Commutative monads, distributions, and differential categories

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Abstract We describe the relationship between the theory of commutative monads on a cartesian closed category, and distribution theory (in the sense of Schwartz) inside this category. We also indicate how differential categories grow out of suitable commutative monads.

The only data assumed for the theory presented is: a strong monad on a cartesian closed category. All the rest depend on properties of this monad, not on any further structural data.

1 The data

The set of extensive quantities of a given type T on a space X is an object $T(X)$ with some additive, or linear, structure. And, crucially, $T(X)$ depends functorially on X in a *covariant* way, (unlike intensive quantities on X, which behave contravariantly on X). The set $T(X)$ may be thought of as the set of possible *distributions* of some kind of quantity (say, mass) on the space X. This is the basic conception in Lawvere's theory of extensive/intensive quantities. Here, X is thus an object in a category $\mathscr{E}$ of *spaces*. I consider the situation where $T(X)$ is, by forgetting the possible linear structure, itself a space; so T is an endofunctor $\mathscr{E} \to \mathscr{E}$ on a category of spaces, which is assumed to be cartesian closed. The desired linear structure on $T(X)$ comes about canonically from certain properties of $\mathscr{E}$ and T.

Since $\mathscr{E}$ is a closed category, we can talk about categories, functors, and natural transformations *enriched* over $\mathscr{E}$.

In particular, $\mathscr{E}$ is enriched over itself; the inner hom $\mathscr{E}(X,Y)$ is for cartesian closed categories usually denoted Y^X (or sometimes $X \Rightarrow Y$); we shall write $X \pitchfork Y$,

$$X \pitchfork Y := Y^X.$$

The data for an enrichment of a functor $T : \mathscr{E} \to \mathscr{E}$ is given by a "combinator" st: for each pair X, Y of objects, there is given a map

$$st_{X,Y} : X \pitchfork Y \to T(X) \pitchfork T(Y)$$

satisfying certain equations. In the context of endofunctors, as here, the term *strength* is also used for such enrichment, hence the notation st.

In my 1970-1972 papers, I proved that the strength could equally well be encoded in "tensorial form" as a combinator

$$t''_{X,Y} : X \times T(Y) \to T(X \times Y),$$

for all X, Y, or in "cotensorial form" as a combinator

$$\lambda_{X,Y} : T(X \pitchfork Y) \to X \pitchfork T(Y).$$

for all X, Y. (The combinators t'' and λ are actually *mates* of each other.)

I shall concentrate on the cotensorial form. For the case where $\mathscr{E}$ is the category of sets, we may reinterpret $X \pitchfork Y$ as $\prod_X Y$, and then $\lambda_{X,Y}$ is the map which for each $x \in X$ makes the following diagram commute

$$\begin{tikzcd}
T(X \pitchfork Y) \arrow[r, equal] & T(\prod_{x\in X} Y) \arrow[r, "\lambda_{X,Y}"] \arrow[dr, "T(pr_x)"'] & \prod_{x\in X} T(Y) \arrow[r, equal] \arrow[d, "pr_x"] & X \pitchfork T(Y) \\
 & & T(Y) &
\end{tikzcd}$$

where pr_x is projection to the xth factor.

If (T, η, μ) is a monad on $\mathscr{E}$, there are rather obvious equations relating the strength with η, and μ, stating that η and μ make (T, η, μ) into a *strong* (= $\mathscr{E}$-enriched) monad, see [K70a] Section 1. (where the strengh is expressed in tensorial form).

The data for the rest of the talk is: a CCC $\mathscr{E}$, a monad $T = (T, \eta, \mu)$ on $\mathscr{E}$, and a strength on T, say, in cotensorial form λ. All the rest is concerned with properties *of these data.*

The first observation is that the category $\mathscr{E}^T$ of (Eilenberg-Moore) algebras is canonically enriched in $\mathscr{E}$ (assuming that $\mathscr{E}$ has equalizers). This goes back to Bunge (1969) [Bu69]. Namely, if (A,α) and (B,β) are algebras, then the $\mathscr{E}$ valued hom-object $(A,\alpha) \pitchfork_T (B,\beta)$ is the equalizer of the two maps

$$A \pitchfork B \overset{\sharp(\beta)}{\underset{\alpha^*}{\rightrightarrows}} T(A) \pitchfork B, \tag{1}$$

where $\sharp(\beta)$ is an instance of the construction which "to a map $X \to B$ extends it to a T-algebra map $T(X) \to B$, using that $T(X)$ is the free T-algebra on X"; more precisely, $\sharp(\beta)$ is here the composite

$$A \pitchfork B \xrightarrow{st_{A,B}} T(A) \pitchfork T(B) \xrightarrow{\beta_*} T(A) \pitchfork B$$

(α^* and β_* denote precomposition with α, respectively postcomposition with β – so for instance α^* is $\alpha \pitchfork B$).

The next observation is that $\mathscr{E}^T$ is cotensored over $\mathscr{E}$; if $X \in \mathscr{E}$ and $(B,\beta) \in \mathscr{E}^T$, then the cotensor $X \underline{\pitchfork} (B,\beta)$ has for its underlying object $X \pitchfork B \in \mathscr{E}$, and for its structure map $T(X \pitchfork B) \to X \pitchfork B$, it has the composite

$$T(X \pitchfork B) \xrightarrow{\lambda_{X,B}} X \pitchfork T(B) \xrightarrow{\beta_*} X \pitchfork B.$$

In the category of sets, this is just the coordinatewise T-structure of $\prod_X B$.

Now to say that $\underline{\pitchfork}$ is a cotensor formation is to say that (for $(B,\beta) \in \mathscr{E}^T$) the contravariant functors $- \pitchfork_T (B,\beta) : \mathscr{E}^T \to \mathscr{E}$ and $-\underline{\pitchfork}(B,\beta) : \mathscr{E} \to \mathscr{E}^T$ are (strongly) adjoint to each other on the right, and therefore produce a strong monad on $\mathscr{E}$, $X \mapsto ((X \underline{\pitchfork} (B,\beta)) \pitchfork_T (B,\beta)$, or simplifying notation by omitting β and the distinction between $X \pitchfork B$ and $X \underline{\pitchfork} (B,\beta)$,

$$X \mapsto (X \pitchfork B) \pitchfork_T B,$$

the "*(restricted) double dualization monad*" with $B = (B,\beta)$ as dualizing object. If we take T to be the identity monad, we obtain an instance of an "unrestricted" double dualization monad, $X \mapsto (X \pitchfork B) \pitchfork B$. The unit $y_{X,B}$ for both is (in the set case) the standard $x \mapsto ev_x$, a "Dirac delta".

2 Semantics of $T(X)$ in a T-algebra $B = (B,\beta)$

It is old wisdom from the 1960s that monads on the category of sets are essentially the same as (infinitary) Lawvere theories (Linton et al. [Li69a], see also [W70] or [M76]). This generalizes to strong monads on CCCs. In terms of "elements", if T is such a monad, an element of $T(X)$ has a semantics as an X-ary operation on any T-algebra (B,β). An X-ary operation on an object Y is a map $X \pitchfork Y \to Y$. Thus, the object of X-ary opertions on Y is $(X \pitchfork Y) \pitchfork Y$. To "interpret" $T(X)$ as X-ary operations on Y therefore means to give a map

$$\tau : T(X) \to (X \pitchfork Y) \pitchfork Y$$

(cf. [K70b]). So to say that $T(X)$ interprets naturally as an object of X-ary operations on B means in element-free terms that, whenever (B,β) is a T-algebra, we have a canonical map $\tau_{(X,(B,\beta))}$, or τ_B for short,

$$\tau_B : T(X) \to (X \pitchfork B) \pitchfork B$$

which is natural in $(B,\beta) \in \mathscr{E}^T$. We do have such a map, namely $\tau_B := \sharp(y_{X,B})$, i.e. the unique T-algebra map which restricts along η_X to $y_{X,B}$. (Recall that $(X \pitchfork B) \pitchfork B$ inherits a T-algebra structure from (the second occurrence of) B, being a cotensor of the algebra $(B,\beta) \in \mathscr{E}^T$.)

Again, in terms of elements: if $P \in T(X)$, we think of P as an "abstract X-ary operation on T-algberas", i.e. a syntactic entity, whereas $\tau_B(P) \in (X \pitchfork B) \pitchfork B$ is the semantics of P in the "model" (B,β) for the "theory" T.

3 The Schwartz paradigm

Recall that a compactly supported *distribution* (in the sense of Schwartz) on a smooth manifold X is a continuous and $\mathbb{R}$-valued linear functional on the topological vector space $C^\infty(X,\mathbb{R})$. In suitable cartesian closed concrete categories $\mathscr{E}$ containing the category of smooth manifolds, this may even be extended to apply to all objects X of $\mathscr{E}$, and is then a particular case of a double dualization monad S, with $C^\infty(X,\mathbb{R}) = X \pitchfork \mathbb{R}$,

$$S(X) = (X \pitchfork \mathbb{R}) \pitchfork_{\text{cont.linear}} \mathbb{R}.$$

This is not apriori of the form $(X \pitchfork \mathbb{R}) \pitchfork_T \mathbb{R}$ for some monad T on $\mathscr{E}$. There are cases where S is of this form, in fact with $T = S$; a general theorem to this effect has recently been proved in [LW12].

Borrowing terminology from the Schwartz paradigm, we may in a restricted double dualization monad $(- \pitchfork B) \pitchfork_T B$ call $X \pitchfork B$ the "object of (B-valued) *test functions* on X", and $(X \pitchfork B) \pitchfork_T B$ the "object of B-valued (compactly supported) Schwartz[1] *distributions* on X". Note that the object of test functions depend contravariantly on X, whereas the object of distributions behaves covariantly. This is a basic distinction in Lawvere's "intensive/extensive-quantities" dialectics.

(In Schwartz' theory, the main emphasis is on test functions of *compact support*; the continuous dual are then formed by distributions of not-necessarily compact support, like the uniform distribution on $X = \mathbb{R}$ (which is Lebesgue measure). But compactly supported *functions* only has functorality w.r.to *proper* maps, so the corresponding distributions also only have such restricted functorality. So, unfortunately, non-compact distributions have a more complicated theory than the one to be presented here.)

A main contention of the present account is that *the notion of distribution is a more basic thing than the distribution notion in the sense of Schwartz*, - e.g. the distribution P of tomato on a pizza X is a notion that does not depend on any double dualization. Such mass distributions should, modulo choice of a unit for mass, have an abstract theory in its own right; this is one of the aims of the monad theoretic formulations[2]. On the other hand, to have a calculus of such "concrete" distributions, it is expedient to have the Schwartz distribution notion available. Our theory amounts to saying that a strong monad T on $\mathscr{E}$ will serve as a theory of distributions if it is *commutative*; then $T(X)$ may be seen as the space of distributions P on X, and at the same time will admit canonical comparison τ with Schwartz distributions:

4 Commutativity of a strong monad

One succinct way of defining when a strong monad T is commutative is to say:

for any T-algebra (B,β), and any $X \in \mathscr{E}$, the map $\sharp(\beta) : X \pitchfork B \to T(X) \pitchfork B$ is a morphism of T-algebras.

(Both the domain and codomain here have "coordinatewise" T-algebra structure, as described above, in terms of the cotensorial strength λ.) There are several other, equivalent formulations of commutativity, see Theorem

[1]terminology of [LW12]: the "natural distributions" on X (relative to the monad T; ([LW12] only considers the case $B = R$.)

[2]Manes also advanced this viewpoint about general distributions in [M82], when $\mathscr{E}$ is the category of sets, and he gave several interesting examples.

9.2 in [K12a]; the first one proposed was the validity of a certain "Fubini Theorem", see [K70a]. Here it was proved (Theorem 3.2) that a commutative monad canonically carries structure

$$\psi_{X,Y} : T(X) \times T(Y) \to T(X \times Y) \tag{2}$$

making (T, η, μ) into a symmetric monoidal monad.

We assume henceforth that T is commutative. Then both the maps whose equalizer is $A \pitchfork_T B$ as described in (1) above, are algebra maps, hence the equalizer object $A \pitchfork_T B$ is a sub*algebra* of $A \pitchfork B$, and the algebras $A \pitchfork B$ equip $\mathscr{E}^T$ with the structure of a symmetric closed category, see [K71a]. So in particular $(X \pitchfork B) \pitchfork_T B$ is a subalgebra of $(X \pitchfork B) \pitchfork B$. Now, $y_{X,B} : X \to (X \pitchfork B) \pitchfork B$ factors through $(X \pitchfork B) \pitchfork_T B$ (whether or not $(X \pitchfork B) \pitchfork_T B$ is a subalgebra); but if it is a subalgebra, it follows that the T-homomorphic extension of y, namely τ, does so as well, so we have a T-homomorphism (diagonal) filling the commutative square

$$\begin{tikzcd}
(X \pitchfork B) \pitchfork_T B \arrow[r, "\subseteq"] & (X \pitchfork B) \pitchfork B \\
X \arrow[u, "y"] \arrow[r, "\eta_X"'] & T(X), \arrow[u, "\tau"'] \arrow[ul]
\end{tikzcd}$$

verbally, "the semantics $\tau(P)$ of any operation P of the theory T is a T-homomorphism", – one of the classical ways of stating commutativity of an algebraic theory.

Thus, for a commutative monad T and for any T-algebra $B = (B, \beta)$, and for any $X \in \mathscr{E}$, we have a canonical comparison map

$$\tau_{X,B} : T(X) \to (X \pitchfork B) \pitchfork_T B,$$

which is a T-homomorphism, and which extends $y : X \to (X \pitchfork B) \pitchfork_T B$. It is natural in $X \in \mathscr{E}$, and natural (in the standard extraordinary sense) in $(B, \eta) \in \mathscr{E}^T$.

In particular, for fixed X, we get a T-homomorphism into the *end* of all the $(X \pitchfork B) \pitchfork_T B$, as $B = (B, \beta)$ ranges over $\mathscr{E}^t$,

$$\overline{\tau} : T(X) \to \int_{(B,\beta) \in \mathscr{E}^T} (X \pitchfork B) \pitchfork_T B, \tag{3}$$

natural in X. It is actually split monic; a retraction is provided by first taking the projection to the factor with $(B,\beta) = (T(X),\mu_X)$, and then applying "evaluation at $\eta_X \in X \pitchfork T(X)$", which is a map

$$(X \pitchfork T(X)) \pitchfork_T T(X) \to T(X).$$

(See the proof of Proposition 11.1 in [K12a].) I conjecture that the right hand side of (3) is the functor part of strong monad $\overline{T}$, which in some sense should be the completion of T.

Here is the example which it is good to keep in mind, in some of its aspects: $\mathscr{E}$ is the category of sets, and $T(X)$ is the free real vector space on X; so the elements of $T(X)$ are the formal real combinations of elements from X (such a formal combination of course only involves finitely many xs.)

The algebras for T are the real vector spaces B. The reader may want to describe the map $\tau : T(X) \to (X \pitchfork B) \pitchfork_T B$ (in particular when B is $\mathbb{R} = T(\mathbf{1})$), but it is not so important, – $T(X)$ is clear enough without this τ.

This is just the point!

There is clearly some important aspect missing in this example; $T(\mathbb{R})$ (= $T^2(\mathbf{1})$) does make sense, but to form it, one starts by considering $\mathbb{R}$ just as a set, thus neglecting all the important cohesion (topology, diffeology, or bornology, ..., say) which makes the geometric continuum $\mathbb{R}$ more than just a heap of points. This illustrates why our theory has to be built on a cartesian closed category $\mathscr{E}$ of *spaces* with cohesion of some kind, encoded by $\mathscr{E}$.

It will appear that the theory of commutative monads, their algebras, and homomorphisms, has so many features in common with concepts from abstract linear algebra that it is convenient to change terminology: T-algebras we shall also call *T-linear spaces*, and T-homomorphisms, we shall also call *T-linear maps*. One of these features is that there is a good notion of *T-bilinearity* (= "bi-T-homomorphisms"). This is partly elaborated in [K71b]; If $(A,\alpha),(B,\beta),(C,\gamma)$ are T-linear spaces (= T-algebras), a map $f : A \times B \to C$ is *T-bilinear* (cf. [K71b]) if its exponential transpose $A \to B \pitchfork C$ is T-linear, and factors through $B \pitchfork_T C$. (The first clause may be stated: the map f is T-linear in the first variable; the second clause may similarly be stated: the map f is T-linear in the second variable.) The notion of T-bilinearity can also be expressed, more symmetrically, in terms of the monoidal structure $\psi_{X,Y} : T(X) \times T(Y) \to T(X \times Y)$ of T, cf. [K71b]; see also [Li69b]. The map $\psi_{X,Y}$ is itself bilinear, and in fact, by Proposition 9.4 in [K12], $\psi_{X,Y}$ is universal with this property.

A coherence result in [DP09] actually implies that there is a multicategory arising, with “k-linear maps ($k = 0, 1, 2, \ldots$)”.

The codomain of a universal T-bilinear map, out of $A \times B$, if it exists, deserves the notation $A \otimes B$. So in particular $T(X \times Y)$ deserves the alternative notation $T(X) \otimes T(Y)$. Under mild assumptions on $\mathcal{E}^T$ (existence of certain coequalizers), for any two T-algebras (A, α) and (B, β), such $A \otimes B$ exist, and they make $\mathcal{E}^T$ into a symmetric monoidal closed category (cf. [K71a] for the “closed” part); the inner hom functor has for its underlying object $A \pitchfork_T B$; as an object in $\mathcal{E}^T$, i.e. equipped with its T-linear structure induced from the one of $A \pitchfork B$, we also denote $A \pitchfork_T B$ by $A \multimap B$. To make the formulations simpler, we assume this weak completeness property so that we can talk about $A \otimes B$, also when A and B are not free T-algebras.

Summarizing, (assuming the requisite coequalizers in $\mathcal{E}^T$):

If T is a commutative monad, then $\mathcal{E}^T$, with $\otimes$ and $\multimap$, is a symmetric monoidal closed category. The unit object is $R := T(\mathbf{1})$.

Since $\mathbf{1}$ trivially is a commutative monoid in $\mathcal{E}$, and ψ is symmetric monoidal, and T-bilinear, it follows that $T(\mathbf{1})$ carries a canonical commutative and T-bilinear multiplication, namely $\psi_{\mathbf{1},\mathbf{1}} : T(\mathbf{1}) \times T(\mathbf{1}) \to T(\mathbf{1} \times \mathbf{1}) \cong T(\mathbf{1})$.

5 Upside down

The category $\mathcal{E}$ is the base, upon which a superstructure $\mathcal{E}^T$ of “linear” (more precisely, T-linear) objects and maps has been built. We now turn things upside down, and discuss what $\mathcal{E}$ looks like when $\mathcal{E}^T$ (= the linear world) is seen as basic. This project, one may call *linear logic*.

First of all, since we have adjoint functors $F : \mathcal{E} \to \mathcal{E}^T$ and $U : \mathcal{E}^T \to \mathcal{E}$ with $F \dashv U$ (and $U \circ F = T$), we have a comonad $! := F \circ U$ on $\mathcal{E}^T$.

(Here, $F(X) := (T(X), \mu_X)$, and $U(A, \alpha) := A$, the standard Eilenberg-Moore factorization of the monad T.)

We consider the coKleisli category $(\mathcal{E}^{(T)})_!$ for the comonad !. So its objects are those of $\mathcal{E}^{(T)}$; the set of arrows in $(\mathcal{E}^{(T)})_!$ from (A, α) to (B, β) may be identified with the maps in $\mathcal{E}$ from A to B; for $!(A, \alpha) = F(U(A, \alpha)) = F(A)$, thus the Kleisli maps from (A, α) to (B, β) are the maps in $\mathcal{E}^T$ from $F(A)$ to (B, β), and by $F \dashv U$, they are in bijective correspondence with maps $A \to U(B, \beta) = B$ in $\mathcal{E}$. Thus, $(\mathcal{E}^{(T)})_!$ is the full image of the forgetful functor $U : \mathcal{E}^{(T)} \to \mathcal{E}$. If we call the arrows in $\mathcal{E}$ “smooth”, and the objects and arrows in $\mathscr{E}^{(T)}$ "linear" (as short for "T-linear"), the coKleisli category $(\mathscr{E}^{(T)})_!$ of ! is thus the category of smooth (but not necessarily linear) maps between linear spaces coming from $\mathscr{E}^{(T)}$.

The coKleisli category will be *cartesian closed*, since $\mathscr{E}^{(T)}$ is cotensored over $\mathscr{E}$, and the full and faithful functor to $\mathscr{E}$ preserves the cartesian closed structure.

The difference between $\mathscr{E}$ and $(\mathscr{E}^{(T)})_!$ is analogous to the difference between differential *geometry* and (coordinate free) differential *calculus*, the latter being the part of differential geometry dealing with smooth (globally defined, but not necessarily linear) maps between linear spaces coming from $\mathscr{E}^{(T)}$. The cartesian closed category of smooth maps between convenient vector spaces is an example of an $(\mathscr{E}^{(T)})_!$; this follows because the forgetful functor from the category of convenient vector spaces to the category $\mathscr{E}$ of Frölicher spaces has a (strong) left adjoint, cf. [FK88] and [BET12].

Proposition 1 *Any object $!(A,\alpha) \in \mathscr{E}^T$ carries a canonical structure of commutative comonoid w.r.to $\otimes$; and any map of the form $!(f)$ is a comonoid homomorphism.*

Proof/construction. We have in $\mathscr{E}$ the diagonal map $A \to A \times A$, and hence in $\mathscr{E}^{(T)}$, we have $F(\Delta) : F(A) \to F(A \times A)$. Since $!(A,\alpha) = F(A)$, and since

$$!(A,\alpha) \otimes !(A,\alpha) = F(A) \otimes F(A) = F(A \times A),$$

we get the comultiplication $!(A) \to !(A) \otimes !(A)$ from $F(\Delta)$ using these identifications. Similarly for the counit, using $A \to \mathbf{1}$ in $\mathscr{E}$. The fact that the laws (associative, etc.) hold follows because they hold for the commutative monoid structure $\Delta_A : A \to A \times A$ in $\mathscr{E}$. The proof of the last (naturality) statement is clear.

6 R-linear structure

We write the (commutative) multiplicative structure of $T(\mathbf{1})$ multiplicatively; for, we would like it to be the multiplication for a *ring*- (or at least, a rig-) structure on $T(\mathbf{1})$. In the example where $T(X)$ is "free real vector space on X", we have $T(\mathbf{1}) = \mathbb{R}$, so we denote from the outset $T(\mathbf{1})$ by R. The additive aspect of such ring structure on $R := T(\mathbf{1})$ comes about canonically from certain *properties* of T and $\mathscr{E}$, as we shall now describe. For $\mathscr{E}$, we assume that it has finite coproducts. For T, we assume $T(\emptyset) = \mathbf{1}$; this object then serves as a zero object $\mathbf{0}$ in $\mathscr{E}^T$, and using $\mathbf{0}$ one can construct canonically a map $T(X+Y) \to T(X) \times T(Y)$, and we require it to be an isomorphism in $\mathscr{E}$. In [CJ10], and in [K11], and presumably also in other places, it is shown that this implies that the category $\mathscr{E}^T$ has biproducts $\oplus$; in particular,

$$T(X+Y) = T(X) \times T(Y) = T(X) \oplus T(Y).$$

As is known from early days (at least since [G62]), this implies that the category $\mathscr{E}^T$ is semiadditive (= enriched in abelian monoids). We shall later need the that the pair ($\mathscr{E}$, T) has the *property* that these abelian monoids are actually abelian groups.

Assuming these properties of $\mathscr{E}$, T, we thus get that R is a (commutative) ring object in $\mathscr{E}$, and that the forgetful functor $\mathscr{E}^T \to \mathscr{E}$ factors through the category R-$\mathrm{Mod}_{\mathscr{E}}$.

$$\mathscr{E}^T \longrightarrow R\text{-}\mathrm{Mod}_{\mathscr{E}} \longrightarrow \mathscr{E};$$

thus, "T-linear is stronger than R-linear".

The multiplication of $R = T(\mathbf{1})$ is not only R-bilinear, but is T-bilinear; for, the multiplication of R is simply $\psi_{\mathbf{1},\mathbf{1}}$.

For a T-algebra (B, β), the fundamental "semantics" map $\tau : T(X) \to (X \pitchfork B) \pitchfork_T B$ is by construction T-linear, and hence its exponential transpose $X \times (X \pitchfork B) \to B$ is T-bilinear, and it deserves a special "pairing" notation;

$$T(X) \times (X \pitchfork B) \xrightarrow{\langle -,- \rangle} B,$$

thus if $P \in T(X)$ (a "distribution on X") and $\phi \in X \pitchfork B$ (a "(B-valued) test function on X"), then $\langle P, \phi \rangle \in B$ is the "value of the distribution on the test function"; and sometimes it is even useful with an integral notation with a dummy variable x ranging over X:

$$\langle P, \phi \rangle = \int_X \phi(x)\, dP(x)$$

("vector valued integration"; recall that B was assumed to be a T-linear space ("vector space")). The naturality of τ w.r.to X can be expressed in terms of the "pairing" notation as follows; for $f : X \to Y$, $P \in T(X)$, and $\phi \in Y \pitchfork B$, we have

$$\langle f_*(P), \phi \rangle = \langle P, f^*(\phi) \rangle; \tag{4}$$

here, $f_* = T(f)$ expresses the covariant functorality of the distribution notion T, whereas $f^* = f \pitchfork B$ expresses the contravariant functorality of the space of test functions (thus $f^*(\phi) = \phi \circ f$).

7 Differential calculus

The ability to differentiate distributions is a prerequisite for some of the most important applications of distribution theory. To deal with it in our context, we shall assume the properties discussed in Section 6, so that any T-linear space has an underlying R-module object in $\mathscr{E}$; and we shall assume that T-linear spaces qua R-modules have the KL property as in [K81]. Furthermore, we shall allow ourselves to talk "synthetically," meaning in terms of elements[3]. We can then describe the "derivative of a distribution $P \in T(A)$ with respect to a vector field on A". We shall only discuss the case where A is a T-linear space, so in particular, A is an R-module. Then any vector v (element in A) gives rise to a constant vector field, by parallel translation. We here only consider derivatives w.r.to such vector fields, and we phrase it without reference to the vector field notion. For any $w \in A$, we have the map "parallel translation by w". It is the map $e^w : A \to A$ given by $e^w(a) = a + w$. It is the identity map if $w = 0$. Let $P \in T(A)$, and let $v \in A$. Then we get a new distribution $d_v(P) \in T(A)$; it is defined by validity, for all $d \in D$, of

$$d \cdot d_v(P) = (e^{d \cdot v})_*(P) - P$$

(recall that f_* denotes $T(f)$). In a similar vein, if $\phi \in A \pitchfork B$ is a B-valued function (with $B = (B, \beta)$ a T-linear space), we may define $d_v(\phi)$ by validity, for all $d \in D$, of

$$d \cdot d_v(\phi) = (e^{d \cdot v})^*(\phi) - \phi.$$

Recall that $f^*(\phi) = \phi \circ f$. Unravelling the definitons, the right hand side here is the function $A \to B$ given by $x \mapsto \phi(x + d \cdot v) - \phi(x)$, so we recover the standard definition of the directional derivative of the function ϕ in the direction of v.

From the naturality property (4) of the pairing, (and cancellation of d), one immediately gets as a consequence that

$$\langle d_v(P), \phi \rangle = \langle P, d_v(\phi) \rangle,$$

which (except for sign) is the classical *definition* of directional derivation in direction v of the distribution P in terms of "test" functions ϕ; however, our description of $d_v(P)$ does not mention test functions.

[3]in these terms, $D \subseteq R$ is the subobject of elements $d \in R$ with $d^2 = 0$, and the KL property for an R-module W says that any $f : D \to W$ is of the form $d \mapsto a + d \cdot b$ for unique a and b in W.

If we record this construction $d_v(P)$ in its dependence both on $v \in A$ and on $P \in T(A)$, we thus get a combinator

$$d_A : A \times T(A) \to T(A),$$

which is T-linear in the second variable P, essentially since $(e^{d \cdot v})_*$ is T-linear. It is also T-linear in the variable v; this apparently does not follow purely formally. It can be proved if one assumes that the two forgetful functors $\mathscr{E}^T \to R\text{-Mod}_{\mathscr{E}}$ and $R\text{-Mod}_{\mathscr{E}} \to (R,\cdot)\text{-act}$ are full. Here, $(R,\cdot)\text{-act}$ is the category of objects of $\mathscr{E}$ equipped with an action by the multiplicative monoid of R. The latter fullness is true in SDG context, where it is usually stated: if a map between R-modules (satisfying KL) is homogeneous (commutes with multiplication of scalars), then it is R-linear, see [K81] Proposition I.10.2. A similar fullness is also known for in the standard smooth world: a smooth and homogeneous map between vector spaces (finite dimensional, say) is necessarily linear. The fullness of $\mathscr{E}^T \to R\text{-Mod}_{\mathscr{E}}$ is related to "density of the free R-modules in the free T-linear spaces" - this still has to be investigated for a precise formulation, and investigation. Let us assume this property.

The exponential adjoint of the combinator d_A is a T-linear

$$T(A) \quad \to \quad A \pitchfork_T T(A). \tag{5}$$

Recall that $T(A)$, with its algebra structure μ_A, is the same thing as the object $!(A) \in \mathscr{E}^T$. Therefore (5) is the same thing as a map in $\mathscr{E}^T$, namely

$$!(A) \quad \to \quad A \multimap !(A)$$

which in turn by the $(- \otimes A) \dashv (A \multimap -)$ adjointness in $\mathscr{E}^T$ corresponds bijectively to a combinator

$$A \otimes !(A) \quad \to \quad !(A),$$

which is the data for a "deriving transform" [BCS06], equipping $\mathscr{E}^T$ with the structural data for a differential category, [BCS06]. I have not checked that the four equations required for a deriving transform (loc.cit. Definition 2.5); the fact that (5) grows out of the SDG situation where precisely the equations of differential calculus can be proved, should, by the intention of [BCS], imply that their four equations could not be stronger than those of ordinary differential calculus.

Remark. We have the space $D \in \mathscr{E}$. Then $T(D)$ (= the free T-linear space on D) is isomorphic to $R \times R$, assuming that all T-linear spaces have the KL property. More precisely, consider the map $\eta : D \to R \times R$ given by $d \mapsto (1, d)$. It has the universal property required for a free T-linear space on D. For, if W is a T-linear space, and $t : D \to W$ an arbitrary map in $\mathscr{E}$, then t is of the form $d \mapsto a + d \cdot b$ for unique a and b in W. We then describe a T-linear map $\hat{t} : R \times R \to W$ with $\hat{t} \circ \eta = t$, namely the one given by the matrix $[a, b]$, i.e. $\hat{t}(x, y) = x \cdot a + y \cdot b$. If $\tilde{t} : R \times R \to W$ is another such extension with matrix $[\tilde{a}, \tilde{b}]$, then since $\tilde{t}$ and $\hat{t}$ agree on D, we have in particular for all $d \in D$ that $\tilde{a} + d \cdot \tilde{b} = \hat{a} + d \cdot \hat{b}$, whence $\tilde{a} = \hat{a}$ (by putting $d = 0$), and then also $\tilde{b} = \hat{b}$ by "cancelling universally quantified ds". This proves the uniqueness of the extension.

The argument internalizes. Also if $\mathscr{E}^{(T)}$ is a full subcategory of $\mathscr{E}^T$ containing all free T-algebras, and whose objects all satisfy KL property, then $R \times R$ will have the universal property required for $T(D)$, with respect to objects in $\mathscr{E}^{(T)}$, or, expressed in slogan form, "T-linear spaces which are KL *perceive* that $T(D)$ is $R \times R$".

So we have here the phenomenon that

$$T(D) \cong R \times R \cong T(1+1),$$

without $D \cong 1 + 1$ (which is well known to be incompatible with KL). The difference between $T(D)$ and $T(1+1)$ reveals itself in that the canonical comonoid structures on them (cf. the proof of Proposition 1) are different; the comonoid structure on $T(D)$, when exported to $R \times R$ via the isomorphism $T(D) \cong R \times R$, gives by dualization (i.e. by applying the functor $(-)^* = - \multimap R$), a monoid structure on $R \times R$, which is the multiplication on $R[\epsilon]$, the "ring of dual numbers over R". In brief

$$T(D)^* = R[\epsilon].$$

8 When is the Schwartz distribution monad commutative ?

Recently, [LW12] has proved a result about commutativity of Schwartz distribution monads relative to a commutative monad T on a CCC $\mathscr{E}$.

He observes first that

$$(X \pitchfork R) \pitchfork_T R \cong (F(X) \pitchfork_T R) \pitchfork_T R = F(X)^{**} \tag{6}$$

(where $X \pitchfork R \cong T(X) \pitchfork_R R$ holds because $T(X)$ is free on X). Here V^* denotes $V \multimap R$, the standard linear dualization functor in $\mathscr{E}^T$.

This $(-)^{**}$ is a monad on $\mathscr{E}^T$ (enriched over the closed category $\mathscr{E}^T$, cf. [K70b]). It is denoted $\mathbb{H}$ in [LW12]. It is usually non-commutative (this notion makes sense not just for CCCs , but for general symmetric monoidal closed categories, [K70a]); and $\tau : T(X) \to (X \pitchfork R) \pitchfork_T R$ identifies via (6) with the usual map into double dual from linear algebra. The monad $\mathbb{H}$ has an Eilenberg-Moore factorization, displayed as the right hand pair of functors in (7) below. If $\mathscr{L} \subseteq \mathscr{E}^T$ is a full subcategory stable under the closed structure $\multimap$, and containing the free T-algebras $F(X)$, we have composable adjoint functors

$$\mathscr{E} \underset{U}{\overset{F}{\rightleftarrows}} \mathscr{L} \underset{G}{\overset{H}{\rightleftarrows}} \mathscr{L}^{\mathbb{H}} \tag{7}$$

where F and U compose to T, and where $\mathbb{H}$ is the (unrestricted) double dualization $(-)^{**}$ on $\mathscr{L}$, and its Eilenberg-Moore factorization H, G. The composite adjoint pair gives the Schwartz monad relative to T, by (6), but is not apriori commutative (= symmetric monoidal, by [K70a] and [K72]), but [LW12] gives conditions when the composite monad is isomorphic to one of the form

$$\mathscr{E} \underset{U}{\overset{F}{\rightleftarrows}} \mathscr{L} \underset{i}{\overset{r}{\rightleftarrows}} \tilde{\mathscr{L}}$$

where $\tilde{\mathscr{L}}$ is a reflexive subcategory of $\mathscr{L}$ with inclusion i and reflection $r \dashv i$. This composite monad is commutative, by general theory. The subcategory $\tilde{\mathscr{L}} \subseteq \mathscr{L}$ consist of "complete" T-algebras in $\mathscr{L}$, and r is "completion". Thus, the Schwartz monad is commutative, and comes about by completing the free T-algebras. The completion notion is: $V \in \mathscr{L}$ is complete if $V \to V^{**}$ is a strong mono, in the sense of factorization theory, see e.g. [Bo94] I.4.3. The main condition for the theory of [LW12] is that cotensors $X \pitchfork R$ are reflexive (isomorphic to its double dual), a condition known to hold for convergence vector spaces of the form $X \pitchfork R$.

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