (by a formula already known in the Renaissance).

The Heron formula has the advantage that it is symmetric w.r.to permutations of the k + 1 vertices of a k-simplex. Also, it does not refer to the vector space or affine structure of the ambient space.

We shall in particular consider the case where the space, in which the k-simplex lives, is a Euclidean space: an affine space E whose associated vector space V is provided with a positive definite inner product. Then the square lengths, square areas, square volumes etc. of the simplices can also be calculated by another well known and simple expression: namely as $(1/k!)^2$ times the Gram determinant of a certain $k \times k$ matrix constructed from the simplex, by choosing one of its vertices as origin. The Gram determinant itself expresses the square volume of the parallelepipedum spanned by k vectors in V that go from the origin to the remaining vertices.

An important difference between the two formulae is the (k+1)!-fold symmetry in the Heron formula, where the Gram formula is apriori only k!-fold symmetric, because of the special role of the chosen origin.

This Gram method of calculating the square-volumes has the advantage that it it is easy to describe algebraically, and in particular, it is easy to describe what happens if one changes the metric; this is needed, when dealing with Riemannian manifolds, where the metric tensor, in any given coordinate chart, changes from point to point.

We begin in Section 1 by recalling the classical case of Euclidean spaces. In particular, we recall the comparison (standard, but nontrivial) between the Heron and Gram calculations. This Section is essentially a piece of standard linear algebra.

In Section 2, we recall or introduce the notions of differential form and square density in the combinatorial versions from synthetic differential geometry (SDG). This leads to synthetic, or combinatorial arguments, based on "infinitesimal" simplices, and their square volume.

In Section 3, we relate (in terms of SDG) the volume form of an n-dimensional Riemannian manifold to the volume of certain infinitesimal n-simplices. This Section contains the main theorem, where we, for a Riemannian manifold of dimension n, compare the square volume of n-simplices given, respectively, by the Heron formula and by the (valuewise) square of the volume form.

Throughout, R denotes "the" number line, a commutative ring with suitable properties to be described when needed. In particular, the notion of *positivity*, and of when a quadratic form over R is *positive definite* is recalled in the beginning of Section 4. I do not know whether positive definiteness plays a role in the algebraic arguments in the first three Sections, except that the use of the phrases "square length", ..., "square volume", etc. are somewhat misleading in the indefinite case. It is useful to think in terms of the quantities occurring as being quantities whose physical dimension is some power of length (measured in meter m, say), so that length is measured in m, area in m^2 , square area in m^4 , etc. Tangent vectors are not used in the following; they would have physical dimension of $m \cdot t^{-1}$ (velocity). The word square-density is used in any dimension. Square length, square area, and square volume are examples, but we do not claim that the square densities considered presently have such geometric significance.

The theory developed here was also attempted in my [5]; I hope that the present account will be less ad hoc.

1 Square volumes in Euclidean spaces

1.1 Heron's formula

The basic idea for the construction of a square k-volume function goes, for the case k = 2, back to Heron of Alexandria (perhaps even to Archimedes); they knew how to express the square of the area of a triangle S (whether located in Euclidean 2-space or in a higher dimensional Euclidean space) in terms of an expression involving only the lengths a, b, c of the three sides:

$$area^{2}(S) = t \cdot (t-a) \cdot (t-b) \cdot (t-c)$$

where $t = \frac{1}{2}(a + b + c)$. Substituting for t, and multiplying out, one discovers (cf. [2] 1.53) that all terms involving an odd number of any of the variables a, b, c cancel, and we are left with

$$area^{2}(S) = -16^{-1}(a^{4} + b^{4} + c^{4} - 2a^{2}b^{2} - 2a^{2}c^{2} - 2b^{2}c^{2}),$$
 (1)

an expression that only involves the squares a^2 , b^2 and c^2 of the lengths of the sides.

The expression in the parenthesis here may be written in terms of the determinant of 4×4 matrix (described in (2) below), which makes it is possible to generalize from 2-simplices (= triangles) to k-simplices, in terms of determinants of certain $(k + 2) \times (k + 2)$ matrices, "Cayley-Menger matrices/determinants"; they again only involve the square lengths of the $\binom{k+1}{2}$ edges of the simplex.

An k-simplex X in a space M is a (k + 1)-tuple of points (vertices) (x_0, x_1, \ldots, x_k) in M. If $g: M \times M \to R$ satisfies g(x, x) = 0 and g(x, y) = g(y, x) for all x and y (like a metric dist(x, y), or its square), one may construct a $(k + 2) \times (k + 2)$ matrix C(X) by the following recipe: first take the $(k + 1) \times (k + 1)$ matrix whose ijth entry is $g(x_i, x_j)$. It has 0s down the diagonal and is symmetric, by the two assumption about g. Enlarge this matrix it to a $(k + 2) \times (k + 2)$ - matrix by bordering it with $(0, 1, \ldots, 1)$ on the top and and on the left. The case k = 2 is depicted here (writing g(ij) for $g(x_i, x_j)$ for brevity; note g(01) = g(10) etc., so that the matrix is symmetric.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & g(01) & g(02) \\ 1 & g(10) & 0 & g(12) \\ 1 & g(20) & g(21) & 0 \end{bmatrix}$$
(2)

(The indices of the rows and columns are most conveniently taken to be -1, 0, 1, 2.)

This is the Cayley-Menger matrix C for the simplex, and its determinant is its Cayley-Menger determinant. Heron's formula then says that the value of this determinant is, modulo the "combinatorial" factor -16^{-1} , the square of the *area* of a triangle with vertices x_0, x_1, x_2 , as expressed in terms of squares $g(x_i, x_j)$ of the *distances* between them. Similarly for (square-) volumes of higher dimensional simplices. Note that no coordinates are used in the construction of this matrix/determinant.

The general formula is that the square of the volume of a k-simplex is $-(-2)^{-k} \cdot (k!)^{-2}$ times the determinant of C, e.g. for k = 1, 2, and 3, the factors are 2^{-1} , -16^{-1} , and 288^{-1} , respectively.

We shall in the following denote the square volume of a k-simplex X, as calculated by the Heron-Cayley-Menger formula, by Heron(X) (provided of course that we have some data giving us the "square distance" $g(x_i, x_j)$ between its vertices).

Proposition 1.1 The Cayley-Menger determinant for a k-simplex is invariant under the (k + 1)! symmetries of the vertices of the simplex.

Proof. Interchanging the vertices x_i and x_j has the effect of first interchanging the *i*th and *j*th column, and then interchanging the *i*th

and *j*th row of the new matrix. Each of these changes will change the determinant by a factor -1.

1.2 Gram's formula

Given a k-simplex $X = (x_0, x_1, \ldots, x_k)$ in a Euclidean space E with associated vector space V with an inner product. If $V = R^n$ with the standard inner product, we may form an $n \times k$ matrix Y with columns $y_i := x_i - x_0$ $(i = 1, \ldots, k)$. The Gram matrix of the simplex X is then the $k \times k$ matrix $Y^T \cdot Y$. Its determinant is the Gram determinant of the simplex: $\operatorname{Gram}(X) := \det(Y^T \cdot Y)$. The determinant itself is coordiante independent, i.e. it only depends on the inner product on V, not on coordinatizing V by R^n .

This determinant likewise has a volume theoretic significance: it gives the square of the volume of the parallelepipedum spanned by the k vectors $y_i := x_i - x_0$ in V (i = 1, ..., k).

The following Proposition is only included for a comparison with the issue of (k + 1)! symmetry of the formulae.

Proposition 1.2 The Gram determinant for a k-simplex is invariant under the (k + 1)! symmetries of the k-simplex.

Proof. It suffices to prove this for the case where $V = R^n$ with standard inner product. Interchanging x_i and x_j , for i and $j \ge 1$ implies an interchange of the corresponding columns in the Y-matrix, and this interchanged matrix comes about by multiplying Y on the right by the $k \times k$ matrix S obtained from the unit matrix by interchanging its *i*th and *j*th column. This S has determinant -1. So $S^T \cdot Y^T \cdot Y \cdot S$ has the same determinant as $Y^T \cdot Y$. Interchanging x_0 and x_j in the simplex corresponds, using $y_i = x_i - x_0$, to multiplying the Y-matrix on the right by the matrix S_j , obtained from the unit $k \times k$ matrix by replacing its *j*th row by the row $(-1, -1, \ldots, -1)$. The matrix S has determinant -1, so $S^T \cdot Y^T \cdot Y \cdot S$ has same determinant as $Y^T \cdot Y$. Similarly for exchanging x_0 by x_j .

1.3 Comparison formula

For a Euclidean space E, it makes sense to compare the values of the Heron and Gram formulas for square volume of a k-simplex X =

 (x_0, x_1, \ldots, x_k) . Let C denote the $(k+2) \times (k+2)$ matrix ((Heron-) Cayley-Menger) formed by the square distances between the vertices, as described above, and let $Y^T \cdot Y$ be the Gram $k \times k$ matrix of the simplex, likewise described above. There is a known relation between their determinants

$$\det(C) = -(-2)^k \det(Y^T \cdot Y).$$
(3)

For a proof, see reference [10].

Note that the left hand hand side in (3) does not make use of the algebraic structure of E and its associated vector space, but only on the (square-) distance function (arising from the inner product). This flexibility will be crucial when we consider Riemannian manifolds.

We denote the square volume of a simplex X, as calculated in terms of the Cayley-Menger matrix C, by Heron(X), and denote the square volume of the corresponding parallelepipedum, as calculated by Gram's method, by Gram(X). But we shall later have occasion to consider different fixed (positive definite) inner products G on one and the same vector space V, in which case we may extend the notation and write $Heron_G$ and $Gram_G$ to specify which inner product we use. The comparison (3) may then be formulated

Proposition 1.3 Let $X = (x_0, ..., x_k)$ be a k-simplex in a Euclidean space. Then $\operatorname{Heron}_G(X) = \frac{1}{k!^2} \operatorname{Gram}_G(X)$.

(The factor $k!^2$ is just because the volume of the parallelepipedum is k! as large as the one of the simplex itself.)

Remark. In terms of physical dimension alluded to in the Introduction: volume of a k-simplex has dimension m^k , so its square volume has dimension $(m^k)^2$; the entries $g(x_i, x_j)$ in the Cayley-Menger matrix have physical dimension m^2 , and expanding its determinant, all terms are products of k copies of these entries. (The entries 0 and 1 in the top line and left column in the matrix are "pure" quantities, i.e. of dimension m^0). So the value of the determinant is of physical dimension $(m^2)^k$. The Heron formula is then meaningful in the sense that it equates quantities of dimension $(m^2)^k$ and $(m^k)^2$.

In particular, the comparison between the square volumes of a k-simplex, as calculated by Heron-Cayley-Menger and by Gram, which is a consequence of (3), is dimensionally meaningful; both have physical dimension m^{2k} .

2 Differential forms and square densities

As in [4], say, we consider the following kind of structure on an object M in a category \mathcal{E} with finite inverse limits M, namely subobjects of $M \times M$,

$$M_{(0)} \subseteq M_{(1)} \subseteq M_{(2)} \ldots \subseteq M \times M,$$

each of the $M_{(r)}$ s being a reflexive and symmetric relation, with $M_{(0)}$ being the equality relation. We have in mind the "rth neighbourhood of the diagonal" of an affine scheme, as considered in algebraic geometry, or the "prolongation spaces" of manifolds as considered in e.g. [6]. Except for $M_{(0)}$, these relations are not transitive. - We are actually only interested in the the cases r = 0, 1, 2.

We use the well known "synthetic" language to express constructions in categories \mathcal{E} with finite limits, in "elementwise" terms¹, in particular we consider, for a natural number k, the object of r-infinitesimal k-simplices in M, meaning the subobject of $M \times M \times \ldots \times M$ (k + 1times) consisting of k + 1-tuples (x_0, x_1, \ldots, x_k) of elements of M with (x_i, x_j) $\in M_{(r)}$ for all $i, j = 0, 1, \ldots, k$; such a k + 1-tuple, we shall call an r-infinitesimal k-simplex; the x_i s are the vertices of the simplex.

Note that the question of whether a k-simplex is r-infinitesimal only depends on the "edges" (x_i, x_j) (face-1-simplices) of the simplex, equivalently, it depends on the 1-skeleton of the simplex.

We shall, as in [4], write $x_i \sim_r x_j$ for $(x_i, x_j) \in M_{(r)}$. In the context of SDG, we have that $x \sim_r y$ in \mathbb{R}^n is equivalent to:

For any r + 1-linear function $\phi : \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}$, we have

$$\phi(x-y,\ldots,x-y) = 0. \tag{4}$$

For r = 1 and r = 2, we shall consider certain maps from the object of *r*-infinitesimal *k*-simplices to *R*, namely maps which have the property that they vanish if $x_i = x_j$ for some $i \neq j$. For r = 1, combinatorial differential *k* forms ω have this property. (In the context of SDG, such maps are automatically alternating with respect to the (k + 1)! permutations of the x_i s, see [4] Theorem 3.1.5.)

¹Recall that a generalized element of an object M in a category \mathcal{E} is just an arbitrary map in \mathcal{E} with codomain M; see [3] II.1, [8] V.5, or [9] 1.4.

For r = 2, such maps have not been considered much², except for the case where k = 1, where (pseudo-) Riemannian metrics g, in the combinatorial sense (recalled after Definition 2.3 below), are examples of such maps; for this case, we think of $g(x_0, x_1)$ as the square of the distance between x_0 and x_1 . The gs of interest are symmetric, $g(x_0, x_1) = g(x_1, x_0)$. For manifolds M, we have

Proposition 2.1 Given $g: M_{(2)} \to R$ with g(x, x) = 0 for all x. Then g is symmetric iff it vanishes on $M_{(1)} \subseteq M_{(2)}$.

Proof. It suffices to consider an \mathbb{R}^n chart around x; we consider the degree ≤ 2 part of the Taylor expansion of g around x. Then g is given as $g(x,y) = C(x) + \Omega(x;x-y) + (x-y)^T \cdot G(x) \cdot (x-y)$, where C(x) is a constant, Ω is linear in the argument after the semicolon, and G(x) is a symmetric $n \times n$ matrix. To say that g vanishes on the diagonal $M_{(0)}$ (i.e. g(x,x) = 0 for all x) is equivalent to saying that C(x) = 0 for all x. We now compare g(x,y) and g(y,x); we claim

$$(x - y)^{T} \cdot G(x) \cdot (x - y) = (y - x)^{T} \cdot G(y) \cdot (y - x).$$
 (5)

Taylor expanding from x the G(y) on the right hand side gives that this the difference between the two sides is $(y-x) \cdot dG(x; y-x) \cdot (y-x)$ which is trilinear in y - x, and therefore vanishes, since $x \sim_2 y$. So we have that if Ω vanishes, then g is symmetric; vice versa, if g is symmetric, its restriction to $M_{(1)}$ is likewise symmetric, and (being a differential 1-form), it is alternating, so the Ω -part vanishes, which in coordinate free terms says: g(x, y) = 0 for $x \sim_1 y$.

For the number line R, $(x_0, x_1) \in R_{(2)}$ iff $(x_0 - x_1)^3 = 0$, and the map g given by $g(x_0, x_1) := (x_0 - x_1)^2$ is a map as described in the Proposition. In fact, it is the restriction of the standard "square-distance" function $R \times R \to R$.

So we recall, respectively pose, the following definitions, corresponding to r = 1 and r = 2. Let M be a manifold.

Definition 2.2 A (combinatorial) differential k-form on M is an R-valued function ω on the set of 1-infinitesimal k-simplices in M, which is alternating with respect to the (k + 1)! permutations of the vertices of the simplex.

²For r = 2 and k = 1, such things were in [4] 8.1 called "quadratic differential forms".

Hence it vanishes on simplices where two vertices are equal.

Definition 2.3 A k-square-density on M is an R-valued function on the set of 2-infinitesimal k-simplices in M, which is symmetric with respect to the (k+1)! permutations of the vertices of the simplex, and which vanishes on simplices where two vertices are equal.

Note that for k = 1, Proposition 2.1 gives that 1-square densities (square lengths) g have the property that they vanish not just on $M_{(0)}$ (the diagonal), but also on $M_{(1)}$: g(x, y) = 0 if $x \sim_1 y$. So the notion of 1-square density agrees with (combinatorial) "differential quadratic form", as considered in [4], Section 8.1. (Combinatorial) differential quadratic 1-forms we shall also call pseudo-Riemannian metrics.

As a bridge between square densities and differential forms, we pose the following auxiliary

Definition 2.4 An extended k-form on M is an R-valued function $\overline{\omega}$ on the set of 2-infinitesimal k-simplices in M, which vanishes on simplices where two vertices are equal.

Such extended *k*-form restricts to a function on 1-infinitesimal *k*-simplices (and the restriction may or may not be a differential 1-form; note that we did not put conditions like "alternating" or "symmetric" on extended *k*-forms).

Proposition 2.5 If two extended k-forms $\overline{\omega}$ and $\overline{\omega}'$ extend the same differential k-form ω , then $\overline{\omega}^2 = \overline{\omega}'^2$.

Proof. We have to prove that

$$\overline{\omega}^2(x_0, x_1, \dots, x_k) = \overline{\omega}^{\prime 2}(x_0, x_1, \dots, x_k),$$

for any 2-infinitesimal k-simplex (x_0, x_1, \ldots, x_k) . It suffices to do this in a coordinate patch around x_0 , which we may assume is $0 \in \mathbb{R}^n$, in which case $\overline{\omega}$ and $\overline{\omega}'$ are functions Ω and $\Omega' : D_2(n) \times \ldots \times D_2(n) \to \mathbb{R}$ (k factors in the product). By the basic axiom scheme of SDG, the ring A of functions $D_2(n) \to \mathbb{R}$ is of the form $A = A_0 \oplus A_1 \oplus A_2$, with A_0 the constant functions $\mathbb{R}^n \to \mathbb{R}$, A_1 the linear functions $\mathbb{R}^n \to \mathbb{R}$, and A_2 the (homogeneous) quadratic functions $\mathbb{R}^n \to \mathbb{R}$. This A is a graded ring (only non-zero in degrees 0,1 and 2). The ideal of functions vanishing on 0 is $A_1 \oplus A_2 \subseteq A$. So the ideal of functions $(D_2(n))^k \to R$ which vanish if any of its arguments is 0 is the *k*-fold (symmetric) tensor product of $(A_1 \oplus A_2)$,

$$(A_1 \oplus A_2)^{\otimes k} \subseteq A^{\otimes k}.$$
 (6)

This ring is k-graded, with e.g. the multidegree (1, ..., 1) consisting of the k-linear functions $(R^n)^k \to R$

By assumption, both Ω and Ω' belong to the ideal (6). The assumption that both Ω and Ω' restrict to the same differential k-form ω implies that Ω and Ω' agree in their component of multidegree $(1, \ldots, 1)$ (this component being the coordinate expression of ω). Thus $\Omega' = \Omega + \theta$ with θ of multidegree $\geq (1, \ldots, 1)$ and of total degree $\geq k + 1$. The required equation is, in these terms, that $(\Omega + \theta)^2 = \Omega^2$, and this is a simple "counting degrees"-argument in the k-graded ring A^k :

$$(\Omega + \theta)^2 = \Omega^2 + 2\Omega \cdot \theta + \theta^2.$$
(7)

Here, θ^2 has total degree $\geq 2 \cdot (k+1) \geq 2k+1$, which is 0 since A^k is 0 in total degrees > 2k; and θ is a linear combination of terms of multidegree of the form $(1, 1, \ldots, 1+p, \ldots 1)$ for $p \geq 1$, so $\theta \cdot \omega$ is a linear combination of terms of multidegree

$$(1, 1, \dots, 1 + p, \dots, 1) + (1, 1, \dots, 1, \dots, 1) = (2, 2, \dots, 2 + p, \dots, 2)$$

which is of total degree $2k + p \ge 2k + 1$. So the two last terms in (7) are 0, and this proves the Proposition.

2.1 k-square-densities from 1-square-densities g

We shall argue that for 2-infinitesimal simplices (x_0, \ldots, x_k) , the Cayley-Menger determinants define square-densities. We already argued above that these determinants are symmetric: the value does not change when interchanging x_i and x_j . We have to argue for the vanishing condition required. If $x_i = x_j$, then $g(x_i, x_m) = g(x_j, x_m)$ for all m, and this implies that the *i*th and *j*th rows in the Cayley-Menger matrix are equal, which implies that the determinant is 0.

We denote the k-square-density corresponding to a 1-square-density g by Heron_g (when k is understood from the context).

2.2 k-square-densities from differential k-forms

Essentially this is the process of squaring (in *R*) the values, so it is tempting to denote the square-density which we are aiming for, by ω^2 . Precisely: we get a well defined *k*-square-density out of a differential *k*-form by a two step procedure: 1) to *extend* the given *k* form ω to a suitable function $\overline{\omega}$, to allow as inputs not just 1-infinitesimal *k*-simplices, but also 2-infinitesimal *k*-simplices; and then 2) squaring $\overline{\omega}$ valuewise. "Suitable" means that $\overline{\omega}$ is an extended form in the sense of Definition 2.4, i.e. that it vanishes on simplices where two vertices are equal.

We shall prove that such an extension $\overline{\omega}$ is possible; it is not unique: it depends on choosing a coordinate chart. But we shall prove that uniqueness holds after squaring.

The question of existence of such $\overline{\omega}$ is local, so let us assume that the manifold M is an open subset of \mathbb{R}^n . Then the k-form ω is given by a function $\Omega: M \times (\mathbb{R}^n)^k \to \mathbb{R}$, such that

$$\omega(x_0, x_1, \dots, x_k) = \Omega(x_0; x_0 - x_1, x_0 - x_2, \dots, x_0 - x_k)$$

where for each $x_0 \in M$, the function $\Omega(x_0; -, \ldots, -) : (R^n)^k \to R$ is *k*-linear and alternating in the *k* arguments; these arguments are arbitrary vectors in R^n , in particular, they may be of the form $x_i - x_0$ for $x_i \sim_2 x_0$, so the restriction of $\Omega(x_0; x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0)$ to the the set of 2-infinitesimal 2-simplexes defines an extension $\overline{\omega}$ of ω , so

$$\overline{\omega}(x_0, x_1, \dots, x_k) := \Omega(x_0; x_0; x_0 - x_1, x_0 - x_2, \dots, x_0 - x_k)$$
(8)

In this form, the fact that $\overline{\omega}$ is alternating w.r.to the k! permutations of the x_i s $(i = 1, \ldots, k)$ can be read of from the fact that $\Omega(x_0; \ldots)$ is alternating. It is also alternating w.r.to permutations involving x_0 , as long as the x_i s are $\sim_1 x_0$; this can be seen from seen from an easy Taylor expansion argument, see the proof of Theorem 3.1.5 in [SGM]. Now if we use Ω to construct the extension of ω to $\overline{\omega}$, defined on 2-infinitesimal k-simplices, the constructed $\overline{\omega}$ will still be alternating w.r.to permutations of the x_i s for i > 0, but the Taylor expansion argument mentioned fails for the interchange of, say, x_0 and x_1 : we cannot conclude that $\Omega(x_1; x_0 - x_1, \ldots) = -\Omega(x_0; x_1 - x_0, \ldots)$. This failure get repaired by valuewise squaring: **Proposition 2.6** For any 2-infinitesimal k-simplex (x_0, x_1, \ldots, x_k) , we have

$$\Omega(x_1; x_0 - x_1, x_2 - x_1, \ldots)^2 = \Omega(x_0; x_1 - x_0, x_2 - x_0, \ldots)^2$$

Proof. We shall only do the case k = 1. (For the more general case, the further argument is essentially the same as in the proof of Proposition 1.2 above.) First, we have by a Taylor expansion from x_0

$$\Omega(x_1; x_0 - x_1) = \Omega(x_0; x_0 - x_1) + d\Omega(x_0; x_1 - x_0; x_0 - x_1)$$

+ a term $d^2\Omega(x_0;...)$, trilinear in $x_1 - x_0$.

The trilinear term vanishes, because $x_1 \sim_2 x_0$. Now we square, and get

$$\Omega(x_1; x_0 - x_1)^2 = \Omega(x_0; x_0 - x_1)^2 + 2 \cdot \Omega(x_0; x_0 - x_1) \cdot d\Omega(x_0; x_1 - x_0, x_0 - x_1)$$

+ a term $(d\Omega(x_0; \ldots))^2$, quadrilinear in $x_1 - x_0$.

The quadrilinear term vanishes because $x_1 \sim_2 x_0$, but also the term $\Omega \cdot d\Omega$ vanishes, because it is trilinear in $x_1 - x_0$. So we get

$$\Omega(x_1; x_0 - x_1)^2 = \Omega(x_0; x_0 - x_1)^2 = (-\Omega(x_0; x_1 - x_0))^2 = \Omega(x_0; x_1 - x_0)^2,$$

as desired.

We conclude that a differential k-form ω can be extended to an $\overline{\omega}$ (whose input are 2-infinitesimal k-simplices), such that $\overline{\omega}^2$ is (k + 1)!-symmetric. (Also, the extension constructed also clearly has the property that it vanishes if $x_i = x_j$ for some $i \neq j$.) Hence $\overline{\omega}^2$ is a square density.

From Proposition 2.5, we therefore conclude that if two extended k-forms extend the same differential k-form ω , the two resulting square-densities agree.

Because of the Proposition, there is a well-defined "squaring" process, leading from differential k-forms to k-square-densities on a manifold M: extend the form ω , and square the result. It is natural to denote this square density by ω^2 , with the understanding that it means $\overline{\omega}^2$ for any extended form $\overline{\omega}$, extending ω .

3 Variable metric tensor

We consider a manifold M which is embedded as an open subset of \mathbb{R}^n (elements of \mathbb{R}^n we write as $n \times 1$ matrices). A 1-square-density g on M can in this case be given by a metric tensor, i.e. by a family of symmetric $n \times n$ matrices G(x) (for $x \in M$), such that for $x \sim_2 y$,

$$g(x,y) = (x-y)^T \cdot G(x) \cdot (x-y)$$
(9)

(which equals $(y - x)^T \cdot G(y) \cdot (y - x)$ by (5)).

We shall also use the notation $G(x; x - y) := (x - y)^T \cdot G(x) \cdot (x - y)$. Thus G(x; -) is quadratic in the argument after the double semicolon.

The letter G is used for the "metric tensor", i.e. for the family of the matrices G(x). So this G suffices to describe a Heron-Cayley-Menger matrix for any 2-infinitesimal k-simplex in M. We write $\operatorname{Heron}_G(X)$ for the determinant of this matrix. This Heron_G defines in fact a k-square density on M, for any k: metric tensors define square densities.

We shall prove (Proposition 3.2) that for a 2-infinitesimal k-simplex, (x_0, x_1, \ldots, x_k) , the $G(x_i)$ s occurring in the Cayley-Menger determinant for this simplex may all be replaced by x_0 , so that, for a given 2-infinitesimal k-simplex, we can use the comparison with the Gram description, available for constant metric tensors.

The terms in the Cayley-Menger determinant for a k-simplex X are linear combinations of k-fold products $g(x_i, x_j)$ with $i \neq j$, in particular the product

$$\pm g(x_0, x_1) \cdot g(x_1, x_2) \cdot \ldots \cdot g(x_{k-1}, x_k)$$
 (10)

is a term. (The other terms in the determinant come about from similar k-chains of adjacent 1-simplices, by permutation of the indices.)

In terms of variable Riemannian tensors G(x) (with the G(x) symmetric $n \times n$ matrices), the product (10) is (possibly modulo sign) the displayed expression in the following Lemma 3.1). It is useful first to intoduce some ad hoc terminology.

A finite sequence of points x_0, x_1, \ldots, x_k in M which are consecutive 2-neighbours i.e. $x_i \sim_2 x_{i+1}$ for $i = 0, \ldots, k-1$, we shall for simplicity call *path* of length k. If \tilde{x} is a path of length k, we get a path of length k-1 by omitting the first of the vertex of the path. Let us denote this truncated path by $|\tilde{x}$. We are interested in such paths in $M \subseteq \mathbb{R}^n$ when M is eqipped with a Riemannian metric g, given by variable symmetric $n \times n$ matrices G(x). So $g(x,y) = (x - y)^T \cdot G(x) \cdot (x - y)$. Then for a path x_0, \ldots, x_k , as above, we write $G(\tilde{x})$ for the product $g(x_0, x_1) \cdot \ldots \cdot g(x_{k-1}, x_k)$, i.e. in coordinates

$$G(\tilde{x}) := G(x_0; ; x_0 - x_1) \cdot G(x_1; ; x_1 - x_2) \cdot \ldots \cdot G(x_{k-1}; ; x_{k-1} - x_k), \quad (11)$$

and we write $\overline{G}(\tilde{x})$ for the similar product, but with all the x_i s appearing before the double semicolon replaced by the G(x) for x the *first* vertex of the path,

$$\overline{G}(\tilde{x}) := G(x_0; ; x_0 - x_1) \cdot G(x_0; ; x_1 - x_2) \cdot \ldots \cdot G(x_0; ; x_{k-1} - x_k)$$

Thus in $\overline{G}(|\tilde{x})$, the constant matrix used is $G(x_1)$ because the first vertex of $|\tilde{x}$ is x_1 .

Lemma 3.1 For any path \tilde{x} , $G(\tilde{x}) = \overline{G}(\tilde{x})$.

Proof. By induction of the length k of the path. The assertion is clearly true for k = 1. Assume that it holds for k - 1. Then

$$G(\tilde{x}) = G(x_0; ; x_0 - x_1) \cdot G(|\tilde{x}) = G(x_0; ; x_0 - x_1) \cdot \overline{G}(|\tilde{x}),$$

by the induction assumption, used for the path $|\tilde{x}$. Now by definition of $\overline{G}(|\tilde{x})$, the equation continues

$$= G(x_0; ; x_0 - x_1) \cdot G(x_1; ; x_1 - x_2) \cdot G(x_1; ; x_i - x_{i+1}) \cdot \dots \cdot G(x_1; ; \cdots + x_{k-1} - x_k).$$

Now we Taylor expand, for fixed *i*, the displayed factor $G(x_1; ; x_i - x_{i-1})$, from x_0 in the direction $x_1 - x_0$: we have

$$G(x_1; x_i - x_{i+1}) = G(x_0; x_i - x_{i+1}) + dG(x_0; x_1 - x_0; x_i - x_{i+1}) + Q$$

Q is quadratic in $x_1 - x_0$. But the linear term $dG(x_0; x_1 - x_0; x_i - x_{i-1})$, as well as the quadratic term Q, get annihilated by being mutiplied with $G(x_0; ; x_1 - x_0)$, since this factor is linear in the dG-term (and, even more so, Q). So altogether, we have an expression (at least) trilinear in $x_0 - x_1$, and therefore it vanishes since $x_0 \sim_2 x_1$. Therefore, in the product (11), each factor $G(x_i; ; \ldots)$ may be replaced by $G(x_0; \ldots)$, and then we have $\overline{G}(\tilde{x})$. **Remark.** The argument simplifies for the case of "restricted" 2-infinitesimal k-simplices, as considered by [1], since there one has that each of the individual $g(x_i, x_j)$ in a simplex (x_0, \ldots, x_k) may be calculated by using $G(x_0)$. In preliminary versions of the present note, I only considered the restricted simplices, but the value of the Heron-Cayley-Menger formula on such simplices is probably not enough for characterizing the volume form, which is our aim.

From the Lemma, we conclude, for any variable metric tensor G:

Proposition 3.2 Given a 2-infinitesimal k-simplex $X = (x_0, \ldots, x_k)$. Then $\operatorname{Heron}_G(X) = \operatorname{Heron}_{G(x_0)}(X)$.

Combining with the comparison in (3), we get

Proposition 3.3 Given a coordinate patch $M \subseteq \mathbb{R}^n$ and a 2-infinitesimal k-simplex $X = (x_0, x_1, \ldots, x_k)$ in M. Then $\operatorname{Heron}_G(X) = (k!)^{-2} \cdot \operatorname{Gram}_{G(x_0)}(X)$.

4 Volume form

The volume form is a differential *n*-form that may be defined on an *n*-dimensional manifold M, which is equipped with a *Riemannian* (not just pseudo-Riemannian) metric g and which is *oriented*.

We need to describe these terms. A Riemannian metric g on M is one which in local coordinate charts is given by *positive definite* matrices G(x), in the following sense:

We need to assume that R is equipped with a subset $P \subseteq R$ (the *positive* elements), stable under addition and multiplication, and such that elements in P are invertible and have unique positive square roots. (The unique positive square root of $a \in P$ is denoted \sqrt{a} .) We also assume the dichomotomy: for every invertible x, either x or -x is positive.

To say that a metric tensor G is positive definite is to say: for each $x \in M$, $\det(G(x))$ is invertible for all x, and $G(x) = H(x)^T \cdot H(x)$ for some $n \times n$ matrix H(x). Then $\det(G(x)) = (\det(H(x))^2$, so $\det(G)$ is positive and therefore has a square root (in fact $\pm \det(H)$ will serve).

An orientation form for an *n*-dimensional manifold M is a differential *n*-form δ , so that any differential *n*-form on M can be written³ $f \cdot \delta$ for a unique $f : M \to R$. In the manifold $M = R^n$, determinantformation is an orientation form. An orientation on M is given by an orientation form, and δ_1 and δ_2 define the same orientation if $\delta_2 = f \cdot \delta_1$ for an $f : M \to P \subseteq R$. An *n*-form ω is positive if it is $f \cdot \delta$ for some $f : M \to P$.

Recall from the last lines of Section 2 the notation ω^2 for the square *k*-volume constructed out of a differential *k*-form ω :

Theorem 4.1 Assume that g is a Riemannian metric on an oriented ndimensional manifold M. Then there exists on M a unique positive differential n-form ω such that Heron_g and ω^2 agree on all 2-infinitesimal n-simplices; it deserves the name volume form for g.

Proof. Since the data and assertions in the statement do not depend on the choice of a (positively oriented) coordinate chart, it suffices to prove the assertion in such. So assume that $M \subseteq \mathbb{R}^n$ is an open subset (with orientation inherited from the canonical one det on \mathbb{R}^n), and G is given in terms of the positive definite $n \times n$ matrices G(x) (for $x \in M$). For the existence of a volume form: Consider the extended *n*-form $\overline{\omega}$, given by the formula

$$\overline{\omega}(x_0, x_1, \dots, x_n) := \frac{\sqrt{\det G(x_0)}}{n!} \cdot \det(x_1 - x_0, \dots, x_n - x_0)$$

for any 2-infinitesimal *n*-simplex $X = (x_0, \ldots, x_n)$. Let Y denote the $n \times n$ matrix with $x_i - x_0$ as its *i*th column. Then squaring the defining equality for $\overline{\omega}$ gives

$$\overline{\omega}^2(X) = \frac{\det G(x_0)}{n!^2} \cdot (\det Y)^2 = \frac{1}{n!^2} \det(Y^T \cdot G(x_0) \cdot Y)$$
(12)

using the product rule for determinants and $det(Y^T) = det(Y)$. By definition of Gram, the equation continues

$$= \frac{1}{n!^2} \operatorname{Gram}_{G(x_0)}(X) = \operatorname{Heron}_{G(x_0)}(X) = \operatorname{Heron}_G(X),$$

³Recall from the theory of combinatorial differential forms ([4] (3.1.7)) that $f \cdot \delta$ is the *n*-form given by $(f \cdot \delta)(x_0, \ldots, x_n) := f(x_0) \cdot \delta(x_0, \ldots, x_n)$.

using the Heron-Gram comparison Proposition 1.3 and Proposition 3.3. This proves the existence of the claimed differential n-form.

For the uniqueness, if we have two positive *n*-forms $\omega_i = f_i \cdot \delta$ (i=1,2) with $f_i : M \to P \subseteq R$, we get for all 2-infinitesimal *n*-simplices $X = (x_0, \ldots, x_n)$ that

$$((f_1(x_0))^2 \cdot \delta(X)) \cdot \delta(X) = \operatorname{Heron}_G(X) = ((f_2(x_0))^2 \cdot \delta(X)) \cdot \delta(X),$$

and cancelling successively the two factors $\delta(X)$ (using the uniqueness of the fs describing n-forms ω_i in terms of δ), we ultimately arrive at $f_1(x_0)^2 = f_2(x_0)^2$, and since $f_i(x) \in P \subseteq R$ for all $x \in M$, we conclude from uniqueness of positive square roots that $f_1(x_0) = f_2(x_0)$. Since this holds for all 2-infinitesimal n-simplices (x_0, \ldots, x_n) , we conclude that $f_1 = f_2$, proving the uniqueness.

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